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# *Quantum theory of many – particle systems*

Lecture notes for TFY4210

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$$|\psi\rangle = a_{\mathbf{p}}^{\dagger}|0\rangle$$

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# Preface

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## First edition

These lecture notes are written to cover the material taught in TFY4210 Applied quantum mechanics which is advanced course for fourth-year students at NTNU. It covers classical mechanics and classical field theory including Noether's theorem, relativistic wave equations, quantization of fields Dilute Bose and Fermi gases, Bose-Einstein condensation and BCS theory. If you find typos or have comments or suggestions, please email the author: [jensoa@ntnu.no](mailto:jensoa@ntnu.no).

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Jens O. Andersen

## Second edition

The text has been expanded and improved several places. Typos have been corrected and If you find more typos or have comments or suggestions, please email the author: [jensoa@ntnu.no](mailto:jensoa@ntnu.no).

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# Chapter 1

## Classical mechanics

In introductory courses, we have been taught classical mechanics and in particular Newton's second law. There are two other formulations of classical mechanics, which are equivalent to the Newtonian formulation, namely the *Lagrangian* and *Hamiltonian* formulations. In the Newtonian formulation, the concept of *force* is essential, while in the other formulations, the concept of *energy* is essential. These formulations are important in the transition to quantum mechanics and in statistical mechanics. We recommend the classic textbook *Classical Mechanics* [1] by Goldstein for a thorough treatment.

In this chapter, we derive Lagrange and Hamilton's equations in a somewhat sloppy fashion. We will do this by considering a simple example, but the results are valid under very general conditions.

### 1.1 Lagrange's equations

Consider a single particle moving in one dimension in a potential  $V(x)$ . Newton's second law becomes

$$m\ddot{x} = -\frac{dV(x)}{dx}, \quad (1.1)$$

where  $m$  is the mass of the particle and a dot means differentiation with respect to time  $t$ . We next define the *Lagrangian*  $L$  of a particle, which is the difference between its kinetic energy  $T$  and its potential energy  $V$ , i.e.,

$$L = T - V. \quad (1.2)$$

and their time derivatives  $\dot{q}_i$ . In our example, the Lagrangian becomes

$$L = \frac{1}{2}m\dot{x}^2 - V(x). \quad (1.3)$$

Note that  $L$  is a function of the coordinate  $x$  and the time derivative  $\dot{x}$ . Taking the partial derivatives of  $L$  with respect to  $x$  and  $\dot{x}$ , we obtain

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad (1.4)$$

$$\frac{\partial L}{\partial x} = -\frac{dV(x)}{dx} . \quad (1.5)$$

By differentiating the first of these equations with respect to  $t$ , we see that Newton's equation (1.1) can be written as

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x} . \quad (1.6)$$

This is the *Lagrange equation* for a particle in one dimension. We have derived it for a very simple mechanical problem, but the result can be generalized to any number of particles and any number of dimensions. If we consider a system of  $N$  particles in three dimensions, we need  $3N$  coordinates to specify the positions of the particles. These coordinates are called *generalized coordinates* and are denoted by  $q_1, q_2, \dots, q_{3N}$ <sup>1</sup>. The Lagrangian is then a function of the coordinates  $q_1, q_2, \dots, q_{3N}$ , their time derivatives  $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_{3N}$ , and possibly  $t$  explicitly, i.e.  $L(q_1, \dots, \dot{q}_{3N}, t)$ . In condensed notation, we write  $L = L(q, \dot{q}, t)$ . There is one equation for each coordinate  $q_i$  and Lagrange equations are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

The Lagrange equations form a set of second-order differential equations.

If the Lagrangian is independent of one of the coordinates,  $q_i$ , i.e. if

$$\frac{\partial L}{\partial q_i} = 0 , \quad (1.7)$$

then the corresponding coordinate  $q_i$  is called a *cyclic coordinate*. Integration of Lagrange's equation then immediately gives

$$\frac{\partial L}{\partial \dot{q}_i} = C . \quad (1.8)$$

Thus a cyclic coordinate leads to a conserved quantity. Conserved quantities result from symmetries of the system.

## Example

We consider a single particle moving in a plane under the influence of a rotationally symmetric potential  $V(r)$ . The Lagrangian in cartesian coordinates is then given by

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - V\left(\sqrt{x^2 + y^2}\right) . \quad (1.9)$$

<sup>1</sup>We can use any coordinates, cartesian coordinates, spherical coordinates...Hence the name generalized coordinates.

Using the relation between  $x$  and  $y$ , and  $r$  and  $\theta$

$$x = r \cos \theta , \quad (1.10)$$

$$y = r \sin \theta , \quad (1.11)$$

differentiation with respect to time yields:

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta , \quad (1.12)$$

$$\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta . \quad (1.13)$$

Substituting Eqs. (1.12) and (1.13) into Eq. (1.9), the Lagrangian in polar coordinates becomes

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) . \quad (1.14)$$

Note that the  $\theta$  is a cyclic coordinate as it does not appear explicitly in the Lagrangian (1.14). The corresponding conserved quantity is

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} , \quad (1.15)$$

which is nothing but the  $z$ -component of the angular momentum.

---

## 1.2 Hamilton's equations

Given the Lagrangian, we next define the *conjugate momentum* or *generalized momentum*  $p_i$  to a coordinate  $q_i$  by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} . \quad (1.16)$$

If  $q_i$  is a usual cartesian coordinate, the generalized momentum is equal to the familiar linear momentum of a particle. If  $q_i$  is an angular coordinate, as in the example above, the generalized momentum is the angular momentum.

The *Hamiltonian*  $H$  is defined as

$$H = \sum_i p_i \dot{q}_i - L . \quad (1.17)$$

In most cases, the Hamiltonian is the sum of the kinetic and potential energy  $H = T + V$ . We next show that  $H$  is a function of  $p_i$  and  $q_i$  and so can be interpreted as Legendre transform of the Lagrangian. To this end, take the differential of the above equation

$$dH = \sum_i p_i d\dot{q}_i + \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \sum_i \frac{\partial L}{\partial q_i} dq_i \quad (1.18)$$

Using the definition of the generalized momentum  $p_i$ , we see that the first and third term cancel and we are left with

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i . \quad (1.19)$$

Thus the differential of  $H$  is given in terms of the differentials of  $p_i$  and  $q_i$ , which shows that  $H = H(p, q)$  in condensed notation. Using this fact, the differential of  $H$  becomes

$$dH = \sum_i \frac{\partial H}{\partial p_i} dp_i + \sum_i \frac{\partial H}{\partial q_i} dq_i . \quad (1.20)$$

Comparing Eqs. (1.19) and (1.20), we obtain *Hamilton's equations*

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} &= -\dot{p}_i \end{aligned}$$

Notice that Hamilton's equations are first order in time, but there are twice as many as compared to Lagrange equations. The two sets of equations represent two equivalent ways of describing the same physics.

Now reconsider the problem of a single particle moving in one dimension in a potential  $V(x)$ . The linear momentum  $p$  is given by

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} . \quad (1.21)$$

This implies that  $\dot{x} = p/m$ . The Hamiltonian is then

$$\begin{aligned} H &= p\dot{x} - L \\ &= \frac{p^2}{m} - \frac{p^2}{2m} + V(x) \\ &= \frac{p^2}{2m} + V(x) \\ &= T + V . \end{aligned} \quad (1.22)$$

Hence the Hamiltonian is the total energy of the particle. We have shown this for a very simple example, but again it holds very generally for systems of many particles where we are using any coordinate system we wish.

---

### Example

Reconsider the previous example. The generalized momenta  $p_r$  and  $p_\theta$  corresponding to the polar coordinates  $r$  and  $\theta$  follow from the Lagrangian (1.14) and the definition (1.16):

$$p_r = m\dot{r} , \quad (1.23)$$

$$p_\theta = mr^2\dot{\theta} . \quad (1.24)$$

The Hamiltonian  $H$  can then be written as

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L . \quad (1.25)$$

Remember that  $H$  must be expressed in terms of  $p_r$ ,  $p_\theta$ ,  $r$ , and  $\theta$ . It is therefore necessary to solve Eqs. (1.23) and (1.24) with respect to  $\dot{r}$  and  $\dot{\theta}$  in order to eliminate these quantities in  $L$  in favor of the  $p$ 's and the  $q$ 's. Using that  $\dot{r} = p_r/m$  and  $\dot{\theta} = p_\theta/mr^2$ , we obtain

$$\begin{aligned} H &= \frac{1}{m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) - \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r) \\ &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r) . \end{aligned} \quad (1.26)$$

The Hamiltonian is again the sum of the kinetic and potential energy.

Hamilton's equation for the the generalized momentum  $p_\theta$  becomes

$$\begin{aligned} \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \\ &= 0 . \end{aligned} \quad (1.27)$$

Thus  $p_\theta$  is independent of time implying the time independence of  $mr^2\dot{\theta}$ . We again identify this as the  $z$ -component of the angular momentum.

Let us finally look at Hamilton's equation for  $p_r$ . We then have

$$\begin{aligned} \frac{\partial H}{\partial r} &= -\frac{p_\theta^2}{mr^3} + \frac{\partial V}{\partial r} \\ &= -\dot{p}_r \end{aligned} \quad (1.28)$$

Substituting the expressions for  $p_r$  and  $p_\theta$ , and rearranging, we obtain

$$m(\ddot{r} - r\dot{\theta}^2) = -\frac{\partial V}{\partial r} . \quad (1.29)$$

In order to interpret this equation, we need to find an expression for the right-hand side. To this end, we write

$$\mathbf{r} = r\mathbf{e}_r , \quad (1.30)$$

where  $\mathbf{e}_r$  is unit vector in the radial direction. This can be written as

$$\mathbf{e}_r = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y . \quad (1.31)$$

Differentiating  $\mathbf{r}$  with respect to  $t$ , we obtain

$$\begin{aligned} \dot{\mathbf{r}} &= \dot{r} \mathbf{e}_r + r \dot{\mathbf{e}}_r \\ &= \dot{r} \mathbf{e}_r + \dot{\theta} r \mathbf{e}_\theta , \end{aligned} \quad (1.32)$$

where

$$\mathbf{e}_\theta = -\sin \theta \mathbf{e}_x + \cos \theta \mathbf{e}_y , \quad (1.33)$$

is a unit vector in the angular direction. Differentiating  $\dot{\mathbf{r}}$  with respect to time, we get

$$\begin{aligned} \ddot{\mathbf{r}} &= \mathbf{e}_r(\ddot{r} - r\dot{\theta}^2) + \mathbf{e}_\theta(2\dot{r}\dot{\theta} + \ddot{\theta}r) \\ &= \mathbf{e}_r(\ddot{r} - r\dot{\theta}^2) + \mathbf{e}_\theta \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta}) . \end{aligned} \quad (1.34)$$

Using the equation of motion for  $p_\theta$ , i.e. that  $r^2\dot{\theta}$  is time independent, we see that the second term vanishes. We then obtain

$$m\ddot{\mathbf{r}} = -\frac{\partial V}{\partial r} \mathbf{e}_r , \quad (1.35)$$

which is Newton's second law in the radial direction.

Note that the dimension of the conjugate momentum depends on the corresponding coordinate. If we are using cartesian coordinates, it has the well known dimension of mass times velocity. In the above example, it has dimension of angular momentum. Hence the term *generalized momentum* is appropriate.

### 1.3 Particle in external electromagnetic fields

Given the vector potential  $\mathbf{A}$  and the scalar potential  $\Phi$ , the electric and magnetic fields can be written as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} , \quad (1.36)$$

$$\mathbf{B} = \nabla \times \mathbf{A} . \quad (1.37)$$

We next show that the Hamiltonian

$$H = \frac{1}{2m} (\mathbf{p} - q\mathbf{A})^2 + q\Phi , \quad (1.38)$$

where  $q$  is the electric charge of the particles, gives the correct equation of motion. Hamilton's equations give

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p_x} \\ &= \frac{p_x - qA_x}{m},\end{aligned}\tag{1.39}$$

$$\begin{aligned}\dot{p}_x &= -\frac{\partial H}{\partial x} \\ &= q \frac{(\mathbf{p} - q\mathbf{A})}{m} \frac{\partial \mathbf{A}}{\partial x} - q \frac{\partial \Phi}{\partial x},\end{aligned}\tag{1.40}$$

and similarly for the coordinates  $y$  and  $z$ , and the generalized momenta  $p_y$  and  $p_z$ . The first equation shows that the generalized momentum  $\mathbf{p}$  is different from the kinetic momentum  $\pi = m\mathbf{v}$ . The acceleration is given by the time derivative of Eq. (1.39):

$$\begin{aligned}m\ddot{x} &= \dot{p}_x - q \frac{dA_x}{dt} \\ &= \dot{p}_x - q \left( \frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} + \frac{\partial A_x}{\partial t} \right).\end{aligned}\tag{1.41}$$

We next rewrite the time derivative of the generalized momentum as

$$\dot{p}_x = q \left( \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right) - q \frac{\partial \Phi}{\partial x}.\tag{1.42}$$

Inserting Eq. (1.42) into Eq. (1.41) and reorganizing, we obtain

$$\begin{aligned}m\ddot{x} &= q \left[ -\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} \right] + q \left[ \dot{y} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - \dot{z} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] \\ &= q\mathbf{E}_x + q(\dot{y}B_z - \dot{z}B_y).\end{aligned}\tag{1.43}$$

This is simply the  $x$ -component of the Lorentz force

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}.\tag{1.44}$$

Hence the Hamiltonian Eq. (1.38) gives Newton's second law for a charged particle in an external magnetic field. The replacement  $\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}$  in the Hamiltonian is called *minimal substitution*. We shall return to it in the chapters on field theory.

## 1.4 Calculus of variations

We will now discuss an alternative formulation of classical mechanics, namely a *variational formulation*. Since calculus of variations is not restricted to classical physics, we discuss it more generally. Ref. [3] provides a good introduction to variational methods.

Let  $F(y(x), y'(x))$  be a function of  $y(x)$  and  $y'$ <sup>2</sup>, where  $y(a) = y_1$  and  $y(b) = y_2$ . Consider the integral  $I$  defined by

$$I[y] = \int_a^b F(y, y') dx . \quad (1.45)$$

$I$  is a function of a function, i.e. it is a *functional*. Generally, a functional receives a function as argument and returns a number. The problem of *variational calculus* is to find a function  $y_0(x)$ , which extremizes the integral  $I[y]$ , and satisfying the boundary conditions  $y(a) = y_1$  and  $y(b) = y_2$ . We next write an arbitrary function  $y(x)$  satisfying the boundary condition as  $y(x) = y_0(x) + \delta y(x)$ , where  $\delta y$  is the deviation. Note that  $\delta y(a) = \delta y(b) = 0$ . We can then write

$$I[y_0 + \delta y] = \int_a^b F(y, y') dx , \quad (1.46)$$

We next expand  $I$  to first order in the deviation  $\delta y$ . This yields

$$I[y_0 + \delta y] \approx \int_a^b \left[ F(y_0, y'_0) + \frac{\partial F}{\partial y} \Big|_{y=y_0} \delta y(x) + \frac{\partial F}{\partial y'} \Big|_{y'=y'_0} \delta y'(x) \right] dx .$$

Using that  $\delta y'(x) = (\delta y)'$  and integrating the last term in the brackets by parts, we obtain

$$I[y_0 + \delta y] \approx I[y_0] + \frac{\partial F}{\partial y'} \Big|_{y'=y'_0} \delta y \Big|_{x_0}^{x_1} + \int_a^b \left[ \frac{\partial F}{\partial y} \Big|_{y=y_0} - \frac{d}{dx} \frac{\partial F}{\partial y'} \Big|_{y=y_0} \right] \delta y(x) dx \quad (1.47)$$

Since the variation is zero at the end points  $a$  and  $b$ , the boundary term vanishes. Since the variation  $\delta y$  is arbitrary, the integrand in  $\delta I$  must vanish identically, i.e.

$$\frac{\partial F}{\partial y} \Big|_{y=y_0} - \frac{d}{dx} \frac{\partial F}{\partial y'} \Big|_{y=y_0} = 0 . \quad (1.48)$$

The function  $y_0(x)$  must therefore satisfy the equation (1.48).

Note: We have derived a differential equation for a function  $F$ , with a single dependent variable  $y(x)$ . It can be easily generalized to a function  $F = F(y_1, y_2, \dots, y_N)$  of an arbitrary number of dependent variables  $y_i(x)$ .

Note: We have extremized the integral  $I$ , but said nothing about the nature of the extremum, i.e. whether it is a minimum, a maximum, or a saddle point. In order to determine the nature of the extremum, we must look at the second-order variation  $\delta^2 I$ , which is found by expanding  $I$  to second order about  $y_0(x)$ :

$$I = I_0 + \delta I + \delta^2 I + \dots \quad (1.49)$$

The sign of  $\delta^2 I$  determines the nature of the extremum.

<sup>2</sup>Occasionally,  $F$  is also a function of  $x$  explicitly. The arguments clearly go through.

### Example

Let  $f(x)$  be an arbitrary function that satisfies  $f(a) = y_1$  and  $f(b) = y_2$ . The infinitesimal arc length of the curve  $f(x)$  is given by  $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2} dx = \sqrt{1 + (f')^2} dx$  and the total arclength between the points  $(a, y_1)$  and  $(b, y_2)$  is therefore

$$\begin{aligned} L &= \int_a^b ds \\ &= \int_a^b \sqrt{1 + (f')^2} dx . \end{aligned} \quad (1.50)$$

The problem is to find the function that extremizes the distance between the two points  $(a, y_1)$  and  $(b, y_2)$ . The function is  $F = \int_a^b \sqrt{1 + (f')^2} dx$ , which yields

$$\frac{\partial F}{\partial f} = 0 , \quad (1.51)$$

$$\frac{\partial F}{\partial f'} = \frac{f'}{\sqrt{1 + (f')^2}} . \quad (1.52)$$

The equation then becomes

$$\frac{d}{dx} \left[ \frac{f'}{\sqrt{1 + (f')^2}} \right] = 0 , \quad (1.53)$$

Integration then gives

$$\frac{f'}{\sqrt{1 + (f')^2}} = C , \quad (1.54)$$

where  $K_1$  is a constant of integration. Solving this equation with respect to  $f'$  gives

$$f' = \pm \left| \frac{C}{\sqrt{1 - C^2}} \right| . \quad (1.55)$$

Integration then yields

$$f(x) = K_1 x + K_2 , \quad (1.56)$$

where  $K_1$  is given in terms of  $C$  and  $K_2$  is a second integration constant.  $K_1$  and  $K_2$  are determined by requiring  $f(a) = y_1$  and  $f(b) = y_2$ . We see that the solution to the problem is straight line.

In the discussion and example above, we wrote  $y = y(x)$ . However, we could also parametrize both  $x$  and  $y$  by a parameter, for example the arc length  $s$ . One would then

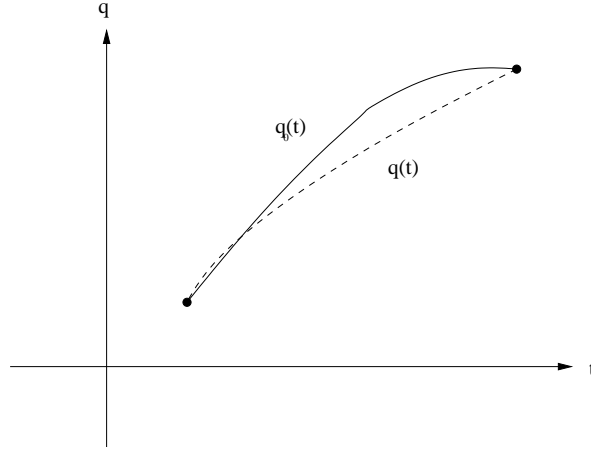


Figure 1.1: Classical trajectory (solid curve -  $q_0(t)$ ) and a deviating curve (dashed curve -  $q(t)$ ).

write  $F(y, x, y', x', s)$ , where  $x = x(s)$  and  $y = y(s)$ . This yields  $ds = \sqrt{(x')^2 + (y')^2} ds$  and requires the simultaneous solution of two coupled equations (see Exercise 1.5.2).

We next use the result (1.48) in classical mechanics by choosing  $F = L$  and  $y(x) = q(t)$ . The boundary conditions  $y(a) = y_1$  and  $y(b) = y_2$  are replaced by  $q(t_0) = q_0$  and  $q(t_1) = q_1$ . The classical path and a deviation is shown in Fig. 1.1 in one dimension with  $q(t) = x(t)$ .

This immediately yields Lagrange's equation<sup>3</sup>

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = \left( \frac{\partial L}{\partial q} \right). \quad (1.57)$$

In classical mechanics the functional  $I$  is denoted by  $S$  and reads

$$S[x(t)] = \int_{t_0}^{t_1} L(q, \dot{q}) dt. \quad (1.58)$$

It is called the *action*. A classical particle is then following a path, which extremizes the action. Since the extremum normally is minimum, the variational principle is also called the *principle of least action*.

## 1.5 Problems

### 1.5.1 Straight line as minimal distance between points I

Show that the straight line is actually a minimum.

---

<sup>3</sup>If there are  $N$  generalized coordinates and momenta, there are  $N$  coupled equations.

### 1.5.2 Straight line as minimal distance between points II

Reconsider the previous problem writing

$$I[y(s), x(s)] = \int_{s_0}^{s_1} F(x(s), y(s), x'(s), y'(s)) ds, \quad (1.59)$$

where  $F = \sqrt{(x')^2 + (y')^2}$  and  $x(s_0) = x_0$ ,  $x(s_1) = x_1$ ,  $y(s_0) = y_0$ , and  $y(s_1) = y_1$ . Find the functions  $x(s)$  and  $y(s)$  that extremize the above functional. Calculate the determinant of the Hessian and conclude that the solution is indeed a minimum in agreement with the previous exercise.

### 1.5.3 Extremal curves on a sphere - Great circles

Consider the two-sphere  $S^2$  in three Euclidean dimensions. The metric on the sphere gives rise to an arclength  $ds$ , which can be written as

$$ds = \sqrt{d\theta^2 + \sin^2 \theta d\phi^2}, \quad (1.60)$$

where  $\theta$  and  $\phi$  are angular variables. Find the curve that extremizes the distance between two points on the sphere. You can either parametrize  $\theta = \theta(s)$  and  $\phi = \phi(s)$  or write  $\phi = \phi(\theta)$  (or vice versa).

### 1.5.4 Rigid rotator

The kinetic energy of a rigid rotator with moment of inertia  $I$  is  $T = \frac{1}{2}I\omega^2$ , where  $\omega$  is the angular frequency.

a) Use spherical coordinates and show that the Lagrangian can be written as

$$L = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta). \quad (1.61)$$

Identify the cyclic coordinate and interpret the constant of motion.

b) Find the generalized momenta and show that the Hamiltonian can be written as

$$H = \frac{1}{2I} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right). \quad (1.62)$$

### 1.5.5 Mathematical pendulum

Consider a mathematical pendulum of length  $l$  and mass  $m$  in a gravitational field where the acceleration is  $g$ .

- a) Choose a suitable generalized coordinate and find the Lagrangian for the system.
- b) Find Lagrange's equation.
- c) Calculate the generalized momentum and the Hamiltonian.
- d) Derive Hamilton's equations for the system and show they are equivalent to Lagrange's equation.

# Chapter 2

## Lorentz transformations

In this chapter, we briefly discuss the most important aspects of Lorentz transformations in the context of field theories. More details can be found in the chapter four in the excellent textbook by J. B. Hartle [4].

### 2.1 Boosts

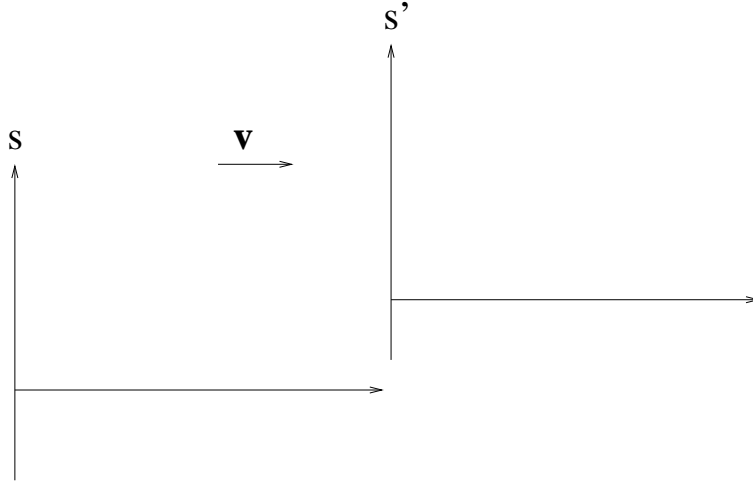


Figure 2.1: A boost from  $S$  to  $S'$  along the  $x$ -axis with speed  $v$ .

Let  $S$  and  $S'$  be two inertial frames where  $S'$  moves with speed  $v$  relative to  $S$  along the  $x$ -axis, see. Fig. 2.1. Let  $(x, y, z, t)$  and  $(x', y', z', t')$  be the spacetime coordinates in the two frames. The clocks are synchronized such that the origin  $O$  of  $S$  coincides with the origin  $O'$  of  $S'$  for the  $t = t' = 0$ . The points in spacetime are *events*, i. e. “something happens” at time  $t$  at a specific point in space. An example of an event is the emission of light at time  $t$  by a source located at  $(x, y, z)$ .

The coordinates in the two inertial frames are related by the following transformations

$$x' = \gamma(x - vt) , \quad (2.1)$$

$$y' = y , \quad (2.2)$$

$$z' = z , \quad (2.3)$$

$$t' = \gamma(t - vx/c^2) , \quad (2.4)$$

where  $c$  is the speed of light and  $\gamma = 1/\sqrt{1 - v^2/c^2}$ . The transformation between  $S$  and  $S'$  is called a *boost* along the  $x$ -axis. There is of course nothing special about the  $x$ -axis and so we can define a boost in any direction given by the velocity  $\mathbf{v}$  of  $S'$  relative to  $S$ . The transformation (2.1) and (2.4) can be conveniently written in matrix form

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh \theta & -\sinh \theta \\ -\sinh \theta & \cosh \theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} , \quad (2.5)$$

where  $\cosh \theta = \gamma$  and  $\sinh \theta = \gamma v/c$ . This gives

$$\tanh \theta = \frac{v}{c} . \quad (2.6)$$

Note that both the coordinates  $x$  and  $t$  are involved in the transformation (2.1) and (2.4). This has important consequences: observers in  $S'$  and  $S$  do not (necessarily) agree that two events are taking place at the same time. The concept of *simultaneous events* is not absolute. More specifically, Lorentz transformations leads to

- **Lorentz contraction:** A rod of length  $L$  at rest is shorter by a factor  $1/\gamma$  when it is moving with speed  $v$  relative to the observer.
- **Time dilation:** A moving clock is ticking at a slower rate than a clock at rest.

Since space and time are intertwined, we introduce the concept of spacetime or Minkowski space with coordinates  $(x, y, z, t)$ . The points in spacetime are thus events.

The determinant of the matrix in Eq. (2.5) is unity. This is the same as for a proper rotation matrix in Euclidean plane  $\mathcal{R}^2$  which reads

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \quad (2.7)$$

If we write  $ct = iT$ , the matrix (2.5) is transformed into the matrix (2.7). Thus a boost can be interpreted as a rotation. Finally, we mention that the set of all boosts defined by the velocity  $v$  and proper rotations around the three axes in space defined by the angles  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$  constitute the set of all homogeneous Lorentz transformations. If we add to these transformations, the translations in space  $\mathbf{x}' = \mathbf{x} + \mathbf{c}$  and time  $t' = t + c_1$ , the transformation are called inhomogeneous Lorentz transformations.

The Newtonian limit of the transformations Eqs. (2.1)-(2.4) is found by considering velocities  $v$  which are small compared to the speed of light, i.e.  $v \ll c$ . Taking the limit  $c \rightarrow \infty$ , i.e. setting  $\gamma = 1$ , we obtain

$$x' = (x - vt) , \quad (2.8)$$

$$y' = y , \quad (2.9)$$

$$z' = z , \quad (2.10)$$

$$t' = t . \quad (2.11)$$

This set of transformation is known as a *Galilean transformation* and is the nonrelativistic limit of special relativity. Note in particular that the idea of simultaneity is absolute as  $t' = t$ . The Galilean transformations are underlying Newtonian mechanics.

## 2.2 Geometry of Minkowski space

Let  $(t_0, x_0, y_0, z_0)$  and  $(t_1, x_1, y_1, z_1)$  be two points in  $S$ . We next introduce the distance  $(\Delta s)^2$  between the two points in Minkowski space by

$$(\Delta s)^2 = c^2(\Delta t)^2 - [(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2] , \quad (2.12)$$

where  $\Delta t = t_1 - t_0$  is the time difference between the events and  $\Delta x = x_1 - x_0$  etc. In the inertial frame  $S'$ , we obtain

$$\begin{aligned} (\Delta s)^2 &= c^2(\Delta t')^2 - [(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2] \\ &= \gamma^2 c^2 \left[ (\Delta t)^2 - 2 \frac{v \Delta x \Delta t}{c^2} + \frac{v^4}{c^4} (\Delta x)^2 \right] - \gamma^2 [(\Delta x)^2 - v^2 (\Delta t)^2 - 2 \Delta x \Delta t] \\ &\quad - (\Delta y)^2 - (\Delta z)^2 \\ &= \gamma^2 c^2 (\Delta t)^2 \left[ 1 - \frac{v^2}{c^2} \right] - \gamma^2 (\Delta x)^2 \left[ 1 - \frac{v^2}{c^2} \right] - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2 (\Delta t')^2 - [(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2] , \end{aligned} \quad (2.13)$$

where we in the last line have used that  $\gamma = 1/\sqrt{1 - v^2/c^2}$ . Thus the quantity  $(\Delta s)^2$  is the same in the two frames. In fact one can show that this quantity is invariant under all Lorentz transformations and it can therefore be viewed as a *geometric* quantity independent of coordinate system<sup>1</sup>. In differential form, we write

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 . \quad (2.14)$$

We next define  $x^\mu = (ct, \mathbf{x})$ , where  $\mu = 0, 1, 2, 3$ . In other words  $x^0 = ct$ ,  $x^1 = x$ ,  $x^2 = y$ , and  $x^3 = z$ . Moreover, we introduce the *metric tensor*  $g_{\mu\nu}$  which can be written as a  $4 \times 4$

---

<sup>1</sup>It is basically the same concept of invariant distance between two points that you know from Euclidean space.

matrix

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (2.15)$$

The shorthand notation for the metric tensor is  $\text{diag}(1, -, 1, -1)$ . We can then write

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (2.16)$$

where we are using the Einstein summation convention: One is summing over repeated indices where one index is upstairs and one is downstairs. Finally note that some textbooks define the invariant distance as minus Eq. (2.14). The metric tensor is then defined as  $\text{diag}(-1, , 1, 1, 1)$ . This convention is just a convention and has no physical significance.

We next introduce the concept of a *contravariant* vector which has four components  $A^\mu$  that transform like  $dx^\mu$  under Lorentz transformations. Moreover, the *covariant* vector  $A_\mu$  is defined by

$$A_\mu = g_{\mu\nu} A^\nu. \quad (2.17)$$

For example,  $x^\mu = (ct, \mathbf{x})$  is a contravariant vector and  $x_\mu = (ct, -\mathbf{x})$  is the corresponding covariant vector. The invariant distance  $ds^2$  can now be written as

$$ds^2 = dx_\mu dx^\mu, \quad (2.18)$$

and can be considered as a *scalar product* between a contravariant and a covariant vector and has the same value in all inertial frames, i.e. it is a scalar.

Note that the distance  $ds^2$  between two points can be positive, negative, or zero. The distance is classified according to

$$(\Delta s)^2 > 0 \quad \text{timelike interval}, \quad (2.19)$$

$$(\Delta s)^2 = 0 \quad \text{lightlike interval}, \quad (2.20)$$

$$(\Delta s)^2 < 0 \quad \text{spacelike interval}. \quad (2.21)$$

In the case of a timelike interval, there is enough time for a signal to propagate the two points in space, i.e. they are in causal contact. In the case of a spacelike interval, there is not enough time and so the points are not causally connected. Light rays are always travelling along paths in spacetime where  $ds^2 = 0$  - hence the name. This is shown in Fig. 2.2, where the two straight lines indicate the *light cone*, the paths at which light is moving. The point A is inside the light cone and is in causal contact with the origin. The point B is connected to the origin by a light signal, while the point C is outside the light cone.

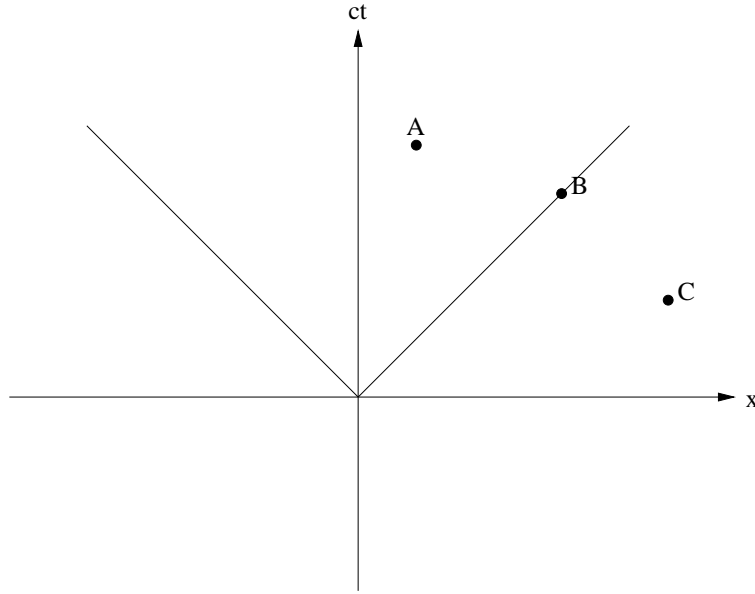


Figure 2.2: Light cone with points inside (A), on (B), and outside (C).

## 2.3 Differential operators

We next discuss how the differential operator  $\partial/\partial x^\mu$  transforms under a boost along the  $x$ -axis. Using the chain rule, we can write

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'} \frac{\partial}{\partial x} + \frac{\partial t}{\partial x'} \frac{\partial}{\partial t}, \quad (2.22)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \quad (2.23)$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}, \quad (2.24)$$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'} \frac{\partial}{\partial x} + \frac{\partial t}{\partial t'} \frac{\partial}{\partial t}. \quad (2.25)$$

In order to calculate the partial derivatives, we need the inverse of the transformations (2.1)-(2.4) which are given by

$$x = \gamma(x' + vt'), \quad (2.26)$$

$$y = y', \quad (2.27)$$

$$z = z', \quad (2.28)$$

$$t = \gamma(t' + vx'/c^2). \quad (2.29)$$

This yields

$$\frac{\partial}{\partial x'} = \gamma \left[ \frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right], \quad (2.30)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y} , \quad (2.31)$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z} , \quad (2.32)$$

$$\frac{\partial}{\partial t'} = \gamma \left[ \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right] . \quad (2.33)$$

Comparing this with the transformation rule of  $x_\mu = (ct, -\mathbf{x})$

$$-x' = \gamma(-x + vt) , \quad (2.34)$$

$$-y' = -y , \quad (2.35)$$

$$-z' = -z , \quad (2.36)$$

$$t' = \gamma(t + vx/c^2) , \quad (2.37)$$

we conclude that the differential operator  $\partial/\partial x^\mu$  transforms as a covariant vector and we therefore denote it by  $\partial_\mu$ .

## 2.4 Proper time and rate of clocks

The invariant distance between two events is

$$\Delta s^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 . \quad (2.38)$$

This can be rewritten as

$$ds^2 = \sqrt{1 - v^2/c^2} dt^2 , \quad (2.39)$$

where  $v$  is the velocity of  $S'$  in  $S$ . It is important to emphasize that Eq. (2.39) is valid also when the velocity is not constant. Assume an observer is moving with velocity  $v$  in the frame  $S$  and carries a clock with her. The time between two events measured by this observer is denoted by  $d\tau$  and called *proper time*. For such an observer, Eq. (2.39) reduced to

$$ds^2 = d\tau^2 , \quad (2.40)$$

and so the equation that relates laboratory time and proper time is

$$d\tau = \sqrt{1 - v^2/c^2} dt . \quad (2.41)$$

This shows that moving clocks run at a slower rate.

## 2.5 Problems

### 2.5.1 Interpretation of boosts

Define  $T$  by  $ct = iT$  and show that the matrix (2.5) is transformed into the matrix (2.7).

### 2.5.2 D'Alembertian under boosts

Show that  $\partial^\mu = \partial/\partial x_\mu$  transforms as a contravariant vector under boosts. How does the operator

$$\square = \partial_\mu \partial^\mu \quad (2.42)$$

transform?

### 2.5.3 Rotation matrix

Consider a rotation in the  $xy$ -plane by an angle  $\theta$ . The rotation is represented by the matrix.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.43)$$

Using the chain rule, show that

$$\begin{aligned} \frac{\partial}{\partial x'} &= \frac{\partial x}{\partial x'} \frac{\partial}{\partial x} + \frac{\partial y}{\partial x'} \frac{\partial}{\partial y} \\ &= \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \end{aligned} \quad (2.44)$$

$$\begin{aligned} \frac{\partial}{\partial y'} &= \frac{\partial x}{\partial y'} \frac{\partial}{\partial x} + \frac{\partial y}{\partial y'} \frac{\partial}{\partial y} \\ &= -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}. \end{aligned} \quad (2.45)$$

Use this to show that  $\nabla^2$  is invariant under proper rotations. This result can be easily generalized to rotations around any axis.

### 2.5.4 Spacelike and timelike intervals

Show that if two events are separated by a spacelike interval, we can always find an inertial frame such that  $\Delta t' = 0$ , i.e. they happen at the same time (but at different points in space). Similarly, show that if two events are separated by a timelike interval, we can always find a frame such that  $\Delta x' = 0$ , i.e. they happen at the same point in space (but at different times).

### 2.5.5 Lorentz contraction

The length of a rod in an inertial frame is defined by two simultaneous events at its ends. The length of a rod is  $L^*$  at rest. Assume the rod is moving with speed  $v$  along the  $x$ -axis in the inertial frame  $S$ . Show that its length in  $S$  is  $L = L^*/\gamma$ , i.e. it is Lorentz contracted.

### 2.5.6 Compositions of boosts

Let  $S$ ,  $S'$ , and  $S''$ , be three inertial frames.  $S$  and  $S'$  are connected by a boost along the  $x$ -axis with velocity  $v$ , and  $S'$  and  $S''$  are connected by a boost along the  $x$ -axis with velocity  $v'$ . Show that  $S$  and  $S''$  are connected by a boost along the  $x$ -axis and find the velocity  $v''$ .

# Chapter 3

## Classical field theory

In this chapter we discuss some aspects of classical field theory. A more thorough treatment can be found in Ref. [5].

A field  $\phi$  is a mapping from spacetime  $\mathcal{M}$  with coordinates  $x$ ,  $y$ ,  $z$ , and  $t$  to a *field space* or a *target space*  $\mathcal{T}$ :

$$\phi : \mathcal{M} \rightarrow \mathcal{T} . \quad (3.1)$$

There are many examples of a field. Below we list a few examples:

- $\rho = \rho(x, y, z, t)$  is the density of a liquid.  $\rho$  is a scalar function.
- The Maxwell field  $A_\mu(x, y, z, t)$ . The field space is that of four-vector fields.
- Quantum mechanics:  $\psi(x, y, z, t)$  is the Schrödinger wavefunction. The field space is that of complex functions.

For each  $\vec{r}$ , we can view  $\phi = \phi(\vec{r}, t)$  as a function of time  $t$  and  $\phi$  therefore represents one degree of freedom, just like  $q = q(t)$  for point particles. A field thus has infinitely many degrees of freedom and the theory of fields replaces the theory of point particles when it is necessary to describe the system with a continuum of degrees of freedom. Moreover, we often speak of field theory in  $D = d + 1$  dimensions, where  $d$  is the number of spatial dimensions and  $D$  is the dimension of space-time. In particular quantum mechanics can be viewed as field theory in zero spatial and one time dimension, i.e. field theory in  $0 + 1$  dimensions.

### 3.1 Action and equation of motion

We first discuss the Lagrangian density  $\mathcal{L}$ , which is a function of the fields  $\phi_i$ , their derivatives and possibly explicitly of the coordinates  $x^\mu$ . The Lagrangian density is the generalization of  $L = L(q_i(t)\dot{q}_i(t), t)$  to a system of with infinitely many degrees of freedom. In most cases, it is a function of  $x^\mu$  only through  $\phi_i$  and  $\partial_\mu \phi_i$ . Thus we can write

$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ , where we for notational simplicity have suppressed the index field index  $i$ . We will often do that in the following.

The Lagrangian  $L$  is defined by

$$L = \int_{\Omega_3} d^3x \mathcal{L}(\phi, \partial_\mu \phi) , \quad (3.2)$$

where  $\Omega_3$  is a region in space. Typically  $\Omega_3$  is the whole space, i.e.  $\Omega_3 = \mathcal{R}^3$ . The action  $S$  of a field theory with Lagrangian density  $\mathcal{L}$  is defined by

$$\begin{aligned} S[\phi] &= \int_{t_0}^{t_1} dt L \\ &= \int_{\Omega_4} d^4x \mathcal{L}(\phi, \partial_\mu \phi) , \end{aligned} \quad (3.3)$$

where the region in Minkowski space<sup>1</sup> is  $\Omega_4 = \Omega_3 \times [t_0, t_1]$ . Typically  $\Omega_4$  is whole space-time. The action  $S$  is again a functional of the field  $\phi$ . In Chapter one, we were looking for a trajectory  $q(t)$  that extremizes the action. Now we are looking for a *field configuration*  $\phi$  that extremizes the action. The field is written as

$$\phi' = \phi + \delta\phi , \quad (3.4)$$

where  $\phi$  is the field configuration that extremizes the action and  $\delta\phi$  is a variation which vanishes on the boundary  $\Delta\Omega_4$ . Inserting Eq. (3.4) into the action (3.3) we obtain

$$S[\phi + \delta\phi] = \int_{\Omega_4} d^4x \mathcal{L}(\phi + \delta\phi, \partial_\mu(\phi + \delta\phi)) . \quad (3.5)$$

The action  $S$  is expanded to first order in  $\delta\phi$ . This yields

$$\begin{aligned} S[\phi + \delta\phi] &= \int_{\Omega_4} d^4x \mathcal{L}(\phi, \partial_\mu \phi) + \int_{\Omega_4} d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \delta(\partial_\mu \phi) \right] \\ &= S[\phi] + \int_{\Omega_4} d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu(\delta\phi) \right] , \end{aligned} \quad (3.6)$$

where we in the last line have used that  $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ . The second term in Eq. (3.6) is denoted by  $\delta S$  and reads

$$\begin{aligned} \delta S &= \int_{\Omega_4} d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu(\delta\phi) \right] \\ &= \int_{\Omega_4} d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right] \delta\phi + \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \Big|_{\Delta\Omega_4} , \end{aligned} \quad (3.7)$$

where we in the last term have integrated by parts. Since we allow no variation on the surface  $\Delta\Omega_4$ , the boundary term is zero. The variational principle says that  $\delta S = 0$  and so we obtain

$$\int_{\Omega_4} d^4x \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \right] \delta\phi = 0 . \quad (3.8)$$

---

<sup>1</sup>We restrict ourselves to flat spacetime, but the formalism can be generalized to curved spacetime.

Since the variation  $\delta\phi$  is arbitrary, the term in the paranthesis must vanish, i.e. we find

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) . \quad (3.9)$$

This is the Euler-Lagrange equation for the field  $\phi$ . If the Lagrangian is a function of several fields  $\phi_i$ , there is an equation for each - just like we have one equation for each  $q_i(t)$ .

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## Example

The Lagrangian for a real scalar field is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}g_{\mu\nu}(\partial^\nu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 . \end{aligned} \quad (3.10)$$

This yields

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi , \quad (3.11)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} = \partial^\mu \phi . \quad (3.12)$$

The equation of motion then becomes

$$\left[ \partial_\mu \partial^\mu + m^2 \right] \phi = 0 . \quad (3.13)$$

This can be written as

$$\left[ \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] \phi = 0 . \quad (3.14)$$

This is the Klein-Gordon equation that describes spin-zero neutral particles of mass  $m$  as we shall see later. Note that the operator  $\partial_\mu \partial^\mu$  is called the D'Alembertian and is sometimes denoted by  $\square$ .

We can also obtain the Klein-Gordon equation from a different path. Consider the energy-momentum relation of a relativistic particle with mass  $m$ :

$$E^2 = m^2 + p^2 , \quad (3.15)$$

In quantum mechanics, we replace the energy  $E$  and momentum  $\mathbf{p}$  by the operators  $i\partial_0$  and  $-i\nabla$ . Inserting this into Eq. (3.15), we obtain

$$-\frac{\partial^2}{\partial t^2} = m^2 - \nabla^2 , \quad (3.16)$$

which is equivalent to the Klein-Gordon equation (3.14).

---

The above example shows that by choosing  $\mathcal{L}$  as in Eq. (3.10) we obtain Eq. (3.16) which is motivated by the substitutions  $E \rightarrow i\partial_0$  and  $\mathbf{p} \rightarrow -i\nabla$ . In general, we have to write down a Lagrangian whose equation of motion is the correct one. A general requirement to a Lagrangian is that it be Lorentz invariant, i.e. it is a scalar under Lorentz transformations. Since  $\phi$  is a scalar, we know that  $\partial_\mu\phi$  is a covariant vector and  $\partial^\mu\phi$  is a contravariant vector. The kinetic term is therefore a scalar since it is the scalar product between a covariant vector and a contravariant vector. The requirement that the Lagrangian is Lorentz invariant also forbids a term like

$$\phi^3\nabla\phi, \quad (3.17)$$

since it is not invariant under rotations. However, it does admit a term like  $\phi^2(\nabla\phi)^2$  and is an interaction term since it is fourth order in the field.

Is the Lagrangian density for a field theory unique? By that, we mean are there more than one  $\mathcal{L}$  that gives rise to the same equation of motion? The answer is no. We can obviously always add a constant to the Lagrangian without changing the field equations. We can also multiply the Lagrangian by a constant  $C$  since this implies that  $\delta S' = C\delta S$  and that the equation of motion is not altered. Finally, we can always add the integral of a total divergence  $\partial_\mu M^u$  to the action. If

$$S' = S + \int_{\Omega_4} d^4x \partial_\mu M^u. \quad (3.18)$$

the second term on the right-hand side can be rewritten using Gauss' theorem

$$\int_{\Omega_4} d^4x \partial_\mu M^u = \int_{\partial\Omega_4} d^3x \mathbf{n} \cdot \mathbf{M}, \quad (3.19)$$

where  $\mathbf{n}$  is a normal vector to the surface  $\Omega_3$ . Since  $M^u$  is not supposed to vary on the surface  $\Omega_3$ , we have  $\delta S' = \delta S$  and so the equation of motion is not changed.

## 3.2 Conjugate momentum and Hamiltonian

The conjugate momentum density  $\pi_i$  of a field  $\phi_i$  is defined by

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}. \quad (3.20)$$

The Hamiltonian density is then defined in analogy with the particle case

$$\mathcal{H} = \pi_i \dot{\phi}_i - \mathcal{L}. \quad (3.21)$$


---

### Example

Reconsider the Lagrangian of a real scalar field. The conjugate momentum follows directly from Eq (3.12) setting  $\mu = 0$ :

$$\pi = \dot{\phi} \quad (3.22)$$

The Hamiltonian density then becomes

$$\begin{aligned} \mathcal{H} &= \pi \dot{\phi} - \mathcal{L} \\ &= \dot{\phi}^2 - \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2}m^2 \phi^2 \\ &= \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2 \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2, \end{aligned} \quad (3.23)$$

where we have eliminated  $\dot{\phi}$  in favor of  $\pi$ . The Hamiltonian density is often referred to as the energy density. Note that all terms in  $\mathcal{H}$  are positive and so the energy density is too.

---

## 3.3 Symmetries and conservation laws

A coordinate  $q_i$  is *cyclic* if the Lagrangian  $L$  does not depend explicitly on it, i.e. if

$$\frac{\partial L}{\partial q_i} = 0. \quad (3.24)$$

Lagrange equation for  $q_i$  then yields

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0. \quad (3.25)$$

Integrating with respect to time gives

$$\frac{\partial L}{\partial \dot{q}_i} = C, \quad (3.26)$$

where  $C$  is an integration constant. In other words, a cyclic coordinate implies a conserved quantity.

We are now going to consider conserved quantities in general. More specifically we are going to investigate the relationship between *continuous* symmetries of a system and conservation laws in classical field theory. This is summarized in *Nöther's theorem*.

An example of a *discrete* field transformation is  $\phi \rightarrow -\phi$ . The Lagrangian for a real scalar field is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2 \phi^2$$

is clearly invariant under this transformation. An example of a continuous field transformation is  $\Phi \rightarrow \Phi e^{i\alpha}$ , where  $\Phi$  is a complex field and  $\alpha$  is a continuous parameter. This transformation is called a *global*<sup>2</sup> phase transformation. The Lagrangian

$$\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi, \quad (3.27)$$

is invariant under global phase transformations.

Assume the Lagrangian  $\mathcal{L}$  is invariant under a transformation of the field  $\phi$ . In infinitesimal form this transformation can be written as

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x), \quad (3.28)$$

where  $\Delta \phi$  is a deformation of the field  $\phi$  and  $\alpha$  is an infinitesimal parameter. The Lagrangian is changing under the transformation (3.28)

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \alpha \Delta \mathcal{L}, \quad (3.29)$$

where

$$\begin{aligned} \Delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi} \Delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu (\Delta \phi) \\ &= \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right] + \left[ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \Delta \phi. \end{aligned} \quad (3.30)$$

The second term vanishes due to the equation of motion (3.9) and so we can write

$$\begin{aligned} \Delta \mathcal{L} &= \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right] \\ &= 0. \end{aligned} \quad (3.31)$$

This implies a continuity equation

$$\partial_\mu j^\mu = 0, \quad (3.32)$$

where the four-current  $j^\mu$  is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi. \quad (3.33)$$

---

<sup>2</sup>It is a global phase transformation since it is the same at every point in spacetime.

Integrating Eq. (3.32) over all space, we find

$$\begin{aligned} \int \partial_\mu j^\mu d^3x &= \frac{d}{dt} \int j^0 d^3x + \int \nabla \cdot \mathbf{j} d^3x \\ &= 0 . \end{aligned} \quad (3.34)$$

Using Gauss' theorem, the last term can be converted into a surface integral that vanishes<sup>3</sup>. We can then write

$$\frac{dQ}{dt} = 0 , \quad (3.35)$$

where the charge  $Q$  is defined by the integral over space of the charge density  $\rho = j^0$ :

$$Q = \int j^0 d^3x . \quad (3.36)$$

Eq. (3.36) is the essence of Noether's theorem: A continuous symmetry gives rise to the charge  $Q$  being constant in time. The conserved charge can be e.g. electric charge, baryon charge, isospin charge or simply particle number.

## Example

Consider the Lagrangian for a complex scalar field

$$\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi . \quad (3.37)$$

The Lagrangian is invariant under global phase transformations, i.e.  $\phi \rightarrow \phi e^{i\alpha}$ , where  $\alpha$  is constant. In infinitesimal form, we have

$$\Phi \rightarrow \Phi + i\alpha \Phi , \quad (3.38)$$

$$\Phi^* \rightarrow \Phi^* - i\alpha \Phi^* . \quad (3.39)$$

Moreover

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi)} = \partial^\mu \Phi^* \quad (3.40)$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \Phi^*)} = \partial^\mu \Phi . \quad (3.41)$$

This yields

$$j^\mu = i [\Phi (\partial^\mu \Phi^*) - \Phi^* (\partial^\mu \Phi)] . \quad (3.42)$$

<sup>3</sup>We assume the fields and their derivatives vanish in spatial infinity.

You can use the Klein-Gordon equation for  $\Phi$  directly to confirm that the  $j^\mu$  above satisfies a continuity equation.

---

We can generalize the result (3.31) by noting that the Euler-Lagrange equations are not affected by adding a total divergence to the Lagrangian<sup>4</sup>. If the change of the Lagrangian  $\Delta\mathcal{L}$  can be written as a total divergence  $\alpha\partial_\mu\mathcal{J}^\mu$ , the transformation  $\phi \rightarrow \phi + \alpha\Delta\phi$  still defines a symmetry. We can then write

$$\partial_\mu \left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \Delta\phi \right] = \partial_\mu\mathcal{J}^\mu, \quad (3.43)$$

and so

$$\partial_\mu j^\mu = 0, \quad (3.44)$$

where

$$j^\mu = \left[ \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \Delta\phi \right] - \mathcal{J}^\mu, \quad (3.45)$$


---

## Example

Let us consider translations in space-time, which can be written as

$$\begin{aligned} x^\mu &\rightarrow (x^\mu)' \\ &= x^\mu + a^\mu, \end{aligned} \quad (3.46)$$

where  $a^\mu$  is infinitesimal. To first order in  $a^\mu$ , we obtain

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) \\ &= \phi(x + a) \\ &= \phi(x) + a^\mu \partial_\mu \phi, \end{aligned} \quad (3.47)$$

$$\partial_\mu \phi \rightarrow \partial_\mu \phi + a^\nu \partial_\mu \partial_\nu \phi. \quad (3.48)$$

Inserting these expressions into the Lagrangian (3.10). To first order in  $a^\mu$ , the Lagrangian changes as

$$\begin{aligned} \mathcal{L} &\rightarrow \mathcal{L} + \Delta\mathcal{L} \\ &= \mathcal{L} + a^\nu \partial_\mu (\delta_\nu^\mu \mathcal{L}). \end{aligned} \quad (3.49)$$

---

<sup>4</sup>In other words, changing the action by a surface term does not change the equation of motion.

The change is therefore a total divergence and this gives rise to four conserved quantities<sup>5</sup> The components are denoted by

$$\mathcal{T}_\nu^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \delta_\nu^\mu \mathcal{L} , \quad (3.50)$$

where we have used that  $\Delta\phi = a^\mu \partial_\mu \phi$ , cf. Eq. (3.47).  $\mathcal{T}_\nu^\mu$  is called the *energy-momentum tensor*. The four conserved quantities are energy and momentum and correspond to translational invariance in time and space. Field theories whose Lagrangian only depend on the coordinates  $x^\mu$  implicitly via  $\phi$  and  $\partial_\mu \phi$  are translationally invariant just like the Lagrangian (3.10).

Let us calculate the various components of  $\mathcal{T}_\nu^\mu$  for the Lagrangian (3.10).. For  $\nu = 0$ , we obtain

$$\mathcal{T}_0^\mu = (\partial^\mu \phi)(\partial_0 \phi) - \delta_0^\mu \mathcal{L} . \quad (3.51)$$

In particular

$$\begin{aligned} \mathcal{T}_0^0 &= (\partial^0 \phi)(\partial_0 \phi) - \mathcal{L} \\ &= \frac{1}{2}(\partial^0 \phi)(\partial_0 \phi) + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2 . \end{aligned} \quad (3.52)$$

This is the Hamiltonian density! The conservation of energy results from the invariance of the Lagrangian under translation in time. The spatial components are

$$\mathcal{T}_0^i = (\partial^i \phi)(\partial_0 \phi) . \quad (3.53)$$

The continuity equation then becomes

$$\begin{aligned} \partial_0 \mathcal{T}_0^0 + \partial_j \mathcal{T}_0^j &= \partial_0 \phi \left[ \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right] \phi \\ &= 0 , \end{aligned} \quad (3.54)$$

by virtue of the equation of motion. Furthermore, for  $\nu = j$ , we obtain

$$\mathcal{T}_j^0 = (\partial^0 \phi)(\partial_j \phi) , \quad (3.55)$$

which is the momentum density in the  $j$ -direction. Thus, momentum conservation results from the invariance of the Lagrangian under spatial translations. Finally, the spatial components are easily computed and we obtain

$$\mathcal{T}_j^i = (\partial^i \phi)(\partial_j \phi) - \delta_j^i \mathcal{L} . \quad (3.56)$$

---

<sup>5</sup>The four quantities  $a^\mu$  can varied independently.

### 3.4 Complex Klein-Gordon equation and gauge transformations

The Lagrangian is given by

$$\mathcal{L} = (\partial_\mu \Phi^*) (\partial^\mu \Phi) - m^2 \Phi^* \Phi, \quad (3.57)$$

where  $\Phi$  is a complex field. In the previous chapter we showed that the Lagrangian for a complex scalar field is invariant under global phase transformations i.e. transformations of the form

$$\Phi \rightarrow e^{i\alpha} \Phi, \quad (3.58)$$

$$\Phi^* \rightarrow e^{-i\alpha} \Phi^*, \quad (3.59)$$

where  $\alpha$  is a constant phase, i. e. independent of spacetime coordinates. This symmetry implies the conservation of particle number or conservation of charge.

Let us now generalize the transformations to *local* phase transformations, i.e. transformations where  $\alpha = \alpha(x)$ . The Lagrangian then transforms as

$$\mathcal{L} \rightarrow \mathcal{L} - i\Phi^* (\partial^\mu \Phi) (\partial_\mu \alpha) + i\Phi (\partial^\mu \Phi^*) (\partial_\mu \alpha) + \Phi^* \Phi (\partial^\mu \alpha) (\partial_\mu \alpha). \quad (3.60)$$

The Lagrangian is therefore not invariant under local phase transformations. However, this can be remedied by introducing another field  $A_\mu$ , called a *gauge field*. The Lagrangian must be modified and is given by

$$\mathcal{L} = (\partial_\mu - iqA_\mu) \Phi^* (\partial^\mu + iqA^\mu) \Phi - m^2 \Phi^* \Phi, \quad (3.61)$$

where  $q$  is a constant (the charge) and  $D_\mu = \partial_\mu + iqA_\mu$  is called the *covariant derivative*. The field  $A^\mu$  must transform as

$$A_\mu \rightarrow A_\mu - \frac{1}{q} \partial_\mu \alpha. \quad (3.62)$$

The transformation (3.62) is called a *gauge transformation*. The Lagrangian is invariant under the simultaneous transformation (3.62) and

$$\Phi \rightarrow e^{i\alpha(x)} \Phi, \quad (3.63)$$

$$\Phi^* \rightarrow e^{-i\alpha(x)} \Phi^*. \quad (3.64)$$

This can be seen by inserting the transformed field into the Lagrangian:

$$\begin{aligned} \mathcal{L} &= (\partial_\mu - iqA_\mu) \Phi^* (\partial^\mu + iqA^\mu) \Phi - m^2 \Phi^* \Phi \\ &\rightarrow \mathcal{L}' \\ &= \left\{ \left[ \partial_\mu - iq \left( A_\mu - \frac{1}{q} \partial_\mu \alpha \right) \right] e^{-i\alpha(x)} \Phi^* \left[ \partial^\mu + iq \left( A^\mu - \frac{1}{q} \partial^\mu \alpha \right) \right] e^{i\alpha(x)} \Phi - m^2 \Phi^* \Phi \right\} \\ &= \mathcal{L}. \end{aligned} \quad (3.65)$$

## Electromagnetism

We next consider the Maxwell field. The electric and magnetic fields can be expressed in terms of the gauge field  $A^\mu = (\phi, \mathbf{A})$  as

$$\mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \quad (3.66)$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3.67)$$

We recall that the physical fields  $\mathbf{E}$  and  $\mathbf{B}$  are gauge invariant, i.e. invariant under the transformation (3.62) of 4-vector field  $A^\mu$ . Maxwell's equations are

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad (3.68)$$

$$\nabla \cdot \mathbf{E} = \rho, \quad \nabla \times \mathbf{B} - \frac{\partial\mathbf{E}}{\partial t} = \mathbf{j}, \quad (3.69)$$

The first homogeneous equation simply states that we can express  $\mathbf{B}$  as the curl of a vector potential  $\mathbf{A}$ . Using this, the second homogeneous equation states that  $\mathbf{E} + \partial\mathbf{A}/\partial t$  can be written as the gradient of a scalar function  $\Phi = A^0$ <sup>6</sup>. The Lagrangian we are interested in must give rise to Maxwell inhomogeneous equations as field equations. The Lagrangian can not simply be found by replacing  $\phi$  by  $A^\mu$  in the Lagrangian for the scalar field. Doing so, yields<sup>7</sup>

$$\mathcal{L} = \frac{1}{2} (\partial_\mu A_\nu) (\partial^\mu A^\nu) - \frac{1}{2} m^2 A_\mu A^\mu. \quad (3.70)$$

The problem is that none of these terms are gauge invariant. In particular the mass term is not gauge invariant and this explains why the photon is massless: Maxwell theory is gauge theory and gauge invariance forbids a mass term in the Lagrangian. However, the *field strength*  $F^{\mu\nu}$  defined by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (3.71)$$

is gauge invariant (show it!). The field strength is an antisymmetric tensor of rank two. Now recall that a Lagrangian must be a Lorentz scalar and therefore we must contract the two indices upstairs. We can contract it with  $F_{\mu\nu}$  and try

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (3.72)$$

This Lagrangian is Lorentz invariant and gauge invariant. The field strength tensor reads

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad (3.73)$$

---

<sup>6</sup>The curl of a divergence of a vector field and the gradient of the curl of a vector field always vanish.

<sup>7</sup>Note that  $A_\mu = (\Phi, -\mathbf{A})$ .

Maxwell's equations can then be written as (Exercise 3.6.3)

$$\partial_\mu F^{\mu\nu} = j^\nu . \quad (3.74)$$

Using the fact that  $A_\mu = (\phi, -\mathbf{A})$  it is straightforward to express  $F_{\mu\nu}$  in terms of  $\mathbf{E}$  and  $\mathbf{B}$ . It can then be shown that

$$F^{\mu\nu} F_{\mu\nu} = -2(\mathbf{E}^2 - \mathbf{B}^2) . \quad (3.75)$$

In other words, the Lagrangian can be written as follows in the terms of the physical fields  $\mathbf{E}$  and  $\mathbf{B}$ .

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) . \quad (3.76)$$

Finally, the Lagrangian for a complex scalar field coupled to the electromagnetic field is then given by

$$\mathcal{L} = (D_\mu \Phi^*)(D^\mu \Phi) - m^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} . \quad (3.77)$$

### 3.5 Nonrelativistic limit of fields

Consider a real selfinteracting relativistic field  $\phi$  whose Lagrangian is

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{24} \phi^4 . \quad (3.78)$$

In order to find the nonrelativistic limit, we write the field as

$$\phi = \frac{1}{\sqrt{2m}} (e^{imt} \psi + e^{-imt} \psi^\dagger) . \quad (3.79)$$

Inserting the representation Eq. (3.79) into Eq. (3.78) and omit the oscilating terms (i.e. the terms that go like  $e^{\pm 2imt}$ ), we obtain

$$\begin{aligned} \frac{1}{2} m^2 \phi^2 &= \frac{1}{4} m (2\psi^\dagger \psi + e^{2imt} \psi \psi + e^{-2imt} \psi^\dagger \psi^\dagger) \\ &\approx \frac{1}{2} m \psi^\dagger \psi , \end{aligned} \quad (3.80)$$

where  $\approx$  implies that we have omitted the oscillating terms. Similarly, we obtain

$$\frac{1}{2} (\partial_0 \phi) (\partial^0 \phi) \approx \frac{1}{2} m \psi^\dagger \psi + \frac{i}{2} (\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger) \quad (3.81)$$

$$\frac{1}{2} (\partial_i \phi) (\partial^i \phi) \approx -\frac{1}{2m} (\nabla \psi^\dagger) \cdot (\nabla \psi) \quad (3.82)$$

$$\frac{\lambda}{24} \phi^4 \approx \frac{\lambda}{(4m)^2} (\psi^\dagger \psi)^2 , \quad (3.83)$$

Putting the pieces together, we obtain

$$\mathcal{L}_{\text{nr}} = \frac{i}{2} (\psi^\dagger \partial_0 \psi - \psi \partial_0 \psi^\dagger) - \frac{1}{2m} (\nabla \psi^\dagger) \cdot (\nabla \psi) - \frac{1}{2} g (\psi^\dagger \psi)^2, \quad (3.84)$$

where we have defined  $g = \lambda/8m^2$ . Integrating the first term by part, we obtain the standard form of a nonrelativistic field theory:

$$\mathcal{L}_{\text{nr}} = i\psi^\dagger \partial_0 \psi - \frac{1}{2m} (\nabla \psi^\dagger) \cdot (\nabla \psi) - \frac{1}{2} g (\psi^\dagger \psi)^2. \quad (3.85)$$

The Lagrangian density Eq. (3.85) is used as a starting point for nonrelativistic selfinteracting field theories.

## 3.6 Problems

### 3.6.1 Nonrelativistic field theory

The Lagrangian density for the nonrelativistic field  $\psi$  is given by

$$\mathcal{L} = -\frac{1}{2} i\hbar (\dot{\psi}^* \psi - \psi^* \dot{\psi}) - \frac{\hbar^2}{2m} (\nabla \psi^*) \cdot (\nabla \psi). \quad (3.86)$$

- 1) Find the equation of motion for  $\psi$ .
- 2) Find the Hamiltonian density  $\mathcal{H}$ .
- 3) Show that the Lagrangian density is invariant under a global phase transformation.
- 4) Find the conserved current that corresponds to the symmetry and interpret the continuity equation.
- 5) How can you make the Lagrangian density invariant under *local* phase transformations?

### 3.6.2 Dual field tensor

The dual tensor  $\tilde{F}^{\mu\nu}$  is defined by

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \quad (3.87)$$

where  $\varepsilon^{\mu\nu\alpha\beta}$  is the Levi-Civita symbol.  $\varepsilon^{\mu\nu\alpha\beta}$  is totally antisymmetric under permutation of indices and  $\varepsilon^{0123} = +1$ . Find the matrix  $\tilde{F}^{\mu\nu}$  and show that

$$\tilde{F}^{\mu\nu} F_{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B}. \quad (3.88)$$

Finally, find an expression for  $\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}$ .

### 3.6.3 Maxwell's equations

The Lagrangian for the electromagnetic field with sources is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu, \quad (3.89)$$

where  $j_\mu = (\rho, -\mathbf{j})$  is the external four-current. Is  $\mathcal{L}$  gauge invariant? Find Euler-Lagrange's equations and express them in terms of the electromagnetic field, i.e. in terms of  $\mathbf{E}$  and  $\mathbf{B}$ . Comments?

### 3.6.4 Free field theory for fermions

The Lagrangian for the free electron-positron field is

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \quad (3.90)$$

1) Calculate the components of the energy-momentum tensor. Find the energy and momentum densities and show explicitly current conservation.

2) The Lagrangian is invariant under global phase transformations:

$$\psi \rightarrow e^{i\alpha}\psi \quad (3.91)$$

$$\bar{\psi} \rightarrow e^{-i\alpha}\bar{\psi} \quad (3.92)$$

This gives rise to the well-known conservation of charge. Consider instead the transformation

$$\psi \rightarrow e^{i\gamma^5\alpha}\psi \quad (3.93)$$

$$\bar{\psi} \rightarrow \bar{\psi}e^{i(\gamma^5)^\dagger\alpha}, \quad (3.94)$$

where  $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$  is a *matrix*. The transformation is called an *chiral* transformation. Show that

$$(\gamma^5)^\dagger = \gamma^5, \quad (3.95)$$

$$(\gamma^5)^2 = 1, \quad (3.96)$$

$$\{\gamma^5, \gamma^\mu\} = 0. \quad (3.97)$$

3) For what values of  $m$  is the chiral transformation a symmetry of the Dirac Lagrangian? Find the corresponding 4-current (called an axial vector current since it changes sign under parity,  $\mathbf{x} \rightarrow -\mathbf{x}$ ).

# Chapter 4

## Relativistic Wave equations

### 4.1 Klein-Gordon equation

See Hemmer chapter 17.

#### 4.1.1 Relativistic corrections

In this chapter, we briefly discuss the leading relativistic corrections to the energy of a scalar particle in a static Coulomb field.

The energy  $E$  of a particle with mass  $m$  and momentum  $p$  is given by

$$\begin{aligned} E &= \sqrt{m^2 c^4 + p^2 c^2} \\ &= mc^2 \sqrt{1 + \left(\frac{p}{mc}\right)^2} . \end{aligned} \quad (4.1)$$

For small momenta, i.e. for momenta  $p \ll mc$ , we can expand Eq. (4.1) in a power series around  $p = 0$ . To second order, we obtain

$$\begin{aligned} E &\approx mc^2 \left[ 1 + \frac{1}{2} \left(\frac{p}{mc}\right)^2 - \frac{1}{8} \left(\frac{p}{mc}\right)^4 + \dots \right] \\ &= mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} . \end{aligned} \quad (4.2)$$

The first term is the energy associated with the rest mass of the particle. The second term is the usual kinetic term and is included in the standard nonrelativistic Hamiltonian  $H_0$ . For a particle in a static Coulomb field, this is

$$H_0 = -\frac{\hbar^2}{2m} \nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r} . \quad (4.3)$$

The third term is the first relativistic correction to the energy. We will denote this term by  $H'$ . In the coordinate representation, we have  $p = -i\hbar\nabla$  and is

$$H' = -\frac{\hbar^4}{8m^3c^2}\nabla^4. \quad (4.4)$$

We know that  $T = \frac{p^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$  commutes with the total angular momentum  $L^2$  as well as the  $x$ -component  $L_z$ , i.e.

$$[T, L^2] = [T, L_z] = 0. \quad (4.5)$$

Since  $H' \sim T^2$ , this implies

$$[H', L^2] = [H', L_z] = 0. \quad (4.6)$$

Furthermore,  $H'$  does not act in spin space and so

$$[H', S^2] = [H', S_z] = 0. \quad (4.7)$$

This in turn implies that  $H'$  is diagonal in the quantum numbers  $l, m_l, s$ , and  $m_s$ . Since the energy depends only on the principal quantum number  $n$ , we must in general use degenerate perturbation theory but due to vanishing commutators, this is not necessary for the perturbation  $H'$ . The first-order energy shift  $E_1$  of the state  $|\psi\rangle$  is then given

$$\begin{aligned} E_1 &= \langle\psi|H'|\psi\rangle \\ &= -\frac{1}{2mc^2}\langle\psi|T^2|\psi\rangle. \end{aligned} \quad (4.8)$$

We can now write

$$T = H_0 + \frac{Ze^2}{4\pi\epsilon_0 r}, \quad (4.9)$$

where  $H_0$  is the unperturbed Hamiltonian. This yields

$$\begin{aligned} E_1 &= -\frac{1}{2mc^2}\langle\psi|\left(H_0 + \frac{Ze^2}{4\pi\epsilon_0 r}\right)^2|\psi\rangle \\ &= -\frac{1}{2mc^2}\left[E_0^2 + 2E_0\frac{Ze^2}{4\pi\epsilon_0}\left\langle\frac{1}{r}\right\rangle + \left(\frac{Ze^2}{4\pi\epsilon_0}\right)^2\left\langle\frac{1}{r^2}\right\rangle\right], \end{aligned} \quad (4.10)$$

where  $E_0$  is the unperturbed energy:

$$E_0 = -\frac{mc^2Z^2\alpha^2}{2}\frac{1}{n^2}. \quad (4.11)$$

In order to proceed, we need to know the expectation values of  $1/r$  and  $1/r^2$  in the unperturbed state. These are given by<sup>1</sup>

$$\left\langle \frac{1}{r} \right\rangle = \frac{mcZ\alpha}{2\hbar} \frac{1}{n^2}, \quad (4.12)$$

$$\left\langle \frac{1}{r^2} \right\rangle = \frac{m^2c^2Z^2\alpha^2}{\hbar^2} \frac{1}{n^3(l + \frac{1}{2})}. \quad (4.13)$$

We can then write the energy to first order in perturbation theory

$$\begin{aligned} E &= mc^2 + E_0 + E_1 \\ &= mc^2 - \frac{mc^2Z^2\alpha^2}{2} \frac{1}{n^2} + \frac{mc^2Z^4\alpha^4}{n^4} \left( \frac{3}{8} - \frac{n}{2l+1} \right), \end{aligned} \quad (4.14)$$

where we have used  $\alpha = e^2/(4\pi\epsilon)\hbar c$ . Note that the shifted energy depends on the angular quantum number  $l$  and so the degeneracy in  $l$  from Schrödinger theory is lifted once one includes relativistic corrections. Eq. (4.14) is obtained by expanding the relativistic energy

$$E = \frac{mc^2}{\sqrt{1 + \frac{Z^2\alpha^2}{(n-l-1/2 + \sqrt{[l+1/2]^2 - Z^2\alpha^2})^2}}} \quad (4.15)$$

to second order in  $Z^2\alpha^2$ .

## 4.2 Dirac equation

In this section, we discuss some extra aspects of the Dirac equation not covered in the textbook.

### 4.2.1 Representation independence

Given two sets of  $\gamma$ -matrices which both satisfy

$$\{\gamma^\mu \gamma^\nu\} = 2g^{\mu\nu} \quad (4.16)$$

Then there exists a nonsingular matrix  $S$  such that <sup>2</sup>

$$(\gamma')^\mu = S\gamma^\mu S^{-1}. \quad (4.17)$$

Next assume that  $\psi$  is a solution to the Dirac equation, i.e.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0. \quad (4.18)$$

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<sup>1</sup>See e.g. Hemmer Sec. 5.7.

<sup>2</sup>Have confidence in this statement!

Inverting Eq. (4.17), we obtain  $\gamma^\mu = S^{-1}(\gamma')^\mu S$ . This yields

$$(iS^{-1}(\gamma')^\mu S - m)\psi = 0. \quad (4.19)$$

Multiplying this equation with  $S$  from the left, we obtain

$$(i(\gamma')^\mu - m)S\psi = 0. \quad (4.20)$$

In other words,  $\psi' = S\psi$  is a solution to the Dirac equation with the new  $\gamma$ -matrices. This shows the representation independence. Note: The components in the two wavefunctions  $\psi$  and  $\psi'$  are not the same since the matrix  $S$  shuffle them around,  $\psi' = S\psi$ .

### 4.2.2 Relativistic corrections

We now consider the Dirac equation in an external electromagnetic field given by  $A^\mu$  and its nonrelativistic limit. A complete discussion can be found in Refs. [8, 17]. The Dirac equation for stationary states in an external field  $A_\mu$  reads

$$i(\gamma^\mu \partial_\mu + iqA_\mu)\psi - m\psi = 0, \quad (4.21)$$

where we have replaced the partial derivative by the covariant derivative. This can be rewritten as

$$\gamma^0(i\partial_0 - qA_0)\psi = \gamma^j(-i\partial_j + qA_j)\psi + m\psi. \quad (4.22)$$

Assuming the time-dependence  $e^{-iEt}$  for the stationary state  $\psi$  and using  $\mathbf{p} = -i\nabla$ , we can write

$$\gamma^0(E - qA_0)\psi = \boldsymbol{\gamma} \cdot (\mathbf{p} - q\mathbf{A})\psi + m\psi. \quad (4.23)$$

If we introduce two two-component vectors  $\psi_A$  and  $\psi_B$  and using the standard representation of the  $\gamma$ -matrices, we can write the Dirac equation in an external field as follows

$$\begin{pmatrix} E - qA_0 - m & -\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A}) \\ \boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A}) & -(E - qA_0 + m) \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = 0. \quad (4.24)$$

The fields  $\psi_A$  and the field  $\psi_B$  are referred to as the large and small components of the field  $\psi$ , respectively. The reason is that the components in  $\psi_A$  are much larger than those of  $\psi_B$  in the nonrelativistic limit. Eq. (4.24) can be written as

$$-\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})\psi_A = (E - qA_0 + m)\psi_B \quad (4.25)$$

$$-\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})\psi_B = (E - qA_0 - m)\psi_A. \quad (4.26)$$

We can use the second of these equations to eliminate the  $\psi_B$  from the first and the equation for the large component  $\psi_A$  can now be written as

$$[\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})] \left[ \frac{1}{E - qA_0 + m} \right] [\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})] \psi_A = (E - qA_0 - m)\psi_A. \quad (4.27)$$

We next write  $E = m + E_{NR}$  and assume  $E - m = E_{NR} \ll m$  and  $|qA_0| \ll m$ . This is simply the nonrelativistic approximation. We obtain

$$\frac{1}{2m}[\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})]^2 \psi_A = (E_{NR} - qA_0) \psi_A \quad (4.28)$$

$$\psi_B = -\frac{\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})}{E + m - qA_0} \psi_A. \quad (4.29)$$

Reinstating factors of  $c$ , one can show from Eq. (4.29) that  $\psi_B \approx v/c \psi_A$ . Using the identity

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma}(\mathbf{A} \times \mathbf{B}) \quad (4.30)$$

the operator in Eq. (4.28) we obtain

$$\begin{aligned} [\boldsymbol{\sigma} \cdot (\mathbf{p} - q\mathbf{A})]^2 &= (\mathbf{p} - q\mathbf{A})^2 + i\boldsymbol{\sigma} \cdot [(\mathbf{p} - q\mathbf{A}) \times (\mathbf{p} - q\mathbf{A})] \\ &= (\mathbf{p} - q\mathbf{A})^2 - qi\boldsymbol{\sigma} \cdot [\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p}] \\ &= (\mathbf{p} - q\mathbf{A})^2 - qi\boldsymbol{\sigma} \cdot \mathbf{B}, \end{aligned} \quad (4.31)$$

where we have used

$$\begin{aligned} \mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p} &= -i(\nabla \times \mathbf{A}) \\ &= -i\mathbf{B}. \end{aligned} \quad (4.32)$$

This yields the nonrelativistic Hamiltonian

$$H_{NR} = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + qA_0 - \frac{q}{2m}\boldsymbol{\sigma} \cdot \mathbf{B}, \quad (4.33)$$

and the Schrödinger equation

$$H_{NR}\psi_A = E_{NR}\psi_A. \quad (4.34)$$

The first term is obtained from the free part of the nonrelativistic Hamiltonian for a particle by minimal substitution,  $\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}$ . For an electron bound to a nucleus, the second term is the usual electrostatic energy of a point charge. The last term is the interaction of the spin of the electron with the external magnetic field. A magnetic moment  $\boldsymbol{\mu}$  in an external magnetic field is  $E = -\boldsymbol{\mu} \cdot \mathbf{B}$ . If we write  $\boldsymbol{\mu} = g_s q/2m\boldsymbol{\sigma}$  in analogy with  $\boldsymbol{\mu} = q/2m\mathbf{L}$ , we conclude that the magnetic moment of an electron is  $g_s = 2$ .

We can systematically find corrections to the nonrelativistic Hamiltonian in Eq. (4.33). This is basically an expansion in  $v/c$ . We will consider three such terms. This is The first term is

$$H_{c1} = \frac{1}{8m^3}(-i\nabla)^4. \quad (4.35)$$

This correction is arising from expanding the energy-momentum relation:

$$\begin{aligned} E &= \sqrt{m^2c^4 + c^2p^2} \\ &= mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots, \end{aligned} \quad (4.36)$$

to fourth order in  $p$ . These terms are also present in the Klein-Gordon theory. This term is diagonal in  $l$ ,  $m_l$ , and  $m_s$  and so we can use first-order perturbation theory directly.

The second term is

$$H_{c2} = -\frac{q\hbar\boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{p})}{4m^2c^2}. \quad (4.37)$$

For a time-independent potential  $A_0$ , we have

$$\begin{aligned} \mathbf{E} &= -\nabla A_0 \\ &= -\frac{dA_0}{dr} \mathbf{e}_r. \end{aligned} \quad (4.38)$$

Inserting this into (4.37), we obtain

$$\begin{aligned} H_{hc2} &= \frac{q\hbar}{4m^2c^2} \frac{1}{r} \frac{dA_0}{dr} \boldsymbol{\sigma} \cdot (\mathbf{r} \times \mathbf{p}) \\ &= \frac{q\hbar}{4m^2c^2} \frac{1}{r} \frac{dA_0}{dr} \boldsymbol{\sigma} \cdot \mathbf{L}. \end{aligned} \quad (4.39)$$

This is the well-known spin-orbit coupling. This term is not diagonal in  $l$ ,  $m_l$ , and  $m_s$  and so we must change basis.

Finally, the third term we consider is

$$H_{c3} = -\frac{q\hbar^2}{8m^2c^2} \nabla \cdot \mathbf{E}. \quad (4.40)$$

The right-hand side is proportional to the charge density. For a Coulomb potential we have  $\rho = q'\delta(\mathbf{r})$ , where  $q'$  is the charge. and in this case we obtain

$$H_{c3} = -\frac{qq'\hbar^2}{8m^2c^2} \delta(\mathbf{r}). \quad (4.41)$$

This term was first found by Charles G. Darwin (1887-1962) and is called the Darwin term. He was the grandson of *the* Charles Darwin (1809-1882). Note that for an electron in the field of a nucleus of charge  $q' = Z|e|$ , the prefactor is positive. Since it involves the delta-function in the origin, it affects  $s$ -states only (only  $s$ -states have nonvanishing wavefunctions in the origin). We can use nondegenerate perturbation theory since it is diagonal in  $l$ ,  $m_l$ , and  $m_s$ .

## Exact spectrum

The energy spectrum for a Dirac fermion in a Coulomb potential can be calculated exactly [8]. The orbital and spin angular momentum operators do not commute with the Dirac Hamiltonian and are thus not good quantum numbers. However, the sum  $\mathbf{J} = \mathbf{L} + \mathbf{S}$  does commute with  $H$  and so the quantum number  $j$  can be used to label the solutions The spectrum is

$$E_{\text{exact}} = \frac{m}{\sqrt{1 + \frac{Z^2\alpha^2}{n-j-1/2 + \sqrt{(j+1/2)^2 - Z^2\alpha^2}}}}, \quad (4.42)$$

where  $n$  is the usual principal quantum number and  $j$  is the *total* angular momentum quantum number. Upon expansion to second order in  $Z^2\alpha^2$ , we obtain

$$E = m \left[ 1 - \frac{1}{2} \frac{(z\alpha)^2}{n^2} - \frac{1}{2} \frac{(z\alpha)^4}{n^3} \left( \frac{1}{j + \frac{1}{2}} - \frac{3}{4n} \right) \right] . \quad (4.43)$$

This is identical to the sum of the first-order energy shifts calculated using degenerate perturbation theory with the three perturbations considered above.

## 4.3 Problems

### 4.3.1 Operator relations

Show the operator relation

$$(\boldsymbol{\sigma} \cdot \mathbf{A})(\boldsymbol{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\boldsymbol{\sigma} \cdot (\mathbf{A} \times \mathbf{B}) . \quad (4.44)$$

### 4.3.2 Klein-Gordon equation in external fields

The Lagrangian density of a complex scalar field coupled to the Maxwell field is

$$\mathcal{L} = (D_\mu \Phi)^*(D^\mu \Phi) - m^2 \Phi^* \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} .$$

1) Show that the equation of motion for  $\Phi^*$  reads

$$[D_\mu D^\mu + m^2] \Phi = 0 .$$

We next consider a particle with charge  $q$  in two spatial dimensions in an external field where the vector potential is given by  $A^\mu = (0, -By, 0, 0)$ , where  $B$  is a constant.

2) Compute the electric field  $\mathbf{E}$  and the magnetic field  $\mathbf{B}$ .

3) Show that the Hamiltonian density can be written as

$$\begin{aligned} \mathcal{H} = & (\partial_0 \Phi)^* (\partial_0 \Phi) + [(\partial_x - iqBy)\Phi^*][(\partial_x + iqBy)\Phi] + (\partial_y \Phi)^* (\partial_y \Phi) \\ & + m^2 \Phi^* \Phi + \frac{1}{2} B^2 . \end{aligned}$$

Is  $\mathcal{H}$  Lorentz invariant?

4) The eigenfunctions can be written as

$$\Phi = e^{-i(Et - p_x x)} f(y) ,$$

where  $E$  is the energy of the state and  $p_x$  is the  $x$ -component of the momentum. Show by inserting  $\Phi$  into the equation of motion that  $f(y)$  satisfies

$$\left[ -\frac{d^2}{dy^2} - E^2 + m^2 + (p_x - qBy)^2 \right] f(y) = 0 .$$

Use this result to derive the spectrum, i.e. the energy eigenvalues. Hint: Harmonic oscillator.

# Chapter 5

## Field quantization

In this chapter, we discuss quantization of field. As a prelude we briefly review the harmonic oscillator and its quantization.

### 5.1 Harmonic oscillator

The Hamiltonian  $H$  of a one-dimensional harmonic oscillator with frequency  $\omega$  and mass  $m$  is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad (5.1)$$

where  $x$  is the position operator of the oscillator and  $p$  is its linear momentum operator. From  $\hbar$ , the mass and the frequency, we can form quantity  $l = \sqrt{\hbar/m\omega}$  which sets the length scale of the oscillator.

Let us define the dimensionless operators  $a$  and  $a^\dagger$  by

$$a = \frac{1}{\sqrt{2}} \left( \frac{x}{l} + \frac{ip}{\hbar} l \right), \quad (5.2)$$

$$a^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x}{l} - \frac{ip}{\hbar} l \right), \quad (5.3)$$

It is then easy to verify that  $a$  and  $a^\dagger$  satisfy the commutation relation

$$[a, a^\dagger] = 1. \quad (5.4)$$

Inverting the relations (5.2) and (5.3), we obtain

$$x = \frac{l}{\sqrt{2}}(a + a^\dagger), \quad (5.5)$$

$$p = \frac{\hbar}{\sqrt{2}il}(a - a^\dagger). \quad (5.6)$$

We can now express the Hamiltonian Eq. (5.1) in terms of the operators  $a$  and  $a^\dagger$ . This yields

$$\begin{aligned} H &= -\frac{1}{4m} \frac{\hbar^2}{l^2} (a - a^\dagger)^2 + \frac{1}{4} m \omega^2 l^2 (a + a^\dagger)^2 \\ &= \frac{1}{2} \hbar \omega (a a^\dagger + a^\dagger a) \\ &= \hbar \omega \left( \frac{1}{2} + a^\dagger a \right) , \end{aligned} \quad (5.7)$$

where we in the last line have used the commutator (5.4). Note that the creation operator  $a^\dagger$  stands to the left of the annihilation operator  $a$ . Whenever we have an operator  $A$  which is a product of  $a$  and  $a^\dagger$  and all the creation operators stand to the left of the annihilation operators, the operator  $A$  is called *normal ordered*.

If we introduce the operator  $N = a^\dagger a$ , the problem of finding the eigenvalues for  $H$  is the same as finding the eigenvalues of  $N$ , since

$$H = \hbar \omega \left( \frac{1}{2} + N \right) . \quad (5.8)$$


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## Example

The commutator between  $N$  and  $a$  is

$$[N, a] = [a^\dagger a, a] = a^\dagger [a, a] + [a^\dagger a] a = -a . \quad (5.9)$$

This implies that

$$[H, a] = -\hbar \omega a . \quad (5.10)$$

Similarly, one can show that

$$[N, a^\dagger] = a^\dagger . \quad (5.11)$$


---

Let  $|\psi_E\rangle$  be an eigenstate of the Hamiltonian  $H$  with energy  $E$ , i.e.

$$H|\psi_E\rangle = E|\psi_E\rangle . \quad (5.12)$$

If  $a|\psi_E\rangle$  also is a vector in Hilbert space, we must have

$$\langle \psi_E | a^\dagger a | \psi_E \rangle > 0 . \quad (5.13)$$

Moreover

$$H a |\psi_E\rangle = a (H - \hbar \omega) |\psi_E\rangle ,$$

where we have used the commutator between  $a$  and  $H$ , Eq. (5.10). This yields

$$Ha|\psi_E\rangle = a(E - \hbar\omega)|\psi_E\rangle. \quad (5.14)$$

Thus  $a|\psi_E\rangle$  is an eigenstate of  $H$  with energy  $E - \hbar\omega$ . The operator  $a$  is therefore called a *lowering operator*. This implies that

$$\begin{aligned} \langle\psi_E|a^\dagger a|\psi_E\rangle &= \langle\psi_E|H - \frac{1}{2}\hbar\omega|\psi_E\rangle \\ &= E - \frac{1}{2}\hbar\omega \\ &> 0. \end{aligned} \quad (5.15)$$

Thus  $E > \frac{1}{2}\hbar\omega$ . We can continue these arguments *ad infinitum* lowering the energy by applying  $a$ . Thus the energy of the system is unbounded from below and this makes no sense. The only way out is that there exists a state,  $|0\rangle$ , such that  $a$  annihilates it:

$$a|0\rangle = 0. \quad (5.16)$$

We call this state the vacuum state. Moreover, it follows from Eqs. (5.7) and (5.16)

$$\begin{aligned} H|0\rangle &= \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)|0\rangle \\ &= \frac{1}{2}\hbar\omega|0\rangle. \end{aligned} \quad (5.17)$$

Finally, consider the action of  $H$  on the state  $a^\dagger|0\rangle$ . One finds

$$\begin{aligned} Ha^\dagger|0\rangle &= (\hbar\omega a^\dagger + a^\dagger H)|0\rangle \\ &= \frac{3}{2}\hbar\omega a^\dagger|0\rangle, \end{aligned} \quad (5.18)$$

where we have used  $[H, a^\dagger] = \hbar\omega a^\dagger$  and Eq. (5.17). Thus the state  $a^\dagger|0\rangle$  is an eigenstate of  $H$  with eigenvalue  $\frac{3}{2}\hbar\omega$ . More generally, one can show that if  $|\psi_E\rangle$  is an eigenstate of  $H$  with energy  $E$ , then  $a^\dagger|\psi_E\rangle$  is an eigenstate of  $H$  with energy  $E + \hbar\omega$ . The operator  $a^\dagger$  is therefore called a *raising operator*. The whole Hilbert space can then be built from the ground state  $|0\rangle$  and the raising operator  $a^\dagger$ . An excited state can be written as

$$|n\rangle = (a^\dagger)^n |0\rangle, \quad (5.19)$$

whose energy is  $E_n = \hbar\omega(n + \frac{1}{2})$  and contains  $n$  quanta.

## 5.2 Fields

When we are quantizing a system of  $N$  particles, the commutation relations are given by

$$[q_i, p_j] = i\delta_{ij}, \quad [q_i, q_j] = [p_i, p_j] = 0, \quad (5.20)$$

where  $i, j = 1, 2, 3 \dots N$ . In a field theory, we have infinitely many degrees of freedom, which is reflected in the fact that we have introduced the momentum density  $\pi$ , which is the conjugate variable to the field  $\phi$ . For a scalar field  $\phi$  this is

$$\pi(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} . \quad (5.21)$$

The field  $\phi(\mathbf{x}, t)$  and its conjugate momentum  $\pi(\mathbf{y}, t)$  play the role of  $q_i$  and  $p_j$  where  $\mathbf{x}$  and  $\mathbf{y}$  are continuous indices corresponding to  $i$  and  $j$ . Quantization of a field theory is now carried out by postulating the following commutation relations:

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) , \quad [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0 . \quad (5.22)$$

Two remarks are in order. Firstly, the Kronecker delta is replaced by Dirac's delta function, which is natural given the continuous nature of a field. Secondly, note that the commutators are at the same time  $t$  for  $\phi(\mathbf{x}, t)$  and  $\pi(\mathbf{y}, t)$ , whence they are called *equal time commutation relations*.

The Klein-Gordon equation

$$\left( \frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi(\mathbf{x}, t) = 0 , \quad (5.23)$$

admits plane-wave solutions of the form

$$\phi \sim e^{-ipx} , \quad (5.24)$$

where  $px = p_0 t - \mathbf{p} \cdot \mathbf{x}$  and  $p_0 = \sqrt{|\mathbf{p}|^2 + m^2}$ . These solutions form a complete set and we therefore expand the classical field in a Fourier series with Fourier coefficients  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^*$

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^* e^{ipx} \right] , \quad (5.25)$$

where  $E_p = \sqrt{|\mathbf{p}|^2 + m^2}$ . The prefactor  $\frac{1}{\sqrt{2E_p}}$  is simply a convenient normalization of the solutions. The coefficients  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^*$  are classically Fourier coefficients, but since  $\phi(\mathbf{x}, t)$  is a quantum field they are really operators and we therefore write the *field operator* as

$$\phi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{\mathbf{p}} e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx} \right] , \quad (5.26)$$

The momentum density can now be written

$$\pi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left[ a_{\mathbf{p}} e^{-ipx} - a_{\mathbf{p}}^\dagger e^{ipx} \right] . \quad (5.27)$$

The equal-time commutator can now be written as

$$\begin{aligned}
 [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \left( \frac{-i}{2} \right) \sqrt{\frac{E_q}{E_p}} \left[ [a_{\mathbf{p}}, a_{\mathbf{q}}] e^{-i(px+qy)} \right. \\
 &\quad \left. - [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] e^{i(px+qy)} + [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}] e^{i(px-xy)} - [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] e^{-i(px-xy)} \right] \\
 &= i\delta^3(\mathbf{x} - \mathbf{y}) .
 \end{aligned} \tag{5.28}$$

Next we *assume* that the operators above satisfy the commutation relations

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \tag{5.29}$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0 , \tag{5.30}$$

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0 . \tag{5.31}$$

This yields

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i(2\pi)^3 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \sqrt{\frac{E_q}{E_p}} [\delta^3(\mathbf{p} - \mathbf{q})] e^{-i(px-xy)} . \tag{5.32}$$

Integrating over  $\mathbf{q}$  gives

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} . \tag{5.33}$$

The integral on the right-hand-side is the integral representation of the  $\delta$ -function in three dimensions, i.e.  $\delta^3(\mathbf{x} - \mathbf{y})$ . Thus we recover the first equal-time commutation relation in Eq. (5.22). The remaining commutators can be verified in the same manner. Now the commutators above are simply those of the harmonic oscillator. We can therefore interpret our *quantum field* as a collection of independent harmonic oscillators with frequency  $\omega = E_p$ . Let us finally note that we could have started by expanding the scalar field in Fourier series of  $e^{ipx}$  with Fourier coefficients  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^*$  as in Eq. (5.25). Quantization is then carried out directly by *promoting* the Fourier coefficients to operators  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^\dagger$  satisfying the commutation relations for the harmonic oscillator. Working out the commutator of  $\phi(\mathbf{x}, t)$  and  $\pi(\mathbf{x}, t)$  we recover Eq. (5.22).

We next consider the Hamiltonian, which can be written as

$$H = \int d^3 x \mathcal{H} , \tag{5.34}$$

where  $\mathcal{H}$  is the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 . \tag{5.35}$$

For a single oscillator with frequency  $\omega$ , we know the Hamiltonian:

$$H = \omega \left( \frac{1}{2} + a^\dagger a \right) . \tag{5.36}$$

Inserting the expression for  $\phi$  and its derivatives yields

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} E_p \left( \frac{1}{2} [a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] + a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right) \\ &= \frac{1}{2} V \int \frac{d^3p}{(2\pi)^3} E_p + \int \frac{d^3p}{(2\pi)^3} E_p a_{\mathbf{p}}^\dagger a_{\mathbf{p}} , \end{aligned} \quad (5.37)$$

where we have used that  $\delta^3(0) = V/(2\pi)^3$  and  $V$  is the volume of space. Note that the first term is actually divergent since it is the integral of the ground-state energies of the harmonic oscillators with energy  $E_p$ . This suggests that the ground state energy (the vacuum energy) of a quantum field is infinite! However, we can only measure energy differences between states and so it is customary to simply delete this term<sup>1</sup>. Note, however, this may be dangerous if we are dealing with general relativity where all energy density is a source of curvature in Einstein's field equations.

It is now easy to calculate the commutator between  $H$  and  $a_{\mathbf{q}}$ :

$$\begin{aligned} [H, a_{\mathbf{q}}] &= \int \frac{d^3p}{(2\pi)^3} E_p [a_{\mathbf{p}}^\dagger a_{\mathbf{p}}, a_{\mathbf{q}}] \\ &= \int \frac{d^3p}{(2\pi)^3} E_p [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}] a_{\mathbf{p}} \\ &= - \int \frac{d^3p}{(2\pi)^3} E_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) a_{\mathbf{p}} \\ &= -E_q a_{\mathbf{q}} , \end{aligned} \quad (5.38)$$

i.e.  $Ha_{\mathbf{q}} = a_{\mathbf{q}}(H - E_q)$ . This implies that  $H^n a_{\mathbf{q}} = a_{\mathbf{q}}(H - E_q)^n$  and therefore

$$e^{iHt} a_{\mathbf{q}} e^{-iHt} = a_{\mathbf{q}} e^{-iE_q t} . \quad (5.39)$$

By taking the adjoint, we obtain<sup>2</sup>

$$e^{iHt} a_{\mathbf{q}}^\dagger e^{-iHt} = a_{\mathbf{q}}^\dagger e^{iE_q t} . \quad (5.40)$$

We have now found the time dependence of the creation and annihilation operators and so in the time-independent Schrödinger picture, we find

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right] , \quad (5.41)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left[ a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right] . \quad (5.42)$$

---

<sup>1</sup>It is related to the operator-ordering problem in quantum field theory. The expression  $xp$  is the same as  $px$  in classical mechanics since they are functions. They are not identical once  $x$  and  $p$  are promoted to operators.

<sup>2</sup>Recall  $H = H^\dagger$ .

By letting  $\mathbf{p} \rightarrow -\mathbf{p}$  in the second terms inside the bracket, these equations can be conveniently written as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}}, \quad (5.43)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} [a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{-\mathbf{p}}^\dagger] e^{i\mathbf{p}\cdot\mathbf{x}}. \quad (5.44)$$

The vacuum state is denoted by  $|0\rangle$  and is by definition annihilated by  $a_{\mathbf{p}}$ , i.e.

$$a_{\mathbf{p}}|0\rangle = 0. \quad (5.45)$$

A single-particle state that contains a particle with momentum  $\mathbf{p}$  and energy  $E_p$  is given by

$$|\mathbf{p}\rangle = a_{\mathbf{p}}^\dagger |0\rangle. \quad (5.46)$$

If the vacuum state is normalized, i.e. if  $\langle 0|0\rangle = 1$ , the norm of the state  $|\mathbf{p}\rangle$  is

$$\begin{aligned} \langle \mathbf{p}|\mathbf{q}\rangle &= \langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{q}}^\dagger a_{\mathbf{p}} + (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})|0\rangle \\ &= (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}). \end{aligned} \quad (5.47)$$

It can be shown that this normalization is not Lorentz invariant. The problem is that volumes are not invariant. In order to make the normalization of the states invariant, one must define the single-particle state as

$$|\mathbf{p}\rangle = \sqrt{2E_p} a_{\mathbf{p}}^\dagger |0\rangle. \quad (5.48)$$

### 5.3 Nonrelativistic field theories

In the preceding chapter, we have quantized the Klein-Gordon equation which is a relativistic field theory. In this chapter, we take a closer look at nonrelativistic (NR) field theories. The Lagrangian of a free nonrelativistic theory is

$$\mathcal{L} = i\psi^* \partial_0 \psi - \frac{1}{2m} (\nabla \psi)^* (\nabla \psi). \quad (5.49)$$

The canonical momentum is then

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} &= i\psi^* \\ &\equiv \Pi. \end{aligned} \quad (5.50)$$

Note that in the NR case, the conjugate momentum is essentially equal to the complex conjugate of the field and not its time derivative. This is of course due to the fact that  $\mathcal{L}$  is linear and not quadratic in the time derivative. Moreover, we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_i \psi)} = \frac{1}{2m} \partial^i \psi^* . \quad (5.51)$$

The equation of motion then becomes

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = 0 , \quad (5.52)$$

since  $\frac{\partial \mathcal{L}}{\partial \psi} = 0$ . Inserting Eqs. (5.50) and (5.51) into Eq. (5.52), we obtain the Schrödinger equation

$$i\partial_0 \psi^* - \frac{\nabla^2}{2m} \psi^* = 0 . \quad (5.53)$$

The Schrödinger equation admits solution of the form

$$\psi(\mathbf{x}, t) = e^{-i(p_0 t - \mathbf{p} \cdot \mathbf{x})} , \quad (5.54)$$

where  $p_0 = p^2/2m = E_p$ <sup>3</sup>. The field operator  $\psi(\mathbf{x}, t)$  can now be expanded in a complete set of solutions as usual<sup>4</sup>

$$\psi(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} a_{\mathbf{p}} e^{-i(p_0 t - \mathbf{p} \cdot \mathbf{x})} , \quad (5.55)$$

and the adjoint is given by

$$\psi^\dagger(\mathbf{x}, t) = \int \frac{d^3 p}{(2\pi)^3} a_{\mathbf{p}}^\dagger e^{i(p_0 t - \mathbf{p} \cdot \mathbf{x})} , \quad (5.56)$$

Again we assume the equal-time commutation relations

$$[\psi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) , \quad (5.57)$$

while all other commutators vanish. This yields

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = \delta^3(\mathbf{x} - \mathbf{y}) . \quad (5.58)$$

Inserting the expressions for  $\psi$  and  $\psi^\dagger$  into the commutator, we find

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] e^{-i(p_0 - q_0)t} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} . \quad (5.59)$$

---

<sup>3</sup>This relation is found by inserting the ansatz (5.54) into the Schrödinger equation.

<sup>4</sup>Note that we have already promoted the Fourier coefficients to operators

Assuming that

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) , \quad (5.60)$$

the commutator reduces to

$$\begin{aligned} [\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] &= \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} e^{-i(p_0 - q_0)t} e^{i(\mathbf{p} \cdot \mathbf{x} - \mathbf{q} \cdot \mathbf{y})} (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \\ &= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} \\ &= \delta^3(\mathbf{x} - \mathbf{y}) , \end{aligned} \quad (5.61)$$

as it should. The other commutators are worked out in the same manner.

We have seen that the Hamiltonian can be written as

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} . \end{aligned} \quad (5.62)$$

Let  $|0\rangle$  be the vacuum state, i.e. the state that is annihilated by  $a_{\mathbf{q}}$  for all  $\mathbf{q}$ , i.e.  $a_{\mathbf{q}}|0\rangle = 0$ . What is the state  $a_{\mathbf{q}}^\dagger|0\rangle$ ? Letting  $H$  act on it, we find

$$\begin{aligned} H a_{\mathbf{q}}^\dagger|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}^\dagger|0\rangle \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} a_{\mathbf{p}}^\dagger [(2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) + a_{\mathbf{q}}^\dagger a_{\mathbf{p}}] |0\rangle \\ &= \frac{q^2}{2m} a_{\mathbf{q}}^\dagger|0\rangle , \end{aligned} \quad (5.63)$$

where we have used that  $a_{\mathbf{p}}$  annihilates the vacuum and that the delta-function picks out  $\mathbf{p} = \mathbf{q}$  from the integral. Thus the state  $a_{\mathbf{q}}^\dagger|0\rangle$  is an eigenstate of  $H$  with eigenvalue  $q^2/2m$ . Thus it is a single-particle state with energy  $E = q^2/2m$ . Not exactly shocking.

## 5.4 Note on fermions

If we are quantizing fermions, we cannot postulate the equal-time commutator relation (5.57). Instead we postulate the equal-time *anticommutator*<sup>5</sup>

$$\{\psi(\mathbf{x}, t), \Pi(\mathbf{y}, t)\} = i\delta^3(\mathbf{x} - \mathbf{y}) . \quad (5.64)$$

---

<sup>5</sup>In NR field theory it is not obvious that we need the anticommutator in the first place. However, quantizing Dirac theory using the commutator leads to logical inconsistencies.

Working out the details, one can show that the fundamental relation (5.64) is consistent with the anticommutator between the annihilation and creation operators

$$\{a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\} = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q}) , \quad (5.65)$$

while all other anticommutators vanish. The anticommutator relations have some important consequences. If we denote, as usual,  $|0\rangle$  as vacuum state, the state

$$a_{\mathbf{q}}^{\dagger}|0\rangle , \quad (5.66)$$

is a state containing a particle with momentum  $\mathbf{q}$ . What about the state

$$|\psi\rangle = a_{\mathbf{p}}^{\dagger} a_{\mathbf{q}}^{\dagger} |0\rangle . \quad (5.67)$$

Using the anticommutator, we can write

$$|\psi\rangle = -a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}}^{\dagger} |0\rangle .$$

The minus sign shows that the state  $|\psi\rangle$  is antisymmetric under the permutation of the two particles and thus the anticommutator implements the Pauli principle. Moreover, if  $\mathbf{p} = \mathbf{q}$ , we find

$$|\psi\rangle = -|\psi\rangle . \quad (5.68)$$

In other words,  $|\psi\rangle = 0$ . Thus a state *cannot* contain more than a single particle with the same momentum. This is again the Pauli principle stating that the states in Hilbert space cannot accommodate two or more identical particles in the same single-particle state. Thus the state  $|\mathbf{q}\rangle = a_{\mathbf{q}}^{\dagger}|0\rangle$  is annihilated by  $a_{\mathbf{q}}^{\dagger}$  just like  $|0\rangle$  is annihilated by  $a_{\mathbf{q}}$  and

$$\begin{aligned} a_{\mathbf{p}}|\mathbf{q}\rangle &= a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} |0\rangle \\ &= [\{a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\} - a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}}] |0\rangle \\ &= (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) |0\rangle . \end{aligned} \quad (5.69)$$

This implies

$$\begin{aligned} \langle 0|a_{\mathbf{p}}|\mathbf{q}\rangle &= \langle 0|a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} |0\rangle \\ &= (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) , \end{aligned} \quad (5.70)$$

where we have used that the vacuum state is normalized to unity.

## 5.5 Propagator

We next consider the following quantity

$$D(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle . \quad (5.71)$$

The fact that the function  $D$  is a function of the *difference*  $x - y$  follows from Lorentz invariance. Inserting the expressions for  $\phi(x)$  and  $\phi(y)$  into this equation, we obtain

$$D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_p E_q}} e^{iqy - ipx} \langle p|q \rangle, \quad (5.72)$$

where we have used that  $a_p$  annihilates the vacuum. Since the  $\langle p|q \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q})$ , it reduces to

$$D(x - y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{ip(y-x)}. \quad (5.73)$$

We next define the Feynman propagator by

$$D_F(x - y) = \begin{cases} D(x - y), & x^0 > y^0 \\ D(y - x), & y^0 > x^0. \end{cases} \quad (5.74)$$

Using the step function this can be rewritten as

$$\begin{aligned} D_F(x - y) &= \theta(x^0 - y^0) D(x - y) + \theta(y^0 - x^0) D(y - x), \\ &\equiv \langle 0|T[\phi(x)\phi(y)]|0 \rangle, \end{aligned} \quad (5.75)$$

where  $T$  is called a *time-ordered* product. Note that the earliest time stands to the right. We will next show that the Feynman propagator can be written as a four-dimensional integral

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}, \quad (5.76)$$

where  $\epsilon > 0$  is a small quantity. The poles in the complex  $p_0$  plane are given by  $p_0 = \pm(E_p - i\epsilon)$ . The term  $i\epsilon$  ensures that the poles are slightly off the real axis. In order to show that the above expression is correct, we must perform the integration over  $p_0$ . The integral we must evaluate is <sup>6</sup>

$$I = \int \frac{dp_0}{2\pi} \frac{i}{(p_0 + E_p - i\epsilon)(p_0 - E_p + i\epsilon)} e^{-ip_0(x^0 - y^0)} \quad (5.77)$$

When  $x^0 < y^0$ , we can close the contour in the upper half plane, see Fig. 5.1. We then pick up a contribution from the pole  $p_0 = -E_p + i\epsilon$ . The integral then reduces to

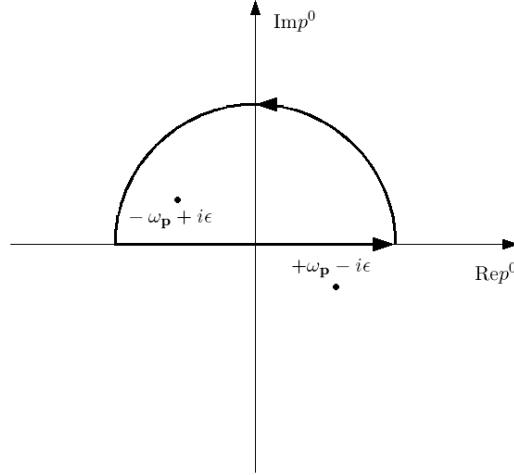
$$I = \frac{1}{2E_p} e^{iE_p(x^0 - y^0)}. \quad (5.78)$$

We next want to show that the contribution to the integral from the semi-circle vanishes in the limit where its radius  $R \rightarrow \infty$ . To this end, we write the complex variable  $p_0$  as

$$\begin{aligned} p_0 &= Re^{i\theta} \\ &= R \cos \theta + iR \sin \theta. \end{aligned} \quad (5.79)$$

---

<sup>6</sup>We are simply considering the  $p_0$ -dependent part of the integral in Eq. (5.76).

Figure 5.1: Contour in the complex  $p_0$ -plane.

This implies that we can write

$$Re[e^{iE_p(x^0-y^0)}] = e^{R \sin \theta(x^0-y^0)}. \quad (5.80)$$

In the upper plane, we have  $Im p_0 > 0$  and since  $x^0 < y^0$ , the exponent is negative. This implies that the integrand evaluated on the semi-circle is vanishing exponentially fast as  $R \rightarrow \infty$  and the contribution to the integral also vanishes. The same argument applies in the case where  $x^0 > y^0$ , with the replacement  $E_p \rightarrow -E_p$  since we in this case close the contour in the lower half-plane

Looking at Eq. (5.76), we conclude that the Fourier transform of  $D_F(x-y)$  is precisely the integrand. Denoting this function by  $D_F(p)$ , the propagator in momentum space is

$$D_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \quad (5.81)$$

The dispersion relation is then given by the poles of the propagator,  $E_p = \sqrt{p^2 + m^2}$ . This property is general.

What is the mathematical interpretation of the propagator? It satisfies

$$\begin{aligned} [-\partial_\mu \partial^\mu + m^2] D_F(x-y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{i(-p^2 + m^2)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \\ &= -i \int \frac{d^4 p}{(2\pi)^4} e^{-ip(x-y)} \\ &= -i \delta^4(x-y). \end{aligned} \quad (5.82)$$

Hence, the free particle propagator is a Green's function for the Klein-Gordon operator. It creates a particle with momentum  $p$  at  $x$  with amplitude  $e^{ipx}$  which propagates to the point  $y$  and is annihilated.

### Example

We already know the Green's function in electromagnetism. Consider Maxwell's equation:

$$\nabla \cdot \mathbf{E} = \rho, \quad (5.83)$$

where  $\rho$  is the charge density. The electric field can be expressed in terms of the scalar potential  $\Phi$  as

$$\mathbf{E} = -\nabla\Phi. \quad (5.84)$$

Combining the two equations, we obtain

$$-\nabla^2\Phi = \rho. \quad (5.85)$$

For a point source  $\rho = q\delta(\mathbf{r})$ , we find

$$\begin{aligned} -\nabla^2\Phi &= -\frac{d^2\Phi}{dr^2} \\ &= q\delta(\mathbf{r}). \end{aligned} \quad (5.86)$$

The function  $\Phi(r)$  and  $\delta(\mathbf{r})$  can be written in terms of their Fourier transforms:

$$\Phi(r) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \Phi(p), \quad (5.87)$$

$$\delta(\mathbf{r}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}}. \quad (5.88)$$

Inserting these expressions into Eq. (5.86), we obtain

$$\begin{aligned} \frac{d^2}{dr^2} \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} \Phi(p) &= - \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}} p^2 \Phi(p) \\ &= -q \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{r}}. \end{aligned} \quad (5.89)$$

This yields

$$\Phi(p) = \frac{q}{p^2}. \quad (5.90)$$

A straightforward calculation shows that its Fourier transform is

$$\begin{aligned} \Phi(r) &= q \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot\mathbf{r}} \frac{1}{p^2} \\ &= \frac{q}{4\pi r}. \end{aligned} \quad (5.91)$$

This is simply the Coulomb potential.

---

## 5.6 Problems

### 5.6.1 Anharmonic oscillator

Consider the *anharmonic oscillator* with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \frac{\lambda}{24}x^4, \quad (5.92)$$

where  $\lambda$  is a constant.

1) The last term can be considered an interaction, i.e. we write

$$H = H_0 + H_1 \quad (5.93)$$

where

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad (5.94)$$

$$H_1 = \frac{\lambda}{24}x^4. \quad (5.95)$$

1) Express  $H_1$  in terms of  $a$  and  $a^\dagger$ . Consider the terms in  $H_1$  with two factors of  $a$  and two factors of  $a^\dagger$ . Write these terms in normal-ordered form.

2) Use first-order perturbation theory to calculate the change in the energy of the ground state. Argue that only the terms found in 1) contribute and use this to simplify your calculations.

### 5.6.2 Momentum operator

Consider the momentum operator for the Klein-Gordon field

$$\mathbf{P} = - \int d^3x \pi(\mathbf{x}) \nabla \phi(\mathbf{x}). \quad (5.96)$$

Express this operator as a momentum integral involving  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^\dagger$  in the same way as we expressed the Hamiltonian.

### 5.6.3 Nonrelativistic fermions

The Lagrangian for a Schrödinger field  $\psi$  is given by

$$\mathcal{L} = -\frac{1}{2}i\hbar(\dot{\psi}^\dagger\psi - \psi^\dagger\dot{\psi}) - \frac{\hbar^2}{2m}(\nabla\psi^\dagger)\cdot(\nabla\psi). \quad (5.97)$$

The Lagrangian is invariant under a global phase transformation

$$\psi \rightarrow e^{i\alpha}\psi, \quad (5.98)$$

$$\psi^\dagger \rightarrow e^{-i\alpha}\psi^\dagger, \quad (5.99)$$

where  $\alpha$  is a constant. In the following, the field describes bosons.

1) Use the field operator  $\psi(\mathbf{x}, t)$  to express the conserved charge operator  $Q$  that corresponds to this symmetry in terms of creation and annihilation operators.

2) Use Heisenberg's equation of motion

$$\frac{dQ}{dt} = i[H, Q], \quad (5.100)$$

to show explicitly that  $Q$  is independent of time.

#### 5.6.4 Integral representation of step function

Prove the integral representation of the step function

$$\theta(t - t') = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega(t-t')}}{\omega + i\eta}, \quad (5.101)$$

where  $\eta$  is a small positive quantity.

#### 5.6.5 Propagator for NR field

Consider a norelativistic field theory with Lagrangian

$$\mathcal{L} = -\frac{1}{2}i\hbar(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi) - \frac{\hbar^2}{2m}(\nabla\psi^\dagger) \cdot (\nabla\psi). \quad (5.102)$$

The equation of motion for  $\psi$  is

$$\left[ -i\partial_0 - \frac{1}{2m}\nabla^2 \right] \psi = 0. \quad (5.103)$$

Find the momentum space propagator  $G(p)$  by using that the propagator  $G(x, y)$  is the Green's function of the differential operator  $\left[ -i\partial_0 - \frac{1}{2m}\nabla^2 \right]$ .

#### 5.6.6 Propagator for the Dirac field

Repeat the previous problem for relativistic fermions.

### 5.6.7 Finite density

In statistical mechanics, we can use different ensembles and their corresponding partition functions. If we consider a system with finite chemical potential, i.e. when the particle number is not fixed, we make the replacement

$$H \rightarrow H - \mu N , \quad (5.104)$$

where  $H$  is the Hamiltonian,  $\mu$  is the chemical potential, and  $N$  is the particle number. More generally, we can replace  $N$  by any conserved quantity  $Q$  (arising from a continuous symmetry), e.g. electric charge and  $\mu$  is the chemical potential that corresponds to the conserved quantity.

The replacement can be written as  $\mathcal{K} \rightarrow \mathcal{H}$ , where

$$\mathcal{K} = \mathcal{H} - \mu j^0 , \quad (5.105)$$

where  $\mathcal{H}$  is the Hamiltonian density and  $j^0$  is the (charge) density.

1) The Lagrangian for relativistic fermions has a global phase symmetry and an associated conserved charge. Introduce a chemical potential for the charge density and use the result derived earlier to calculate  $\mathcal{K}$ .

2) Go “backwards” and calculate the Lagrangian density  $\mathcal{L}$  from the new Hamiltonian density  $\mathcal{K}$ . Show that this corresponds to the replacement recipe

$$\partial_0 \rightarrow \partial_0 - i\mu . \quad (5.106)$$

Conclude that  $\mu$  can be regarded as the zeroth component  $A_0$  of a gauge field!

3) Repeat the calculations for a complex scalar field. Any comments?

### 5.6.8 Complex propagator and Greens’ function

Consider the complex scalar field coupled to a chemical potential.

1) Derive the equation of motion for  $\Phi^\dagger$ .

Write the complex scalar field  $\Phi$  in terms of two real fields,  $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ . The equation of motion can now be written as

$$(A + iB)\phi_1 + (C + iE)\phi_2 = 0 . \quad (5.107)$$

2) Determine  $A, B, C$  and  $E$ .

The equation of motion can be written in matrix form

$$\frac{1}{\sqrt{2}} D^{-1} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 , \quad (5.108)$$

where  $D^{-1}$  is a  $2 \times 2$  hermitean matrix where the entries are differential operators. 3) Determine the matrix  $D^{-1}$ .

3) Find the propagator matrix  $D(p)$  in momentum space, i. e. find the Greens' function for the differential operator  $D^{-1}$  in momentum space.

4) The dispersion relation is given by  $\det D^{-1} = 0$ , i. e. the zeros of the determinant of the inverse matrix. Any comments? (This generalizes the one-component result that the dispersion relation is given by the poles of the propagator, i. e. the zeros of the inverse propagator).

# Chapter 6

## Perturbation theory

In the previous chapter we introduced the propagator, which basically is the correlator between the values of the field  $\phi$  at two points  $x$  and  $y$  in spacetime. It is sometimes called the *two-point correlator*. It gives the amplitude of a particle to propagate from  $x$  to  $y$ .

Richard Feynman <sup>1</sup> introduced a graphical representation of two-point function as well as other  $n$ -point functions - the so-called *Feynman diagrams*. The diagrams can be translated into mathematical expressions by a set of rules that can be derived from the path-integral formalism introduced in later courses. Here we will restrict ourselves to a few examples.

The free propagator is represented by a line joining the points  $x$  and  $y$  in spacetime, see Fig. 6.1.



Figure 6.1: Graphical representation of the free propagator.

### 6.1 Propagator

In this section, we consider perturbation theory for the interacting field theory whose Lagrangian density is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{24}\phi^4. \quad (6.1)$$

The last term in Eq. (6.1) is an interaction. It is the equivalent in field theory to the  $x^4$ -term of the anharmonic oscillator discussed in problem 5.6.1. The factor  $1/24 = 1/4!$  is just convention. The Euler-Lagrange equation becomes nonlinear and cannot be solved exactly. It is therefore impossible to expand the quantum field in terms of a complete set of

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<sup>1</sup>Feynman received the Nobel prize in physics in 1965 jointly with Julian Schwinger and Sin-Itiro Tomonaga for their fundamental work on QED. He also invented the path-integral formalism.

solutions. One approach is to do perturbation theory. i.e. we consider the term  $\lambda\phi^4/24$  a “small” quantity and carry out a series expansion in  $\lambda$ . This is simply a Taylor expansion in  $\lambda$  and the hope is that it converges (quickly) if  $\lambda \ll 1$ . It should then be sufficient to keep a few terms only in this expansion <sup>2</sup>.



Figure 6.2: Graphical representation of the free propagator plus the first loop correction.

We next consider the first correction to the propagator. Pictorially, it is given by the right diagram in Fig. 6.2. A particle is created at  $x$ , propagates to  $z$ , where it is annihilated. At the point  $z$ , another virtual <sup>3</sup> particle is created, propagates back to the point  $z$  and is annihilated. At the same point, another particle is created and propagates to the point  $y$ .

The first-order correction to the propagator is given by

$$\sim \lambda \langle 0 | T \left\{ \phi(x) \phi(y) \int d^4 z \phi^4(z) \right\} | 0 \rangle . \quad (6.2)$$

After a lot of algebra, one finds a contribution <sup>4</sup>

$$\sim \lambda \int d^4 z D_F(x - z) D_F(z - z) D_F(y - z) . \quad (6.3)$$

This expression can be obtained from the Feynman diagram by multiplying by  $-i\lambda$  for each *vertex* (the point where four lines meet) and  $D_F(v - w)$  for each propagator line joining the points  $v$  and  $w$ . Inserting the expressions for the propagators, we can write

$$-i \frac{\lambda}{2} \int d^4 z D_F(0) \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-z)}}{p^2 - m^2 + i\epsilon} \int \frac{d^4 q}{(2\pi)^4} \frac{ie^{-iq(z-y)}}{q^2 - m^2 + i\epsilon} . \quad (6.4)$$

Integrating over  $z$  gives a delta function,  $\delta^4(p - q)$  and after integrating over  $q$ , we can write Eq. (6.4) as

$$\frac{\lambda}{2} D_F(0) \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{(p^2 - m^2 + i\epsilon)^2} . \quad (6.5)$$

Adding this correction to the free propagator, one can write

$$D_F(x - y)_{\text{int}} = \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{(p^2 - m^2 + i\epsilon)} + \frac{\lambda}{2} D_F(0) \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{(p^2 - m^2 + i\epsilon)^2} , \quad (6.6)$$

<sup>2</sup>In QED, the expansion parameter is  $\alpha \approx 1/137 \ll 1$  and so perturbative calculations in QED seem to make sense. However, such series are typically *asymptotic series* which have zero radius of convergence(!). See e. g. [11].

<sup>3</sup>A virtual particle does *not* obey the relation  $E^2 = m^2 + p^2$  as real particles must. They are not on the so-called *mass shell*.

<sup>4</sup>This calculation is somewhat simplified since we haven't developed the full machinery of QFT.

where  $D_F(x - y)_{\text{int}}$  is a propagator including interactions. This correction can be absorbed in the expression for the free propagator as follows. We write the interacting propagator as

$$D_F(x - y)_{\text{int}} = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip(x-y)}}{(p^2 - m^2 - \frac{\lambda}{2}D_F(0) + i\epsilon)}. \quad (6.7)$$

Expanding the denominator in Eq. (6.7) to first order in the “small” quantity  $\lambda$ , we obtain Eq. (6.6) - correct to the order we are working. The pole of the propagator is now given by

$$p^2 - m^2 - \frac{\lambda}{2}D_F(0) = 0. \quad (6.8)$$

or equivalently the dispersion relation

$$E_p^2 = \mathbf{p}^2 + m^2 + \frac{\lambda}{2}D_F(0). \quad (6.9)$$

The *physical* mass is no longer equal to the mass parameter in the Lagrangian. What has happened??? Due to interactions, the pole of the propagator has moved. In fact, if we take equation (6.9) at face value, it has become infinite due to quantum fluctuations. The problem is that the correction due to interactions involve virtual particles and quantum fluctuations on all momentum scales since we are integrating over the four-momentum of the particles from zero all the way to infinity. The largest energy scales correspond to the shortest length scales. We know that quantum field theory cannot be the correct description of Nature at all length scales. For example, quantum gravity is expected to break down at the Planck scale  $\sim 10^{19}$  GeV<sup>5</sup>. This scale comes about from combining  $\hbar$ ,  $c$ , and Newton’s constant of gravity  $G_N$ . So we take a practical approach and say that our theory is valid up to some ultraviolet scale  $\Lambda$ . The physics above that scale does not concern us. We then *regularize*<sup>6</sup> the integral in  $D_F(0)$  by imposing a cutoff on the radial part of the integral:

$$\begin{aligned} D_F(0) &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \\ &= \frac{1}{4\pi^2} \int_0^\Lambda \frac{dp}{\sqrt{p^2 + m^2}} \\ &= \frac{1}{8\pi^2} \left[ \Lambda \sqrt{\Lambda^2 + m^2} - m^2 \log \frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m} \right] \\ &\approx \frac{1}{8\pi^2} \left[ \Lambda^2 - \frac{1}{2}m^2 \left( \log \frac{\Lambda^2}{m^2} - 1 \right) \right], \end{aligned} \quad (6.10)$$

---

<sup>5</sup>Yes, one can quantize gravity. However, at the length scales probed so far, ordinary gravity, i.e. classical field theory is in excellent agreement with observations.

<sup>6</sup>To regularize an integral means a procedure to make sense of or deal of integrals that are infinite. There are many different regularization procedures and one of them is to cut off the integral by integrating to an upper limit  $\Lambda$ . You will learn more about this in FY3464 and FY3466!

where we in the last line have used  $m \ll \Lambda$ . At this point, we do not know what the UV cutoff scale is, except that it is huge compared to scales we have probed in accelerators so far. The Large Hadron Collider (LHC) at CERN is constructed for several purposes. One of them is to probe smaller length scales than at previous accelerators.

Summarizing our discussion, we can write

$$m_{\text{phys}}^2 = m_{\text{bare}}^2 + \frac{\lambda}{16\pi^2} \left[ \Lambda^2 - \frac{1}{2} m^2 \left( \log \frac{\Lambda^2}{m^2} - 1 \right) \right], \quad (6.11)$$

where we have renamed the mass parameter  $m^2$  of the Lagrangian to  $m_{\text{bare}}^2$ . More generally, all parameters in the Lagrangian are called *bare* quantities in contrast to the physical ones. They are being *dressed* by quantum fluctuations. Note that the equation (6.11) that relates the bare and the physical mass depends on the scale  $\Lambda$ . Since the physical mass that we measure cannot depend on this scale, the bare mass parameter must depend on  $\Lambda$  in such a manner that  $m_{\text{phys}}$  does. This implies that if we change the cutoff, we must also adjust the bare parameters so that physical measured quantities do not. Physically, increasing the cutoff from  $\Lambda_0$  to  $\Lambda_1$ , say, means including quantum fluctuations between these two scales.

## 6.2 Vacuum energy

We next consider the vacuum energy of the  $\phi^4$ -theory. The problem of the vacuum energy has been dubbed as the most serious problem in theoretical physics<sup>7</sup>. While the present author thinks that this is overselling the problem, it does show the difficulties of local quantum field theories. Some clever string theorist will eventually solve it.

The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{24} \phi^4. \quad (6.12)$$

The Hamiltonian density is written as sum of a zeroth-order term and a perturbation according to  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ , where

$$\mathcal{H}_0 = \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2, \quad (6.13)$$

$$\mathcal{H}_1 = \frac{\lambda}{24} \phi^4. \quad (6.14)$$

We will use first-order perturbation theory to calculate first-order shift of the vacuum energy density due to interactions. This yields

$$\mathcal{E} = \mathcal{E}_0 + \mathcal{E}_1, \quad (6.15)$$

---

<sup>7</sup>As shown below, the vacuum energy goes like  $\Lambda^4$ , where  $\Lambda$  is the ultraviolet cutoff. This could be the Planck scale. However, observations of the curvature of the Universe indicates a cosmological constant (or vacuum energy density), which is some 120 orders of magnitude smaller, i.e. an almost flat Universe.

where

$$\mathcal{E}_0 = \langle 0 | \mathcal{H}_0 | 0 \rangle , \quad (6.16)$$

$$\mathcal{E}_1 = \langle 0 | \mathcal{H}_1 | 0 \rangle . \quad (6.17)$$

We already know the expression for the zeroth-order term. It is given by the standard expression

$$\mathcal{E}_0 = \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} E_p . \quad (6.18)$$

As noted before, this integral is infinite. If we regularize it using an ultraviolet cutoff  $\Lambda$ , we obtain

$$\mathcal{E}_0 \approx \frac{1}{16\pi^2} \left[ \Lambda^4 + m^2 \Lambda^2 - \frac{1}{4} m^4 \left( \log \frac{\Lambda^2}{m^2} - \frac{1}{2} \right) \right] , \quad (6.19)$$

where  $\approx$  sign is used to indicate that this is the approximate value of the integral for  $m \ll \Lambda$ . We know that the expression for  $\mathcal{E}_0$  is in terms of  $m_{\text{bare}}^2$  and we would like to express it in terms of the physical mass  $m^2$ , which differs by a “small” quantity proportional to  $\lambda$ . We do this by writing  $m_{\text{bare}}^2 = m^2 - \delta m^2$  (skipping the subscript phys) on the physical mass, and substituting this into Eq. (6.19). This yields

$$\begin{aligned} \mathcal{E}_0 &\approx \frac{1}{16\pi^2} \left[ \Lambda^4 + (m^2 - \delta m^2) \Lambda^2 - \frac{1}{4} (m^2 - \delta m^2)^4 \left( \log \frac{\Lambda^2}{m^2 - \delta m^2} - \frac{1}{2} \right) \right] \\ &\approx \frac{1}{16\pi^2} \left[ \Lambda^4 + m^2 \Lambda^2 - \frac{1}{4} m^4 \left( \log \frac{\Lambda^2}{m^2} - \frac{1}{2} \right) \right] - \frac{\delta m^2}{16\pi^2} \left[ \Lambda^2 - \frac{1}{2} m^2 \left( \log \frac{\Lambda^2}{m^2} - 1 \right) \right] , \\ &\quad \frac{1}{16\pi^2} \left[ \Lambda^4 + m^2 \Lambda^2 - \frac{1}{4} m^4 \left( \log \frac{\Lambda^2}{m^2} - \frac{1}{2} \right) \right] - \frac{\lambda}{4} [D_F(0)]^2 , \end{aligned} \quad (6.20)$$

where we in the last second line has expanded to first order in  $\delta m^2$  i. e. kept all terms through order  $\lambda$ , and used that  $\delta m^2 = \lambda D_F(0)/2$ .

Let us now look at the first-order shift given by

$$\mathcal{E}_1 = \frac{\lambda}{24} \langle 0 | \phi^4(x) | 0 \rangle . \quad (6.21)$$

Note that one might think that the vacuum energy density is a function of  $x$  due to  $\phi(x)$ . However, since the vacuum is Lorentz invariant, it turns out to be a constant in space and time, as we shall see below. The perturbation is given by products of four field operators and can be written as

$$\begin{aligned} \phi^4(x) &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3} \int \frac{d^3 s}{(2\pi)^3} \sqrt{\frac{1}{2E_p 2E_q 2E_r 2E_s}} \\ &\quad \left[ a_p a_q^\dagger a_r a_s^\dagger e^{-i(p-q+r-s)x} + \dots \right] , \end{aligned} \quad (6.22)$$

where the ellipsis denote the remaining 15 terms you get by multiplying out  $\phi^4(x)$  in products of creation and annihilation operators. The trick is now to normal order all the interaction terms just like we did for the anharmonic oscillator (problem 5.6.1). For example, consider the term  $a_p a_q^\dagger a_r a_s^\dagger$ . Using the commutation relations, we can write

$$\begin{aligned} a_p a_q^\dagger a_r a_s^\dagger &= \left[ (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) + a_q^\dagger a_p \right] \left[ (2\pi)^3 \delta^3(\mathbf{r} - \mathbf{s}) + a_s^\dagger a_r \right] \\ &= (2\pi)^6 \delta^3(\mathbf{p} - \mathbf{q}) \delta^3(\mathbf{r} - \mathbf{s}) + a_q^\dagger a_p a_s^\dagger a_r + \dots \end{aligned} \quad (6.23)$$

Now all the operator products in Eq. (6.23) as well as the other terms in  $\phi^4(x)$  will eventually be normal ordered. They will therefore annihilate the vacuum state  $|0\rangle$ . The only terms that contribute are the constants such as the first term in Eq. (6.23). There are three such identical terms (cf. the factor three in problem 5.6.1), and so we can write

$$\begin{aligned} \mathcal{E}_1 &= \frac{\lambda}{8} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3} \int \frac{d^3 s}{(2\pi)^3} \sqrt{\frac{1}{2E_p 2E_q 2E_r 2E_s}} \\ &\quad \times (2\pi)^6 \delta^3(\mathbf{p} - \mathbf{q}) \delta^3(\mathbf{r} - \mathbf{s}) e^{-i(p-q+r-s)x} . \end{aligned} \quad (6.24)$$

Integrating over  $\mathbf{q}$  and  $\mathbf{s}$  (and letting  $\mathbf{r} \rightarrow \mathbf{q}$  in one of the integrals) yields

$$\mathcal{E}_1 = \frac{\lambda}{8} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{2E_p 2E_q} . \quad (6.25)$$

Recalling that the propagator  $D_F(x)$  can be written as

$$D_F(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} , \quad (6.26)$$

one finds

$$\mathcal{E}_1 = \frac{\lambda}{8} [D_F(0)]^2 . \quad (6.27)$$

Note that we do not distinguish between the bare and the physical mass in Eq. 6.27) since it is already proportional to  $\lambda$ . The difference will be of higher order in  $\lambda$ . If we were to go to second order in perturbation theory, we would have to be careful to calculate consistently to order  $\lambda^2$ . This would also include the order- $\lambda^2$  correction to the physical mass.

If we again represent a propagator  $D_F(x - y)$  by a line with endpoints  $x$  and  $y$ , the first-order shift can be represented by two bubbles starting and ending at the same point. This is another example of a Feynman diagram. This is shown in Fig. 6.3. Since there are no open lines, it is called a *vacuum diagram*.

Adding Eqs. (6.19) and (6.27), we obtain the full first-order result in vacuum energy expressed in terms of the physical mass.

$$\begin{aligned} \mathcal{E} &= \frac{1}{16\pi^2} \left[ \Lambda^4 + m^2 \Lambda^2 - \frac{1}{4} m^4 \left( \log \frac{\Lambda^2}{m^2} - \frac{1}{2} \right) \right] - \frac{\lambda}{8} [D_F(0)]^2 \\ &= \frac{1}{16\pi^2} \left[ \Lambda^4 + m^2 \Lambda^2 - \frac{1}{4} m^4 \left( \log \frac{\Lambda^2}{m^2} - \frac{1}{2} \right) \right] - \frac{\lambda}{8} \frac{1}{64\pi^4} \left[ \Lambda^2 - \frac{1}{2} m^2 \left( \log \frac{\Lambda^2}{m^2} - 1 \right) \right]^2 . \end{aligned} \quad (6.28)$$

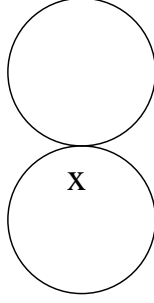


Figure 6.3: Vacuum diagram: two-loop correction to the ground-state energy.

### 6.3 Dimensional regularization

Calculating the quantum correction to the mass from vacuum fluctuations, we found

$$\delta m^2 = \frac{\lambda}{16\pi^2} \left[ \Lambda^2 - \frac{1}{2} m^2 \left( \log \frac{\Lambda^2}{m^2} - 1 \right) \right] .$$

In the limit  $\Lambda \rightarrow \infty$ , the result is divergent. Moreover there are two types of divergences. There are *power divergences* and *logarithmic divergences*. One might argue that the latter are more physical than the former, since a logarithmic divergences always has to match the logarithm of another scale giving rise to the logarithm of a dimensionless quantity <sup>8</sup>.

There is another and generally more convenient and elegant way to regularize divergent integrals and the method is called *dimensional regularization* <sup>9</sup> invented by 't Hooft and Veltman in 1972 [10] <sup>10</sup>. Consider the integral

$$I_n^d = \int \frac{d^d p}{(2\pi)^d} \frac{1}{(\sqrt{p^2 + m^2})^n} , \quad (6.29)$$

where  $d$  is the dimension of space. For  $n = 2$  this integral is divergent in the ultraviolet in integer dimensions  $d = 2, 3, \dots$ , but convergent in  $d = 1$ . In  $d = 2$  it is logarithmically divergent, in  $d = 3$  it is linearly divergent and so forth:

$$I_2^2 \approx \frac{1}{2m} , \quad (6.30)$$

$$I_2^2 \approx \frac{1}{4\pi} \log \frac{\Lambda^2}{m^2} , \quad (6.31)$$

$$I_2^3 \approx \frac{1}{2\pi^2} \left[ \Lambda - \frac{\pi}{2} m \right] , \quad (6.32)$$

where we again have used  $m \ll \Lambda$ .

<sup>8</sup>The logarithm of a dimensionful number is in itself meaningless.

<sup>9</sup>Dimensional regularization respects gauge invariance and Lorentz invariance. The three-dimensional cutoff  $\Lambda$  does not.

<sup>10</sup>It is often called "dimreg" by practitioners.

Assume now that we define it *any* dimension by analytic continuation. For example, if we are interested in the integral in  $d = 2$  dimensions, we write  $d = 2 - 2\epsilon$ , where  $\epsilon$  is a positive number. The integral is then defined by

$$I_n^d = \mu^{2-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(\sqrt{p^2 + m^2})^n} . \quad (6.33)$$

Here we have introduced a new mass scale  $\mu$  so that the integral has mass dimension two, also when  $d \neq 2$ . The first thing we need to do, is to define the angular integral in  $d$  dimensions. For  $d = 1, 2, 3, \dots$ , the “area” of the sphere  $S_d$  is given by

$$S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} , \quad (6.34)$$

where  $\Gamma(x)$  is the Gamma function. We then *define* the area of the sphere in noninteger dimensions  $d$  by Eq. (6.34). The remaining radial integral needs to be generalized to  $d$  dimensions such that it reduces to the standard expression for  $d = 1, 2, 3$ . It can therefore be defined as

$$I_n^d(\text{radial}) = \int_0^\infty \frac{p^{d-1} dp}{(\sqrt{p^2 + m^2})^n} . \quad (6.35)$$

A straightforward calculation yields

$$I_n^d(\text{radial}) = \frac{m^{(d-n)/2} \Gamma(\frac{d}{2})}{2\Gamma(\frac{n}{2})} \Gamma\left(\frac{n-d}{2}\right) . \quad (6.36)$$

Putting the pieces together, one finds

$$I_n^d = \left(\frac{\mu}{m}\right)^{2-d} \frac{m^{2-\frac{d}{2}-\frac{n}{2}} \Gamma(\frac{n-d}{2})}{2^d \pi^{\frac{d}{2}} \Gamma(\frac{n}{2})} . \quad (6.37)$$

We next expand our result in powers of  $\epsilon$  i. e. we expand around  $d = 2$  dimensions. For example, for  $n = 2$  this yields

$$I_2^d = \frac{1}{4\pi} \left[ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + \text{constant} + \mathcal{O}(\epsilon) \right] . \quad (6.38)$$

We first notice that there is a pole in  $\epsilon$ <sup>11</sup> and a logarithmic term associated with. If we compare this result with the one using a cutoff  $\Lambda$ , we notice that the logarithms are the same except that we have traded  $\Lambda$  for the scale  $\mu$ . This is always the case. If an integral is logarithmically divergent using a cutoff, there is pole and a corresponding logarithmic term if one uses dimensional regularization.

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<sup>11</sup>Note that the  $\Gamma(n)$ -function has poles at  $n = 0, -1, -2, \dots$

Returning to the original integral  $I_1^d = 2D_F(0)$  for  $d = 3 - 2\epsilon$ , one finds <sup>12</sup>

$$\begin{aligned} I_1^d &= \left(\frac{\mu}{m}\right)^{3-d} \frac{m^{4-\frac{d}{2}-\frac{1}{2}} \Gamma(\frac{1-d}{2})}{2^d \pi^{\frac{d}{2}} \Gamma(\frac{1}{2})} \\ &= -\frac{m^2}{8\pi^2} \left[ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + \text{constant} + \mathcal{O}(\epsilon) \right]. \end{aligned} \quad (6.39)$$

Again we see that the logarithm matches the original one in Eq. (6.10). Moreover, the power divergence has disappeared. This is also always the case: Power divergences are set to zero with dimensional regularization. If an integral has only such divergences, it will be finite with dimensional regularization. This is hexerei <sup>13</sup>.

Using the expression (6.39), we obtain

$$m^2 = m_{\text{bare}}^2 - \frac{\lambda m_{\text{bare}}^2}{32\pi^2} \left[ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} \right] + \delta m^2, \quad (6.40)$$

where we have added a so-called *counterterm*  $\delta m^2$ , and where we have ignored the constant, which is immaterial for our discussion. The counterterm is now chosen such that the physical mass of the particle is finite. The simplest choice is

$$\delta m^2 = \frac{\lambda m_{\text{bare}}^2}{32\pi^2} \frac{1}{\epsilon}, \quad (6.41)$$

i.e. we simply use it to cancel the pole in  $\epsilon$ . The physical mass then reads

$$m^2 = m_{\text{bare}}^2 + \frac{\lambda m_{\text{bare}}^2}{32\pi^2} \log \frac{\mu^2}{m_{\text{bare}}^2}. \quad (6.42)$$

The procedure of rendering quantities finite by adding appropriate counterterms is called renormalization and is necessary in most field theories *renormalization* <sup>14</sup>. The reason behind these divergences is that interactions in spacetime take place at a single point and is therefore highly singular. *String theory* is an attempt to tame the divergences via interactions that are “blurred”. A beautiful and deep article on renormalization was written by Lepage and can be found at [12]. It is an excellent read.

### 6.3.1 Vacuum energy revisited

The zeroth-order vacuum energy density is given by

$$\mathcal{E}_0 = \frac{1}{2} \mu^{3-d} \int \frac{d^d p}{(2\pi)^3} \sqrt{p^2 + m^2}. \quad (6.43)$$

<sup>12</sup>This integral is divergent in all integer dimension. However, it is still defined the same way in dimensional regularization. Just multiply with the appropriate factor of  $\mu$ .

<sup>13</sup>Hexerei also implies that one must be careful with applying dimreg carelessly. There are a few pitfalls.

<sup>14</sup>Quantum mechanics can be considered a field theory in 0+1 dimensions but is finite.

With dimensional regularization this reads

$$\mathcal{E}_0 = m^4 \left( \frac{\mu}{m} \right)^{2\epsilon} \frac{\Gamma(\frac{n-d}{2})}{2^{d+1} \pi^{d/2} \Gamma(\frac{1}{2})} . \quad (6.44)$$

Expanding in powers of  $\epsilon$ , one obtains

$$\mathcal{E}_0 = -\frac{m^4}{64\pi^2} \left[ \frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} + C + \mathcal{O}(\epsilon) \right] . \quad (6.45)$$

Again we see that the power divergences are set to zero and that we need a counterterm

$$\delta\mathcal{E}_0 = \frac{m^4}{64\pi^2\epsilon} . \quad (6.46)$$

to cancel the divergences in the vacuum energy. Thus the renormalized vacuum energy is

$$\mathcal{E}_0 = -\frac{m^4}{64\pi^2} \log \frac{\mu^2}{m^2} + C . \quad (6.47)$$

The first-order correction to the ground-state energy can likewise be written as

$$\begin{aligned} \mathcal{E}_1 &= \frac{\lambda}{8} [D(0)]^2 \\ &= \frac{\lambda}{8} \frac{m^4}{256\pi^4} \left[ \frac{1}{\epsilon^2} + \frac{2}{\epsilon} \log \frac{\mu^2}{m^2} + \log^2 \frac{\mu^2}{m^2} + K \right] + \delta\mathcal{E}_1 . \end{aligned} \quad (6.48)$$

Expressing the zeroth-order energy density in terms of the physical mass instead of the bare mass is equivalent to the replacement  $\mathcal{E}_0 \rightarrow \mathcal{E}_0 + \frac{\delta m^2}{2} D_F(0)$ <sup>15</sup>. This yields

$$\begin{aligned} \mathcal{E} &= \mathcal{E}_0 + \frac{\delta m^2}{2} D_F(0) + \frac{\lambda}{8} [D_F(0)]^2 \\ &= -\frac{m^4}{64\pi^2} \log \frac{\mu^2}{m^2} - \frac{\lambda}{8} \frac{m^4}{256\pi^4} \left[ \frac{1}{\epsilon^2} - \log^2 \frac{\mu^2}{m^2} + \dots \right] + \delta\mathcal{E}_1 . \end{aligned} \quad (6.49)$$

Note that the term  $\frac{2}{\epsilon} \log \frac{\mu^2}{m^2}$  in Eq. (6.48) is cancelled by a similar term coming from  $\frac{\delta m^2}{2} D_F(0)$ . The remaining divergence, which is a double pole in  $\epsilon$ , is cancelled by the vacuum counterterm

$$\delta\mathcal{E}_1 = \frac{\lambda}{8} \frac{m^4}{256\pi^4\epsilon} . \quad (6.50)$$

The renormalized vacuum energy is then

$$\mathcal{E} = -\frac{m^4}{64\pi^2} \left[ \log \frac{\mu^2}{m^2} + C \right] + \frac{\lambda}{8} \frac{m^4}{256\pi^4} \left[ \log^2 \frac{\mu^2}{m^2} + K_1 \right] . \quad (6.51)$$

---

<sup>15</sup>Perhaps the easiest way to see this, is to realize that the counterterm gives rise to an extra term in the Lagrangian,  $\delta\mathcal{L} = \frac{1}{2}\delta m^2\phi^2$  and this term should be treated as a perturbation since it is of order  $\lambda$ . It then gives a correction to the vacuum energy  $\delta\mathcal{E} = \frac{\delta m^2}{2} D_F(0)$ .

## 6.4 Problems

### 6.4.1 Dimensional regularization

In the theory of Bose-Einstein condensation, the dispersion relation for quasiparticles is given by

$$E_p = \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + 2\mu \right)}, \quad (6.52)$$

where  $m$  is the mass of the particle and  $\mu$  is the chemical potential.

- 1) What is the behavior of  $E_p$  for large values of  $p$ ? And for small values of  $p$ ?
- 2) The ground-state energy of the system is then

$$\mathcal{E} = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} E_p + \text{constant}, \quad (6.53)$$

where the constant is of no interest to us. Calculate the integral in  $d = 1, 2$ , and 3 dimensions using a cutoff  $\Lambda$ . Analyze the divergences.

- 3) Repeat the calculation using dimensional regularization. You might want to use Mathematica, Maple, or Matlab to calculate the radial integral. Compare the poles in  $\epsilon$  with the corresponding divergences using a cutoff  $\Lambda$ .

# Chapter 7

## Dilute Bose gas

### 7.1 Lagrangian and Hamiltonian

The starting point for the discussion on the dilute Bose gas the Lagrangian density

$$\mathcal{L} = i\psi^\dagger \partial_t \psi + \mu \psi^\dagger \psi - \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi - \frac{1}{2} g (\psi^\dagger \psi)^2, \quad (7.1)$$

The last term is an interaction terms, which describes two-particle scattering. The coupling constant  $g$  is basically the  $s$ -wave scattering length. Note that the Lagrangian is invariant under the global phase symmetry  $\psi \rightarrow e^{i\alpha} \psi$  even in the presence of interactions. This gives rise to a conserved current in the usual way. In the nonrelativistic case, it is simply number conservation (of atoms) and the number density is given by  $j^0 = \rho = \langle \psi^\dagger \psi \rangle$ . In order to couple the theory to a chemical potential, we have therefore made the substitution  $\partial_0 \rightarrow \partial_0 - i\mu$ . The Hamiltonian density is then <sup>1</sup>

$$\mathcal{K} = \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi - \mu \psi^\dagger \psi + \frac{1}{2} g (\psi^\dagger \psi)^2. \quad (7.2)$$

We then write

$$\mathcal{K} = \mathcal{H} - \mu \psi^\dagger \psi, \quad (7.3)$$

where we have defined

$$\mathcal{H} = \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi + \frac{1}{2} g (\psi^\dagger \psi)^2. \quad (7.4)$$

The integrals of  $\mathcal{K}$  and  $\mathcal{H}$  are denoted by

$$K = \int d^3x \mathcal{K}, \quad (7.5)$$

$$H = \int d^3x \mathcal{H}. \quad (7.6)$$

---

<sup>1</sup>When we include a chemical potential, it is often denoted by  $\mathcal{K}$ , sometimes called the "Kamiltonian" density.

Taking the derivative of  $K$  with respect to  $\mu$ , we obtain

$$\begin{aligned}\frac{\partial K}{\partial \mu} &= - \int d^3x \psi^\dagger \psi \\ &= -\langle N \rangle .\end{aligned}\tag{7.7}$$

The expectation value of  $\mathcal{K}$  is by definition the Helmholtz free energy density  $\mathcal{F}$  and so we find

$$\begin{aligned}\mathcal{F} &= \langle \mathcal{K} \rangle \\ &= \langle \mathcal{H} - \mu \psi^\dagger \psi \rangle \\ &= \langle \mathcal{H} \rangle - \mu \langle \psi^\dagger \psi \rangle \\ &= \mathcal{E} - \mu \rho\end{aligned}\tag{7.8}$$

where the expectation value of the energy density is  $\mathcal{E} = \langle \mathcal{H} \rangle$  and the expectation value of the number density is  $\rho = \langle \psi^\dagger \psi \rangle$ . This yields

$$\mathcal{E} = \mathcal{F} + \mu \rho .\tag{7.9}$$

The free energy density is a Legendre transform of the energy density.

Consider the ground state of an ideal Bose gas with  $N_0$  particles in a volume  $V$ . The particle density is therefore  $\rho_0 = N_0/V$ . The thermodynamic limit is defined by taking the limit  $N_0, V \rightarrow \infty$  with fixed density  $\rho_0$ . Since the particles are bosons, they all recide in the single-particle ground state with  $\mathbf{p} = 0$ . This state is denoted by  $|\Phi\rangle$  and is given by

$$|\Phi\rangle = \left(a_0^\dagger\right)^{N_0} |0\rangle .\tag{7.10}$$

Let us calculate the charge density  $I$  in the ground state

$$I = \langle \Phi | Q | \Phi \rangle ,\tag{7.11}$$

where  $Q$  is the "charge" density operator that arises from the global phase symmetry

$$Q = \frac{1}{V} \sum_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} ,\tag{7.12}$$

where we for convenience have switched to the discrete case summing over all values of the momentum  $\mathbf{p}$ . This is just to simplify intermediate calculations. We will take the continuum limit at the end. This yields

$$\begin{aligned}I &= \frac{1}{V} \sum_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}}^\dagger a_{\mathbf{p}} | \Phi \rangle \\ &= \frac{1}{V} \sum_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \left(a_0^\dagger\right)_0^N | 0 \rangle \\ &= \frac{1}{V} \sum_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}}^\dagger \left[ a_0^\dagger a_{\mathbf{p}} + \delta_{\mathbf{p},0} \right] \left(a_0^\dagger\right)^{N_0-1} | 0 \rangle , \\ &= \frac{1}{V} \langle \Phi | \Phi \rangle + \frac{1}{V} \sum_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}}^\dagger a_0^\dagger a_{\mathbf{p}} \left(a_0^\dagger\right)^{N_0-1} | \Phi \rangle .\end{aligned}$$

We can now use the commutation relation to move  $a_{\mathbf{p}}$  one more step to the right. Every time we do it, we pick up a matrix element  $\langle \Phi | \Phi \rangle / V$ . Doing it  $N_0$  times, it brings  $a_{\mathbf{p}}$  to the right of all the  $a_0^\dagger$ 's and gives zero since it annihilates the vacuum. We are then left with

$$I = \frac{N_0}{V} \langle \Phi | \Phi \rangle . \quad (7.13)$$

Since the ground state is assumed to have unit norm, this reduces to the density  $\rho = N_0/V$  of the ground state as expected:

$$I = \rho , \quad (7.14)$$

In the same manner we can calculate the matrix element

$$J = \frac{1}{V} \sum_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}} a_{\mathbf{p}}^\dagger | \Phi \rangle . \quad (7.15)$$

Using the commutation relations, it is not difficult to show that

$$J = \frac{(N_0 + 1)}{V} \langle \Phi | \Phi \rangle . \quad (7.16)$$

When  $N_0$  is a macroscopic number,  $N_0 \gg 1$ , we can therefore ignore the commutator between  $a_0$  and  $a_0^\dagger$  and simply treat the operators as numbers<sup>2</sup>. This is exact in the thermodynamic limit when  $N_0 \rightarrow \infty$ . The replacement of the operators  $a_0$  and  $a_0^\dagger$  by  $\sqrt{\rho}$  is called the Bogoliubov replacement and is the basis of the Bogoliubov approximation which we discuss next.

## 7.2 Bogoliubov approximation

We next rewrite the field operators by separating out the  $\mathbf{p} = 0$  mode from the integral

$$\psi(\mathbf{x}, t) = \sqrt{\rho} + \frac{1}{\sqrt{V}} \sum'_{\mathbf{p}} a_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}} , \quad (7.17)$$

$$\psi^\dagger(\mathbf{x}, t) = \sqrt{\rho} + \frac{1}{\sqrt{V}} \sum'_{\mathbf{p}} a_{\mathbf{p}}^\dagger e^{i\mathbf{p}\cdot\mathbf{x}} , \quad (7.18)$$

where the prime on the sum is a reminder that  $\mathbf{p} = 0$  is excluded and  $V$  is the volume of the system. Inserting the expression for the field operator, we can write

$$H = H_0 + H_{\text{int}} , \quad (7.19)$$

---

<sup>2</sup>By neglecting the operator nature of  $a_0$  and  $a_0^\dagger$ , we treat them as classical variables. We refer to them as *c*-numbers.

where

$$H_0 = \frac{1}{V} \int d^3x \sum'_{\mathbf{p}, \mathbf{q}} \frac{\mathbf{p} \cdot \mathbf{q}}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)x}, \quad (7.20)$$

$$H_{\text{int}} = \frac{1}{2}g \int d^3x \frac{N_0^2}{V^2} + \frac{g\rho_0}{2V} \int d^3x \sum'_{\mathbf{p}, \mathbf{q}} \left[ a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)x} + a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)x} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)x} + \dots \right] \quad (7.21)$$

where the dots indicate the remaining terms, for example terms with more than two operators with nonzero momenta, e.g.  $\sqrt{\rho} a_{\mathbf{p}}^\dagger a_{\mathbf{q}} a_{\mathbf{k}} e^{i(p-q-k)x}$ . Since we expect most particles to be in  $\mathbf{p} = 0$  state also in the interacting case, we will neglect the terms with more than two operators with nonzero momenta<sup>3</sup>. These terms can be viewed as perturbations and can be taken systematically into account using perturbation theory. The first term in Eq. (7.21) is rewritten as follows

$$\begin{aligned} \frac{1}{2}g \int d^3x \frac{N_0^2}{V^2} &= \frac{1}{2}g \int d^3x \frac{1}{V^2} [N - N']^2 \\ &\approx \frac{1}{2}g \int d^3x \frac{1}{V^2} [N^2 - 2NN'] , \\ &= \frac{1}{2}gV\rho^2 - g\rho N' \\ &= \frac{1}{2}gV\rho^2 - g\rho \sum'_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} , \end{aligned} \quad (7.22)$$

where  $N'$  is the number of particles outside the condensate. Note that we have dropped the term  $(N')^2$  as it is expected to be small compared to the terms we kept.

After integrating over  $\mathbf{x}$  and then summing over  $\mathbf{q}$ , the second term in Eq. (7.21) reduces to (cf. Eq. (A.13))

$$\begin{aligned} \frac{g\rho_0}{2V} \int d^3x \sum'_{\mathbf{p}, \mathbf{q}} \left[ a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger e^{i(p+q)x} + a_{\mathbf{p}} a_{\mathbf{q}} e^{-i(p+q)x} + a_{\mathbf{p}}^\dagger a_{\mathbf{q}} e^{i(p-q)x} + \dots \right] \\ = \frac{g\rho}{2} \sum'_{\mathbf{p}} \left[ a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}} a_{-\mathbf{p}} + 4a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right] . \end{aligned} \quad (7.23)$$

where we have replaced  $\rho_0$  by  $\rho$ , which is correct to the order we are calculating. Adding Eqs. (7.20), (7.22) and (7.23), we obtain

$$H = \frac{1}{2}gV\rho^2 + \sum'_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{g\rho}{2} \sum'_{\mathbf{p}} \left[ a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}} a_{-\mathbf{p}} + 2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right] . \quad (7.24)$$

We next divide the Hamiltonian (7.24) to obtain the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}g\rho^2 + \frac{1}{V} \sum'_{\mathbf{p}} \frac{\mathbf{p}^2}{2m} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{g\rho}{2V} \sum'_{\mathbf{p}} \left[ a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}} a_{-\mathbf{p}} + 2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right] . \quad (7.25)$$

---

<sup>3</sup>Interactions are supposed to be weak! (whatever that means at this stage).

We have now reduced the problem to that of a Hamiltonian density which is quadratic in the creation and annihilation operators and is essentially that of a free theory. However, we have some troublesome terms containing creation and annihilation operator. The next step is to introduce new operators  $b_{\mathbf{p}}$  and  $b_{\mathbf{p}}^\dagger$  expressed in term of the old ones such that they satisfy the standard commutation relations and such that the Hamiltonian is "diagonal" in these operators. The problem is then reduced to problem we know how to solve. To this end, we define

$$a_{\mathbf{p}}^\dagger = u_p b_{\mathbf{p}}^\dagger + v_p b_{-\mathbf{p}} \quad (7.26)$$

$$a_{\mathbf{p}} = u_p b_{\mathbf{p}} + v_p b_{-\mathbf{p}}^\dagger . \quad (7.27)$$

This transformation is called a Bogoliubov transformation after the famous Soviet physicist N. N. Bogoliubov. Note that the coefficients  $u_p$  and  $v_p$  are chosen real and spherically symmetric. This choice is without loss of generality. These equations are easily solved with respect to the new operators

$$b_{\mathbf{p}} = \frac{1}{v_p^2 - u_p^2} [v_p a_{-\mathbf{p}}^\dagger - u_p a_{\mathbf{p}}] , \quad (7.28)$$

$$b_{\mathbf{p}}^\dagger = \frac{1}{v_p^2 - u_p^2} [v_p a_{-\mathbf{p}} - u_p a_{\mathbf{p}}^\dagger] . \quad (7.29)$$

Inserting these definitions into the commutator, we find

$$\begin{aligned} [b_{\mathbf{p}}, b_{\mathbf{q}}^\dagger] &= \frac{1}{u_p^2 - v_p^2} \frac{1}{u_q^2 - v_q^2} [u_p u_q [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] + v_p v_q [a_{-\mathbf{p}}^\dagger, a_{-\mathbf{q}}]] \\ &= \frac{1}{u_p^2 - v_p^2} \frac{1}{u_q^2 - v_q^2} [u_p u_q - v_p v_q] \delta_{\mathbf{p}, \mathbf{q}} . \end{aligned} \quad (7.30)$$

We now require that the new operators satisfy the standard commutation relation for bosonic operators. This implies

$$u_p^2 - v_p^2 = 1 , \quad (7.31)$$

for all  $\mathbf{p} \neq 0$ . We can parametrize  $u_{\mathbf{p}}$  and  $v_{-\mathbf{p}}$  as follows

$$u_p = \cosh \alpha_p , \quad v_p = \sinh \alpha_p , \quad (7.32)$$

where  $\alpha_{\mathbf{p}}$  is a real number. We next insert the expressions for  $a_{\mathbf{p}}$  and  $a_{\mathbf{p}}^\dagger$  into the Hamiltonian and collect the various terms. After some tedious algebra, we find

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} g \rho^2 + \frac{1}{V} \sum_{\mathbf{p}}' \left\{ \left[ \left( \frac{p^2}{2m} + g \rho \right) (u_p^2 + v_p^2) + 2g \rho u_p v_p \right] b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \right. \\ &\quad + \left[ \left( \frac{p^2}{2m} + g \rho \right) u_p v_p + \frac{1}{2} (u_p^2 + v_p^2) g \rho \right] [b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger + b_{\mathbf{p}} b_{-\mathbf{p}}] \\ &\quad \left. + \left[ \left( \frac{p^2}{2m} + g \rho \right) v_p^2 + 2g \rho u_p v_p \right] \right\} . \end{aligned} \quad (7.33)$$

In order to diagonalize the Hamiltonian, we require that the coefficients of  $b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger$  and  $b_{\mathbf{p}} b_{-\mathbf{p}}$  vanish, i.e. we require

$$\left[ \frac{p^2}{2m} + g\rho \right] u_p v_p + \frac{1}{2} g\rho [u_p^2 + v_p^2] = 0. \quad (7.34)$$

Using the fact that  $u_p^2 = 1 + v_p^2$ , we can eliminate  $v_p$ , and obtain a quartic equation for  $u_p$

$$\left[ \frac{p^2}{2m} + g\rho \right]^2 u_p^2 (u_p^2 - 1) - \frac{1}{4} g^2 \rho^2 [2u_p^2 - 1]^2 = 0. \quad (7.35)$$

The solution to this equation is easily found and we finally obtain for  $u_p$  and  $v_p$

$$u_p^2 = \frac{1}{2} \left( \frac{p^2/2m + g\rho}{\epsilon(p)} + 1 \right), \quad (7.36)$$

$$v_p^2 = \frac{1}{2} \left( \frac{p^2/2m + g\rho}{\epsilon(p)} - 1 \right), \quad (7.37)$$

where the dispersion relation is

$$\epsilon(p) = \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + 2g\rho \right)}. \quad (7.38)$$

The spectrum (7.38) was first obtained by Bogoliubov [14]. and is referred to as the Bogoliubov spectrum. Inserting the expressions for  $u_p$  and  $v_p$  into the Hamiltonian density (7.33) can now be written as

$$\mathcal{H} = \frac{1}{2} g\rho^2 + \frac{1}{2V} \sum_{\mathbf{p}}' \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho \right] + \frac{1}{V} \sum_{\mathbf{p}}' \epsilon(p) b_{\mathbf{p}}^\dagger b_{\mathbf{p}}. \quad (7.39)$$

We have transformed the Hamiltonian density into one which is the sum of two constant terms plus a term which is quadratic in the creation and annihilation operators. These operators satisfy the standard commutation relation and we therefore have a particle interpretation of the theory. These particles are referred to as *quasiparticles* and are not reminiscent of those of the free theory. Their properties depend on the medium. The first term is a so-called *mean-field* term and is the classical contribution to the energy density. Moreover, the second term is a divergent contribution to the energy density. It is due to quantum fluctuations and is the analog of the zero-point energy  $\hbar\omega/2$  of the harmonic oscillator. However, since it involves the coupling constant, it requires renormalization.

We denote the ground state state by  $|\Phi\rangle$  and it is defined by the absence of quasiparticles, i. e. <sup>4</sup>

$$b_{\mathbf{p}} |\Phi\rangle = 0. \quad (7.40)$$

---

<sup>4</sup>The absence of quasiparticles is *not* the same as absence of real particles.

We next discuss the Bogoliubov dispersion relation (7.38). At long wavelengths,  $p^2/2m \ll g\rho$ , we see that

$$\begin{aligned}\epsilon(p) &\approx \sqrt{\frac{g\rho}{m}}p \\ &= cp ,\end{aligned}\tag{7.41}$$

where  $c = \sqrt{g\rho/m}$ . The dispersion relation is linear at long wavelengths. These are called *phonons* or sound waves. The dispersion relation for small momenta is very different from that of free particles,  $\epsilon(p) = p^2/2m$  and is due to *medium effects*. For large momenta, we  $p^2/2m \gg g\rho$ , we see that

$$\epsilon(p) \approx \frac{p^2}{2m} + g\rho .$$

This is the dispersion relation of a free particle with an additional mean-field energy  $g\rho$ . The dispersion relation normalized to  $g\rho$  is shown in Fig.7.1.

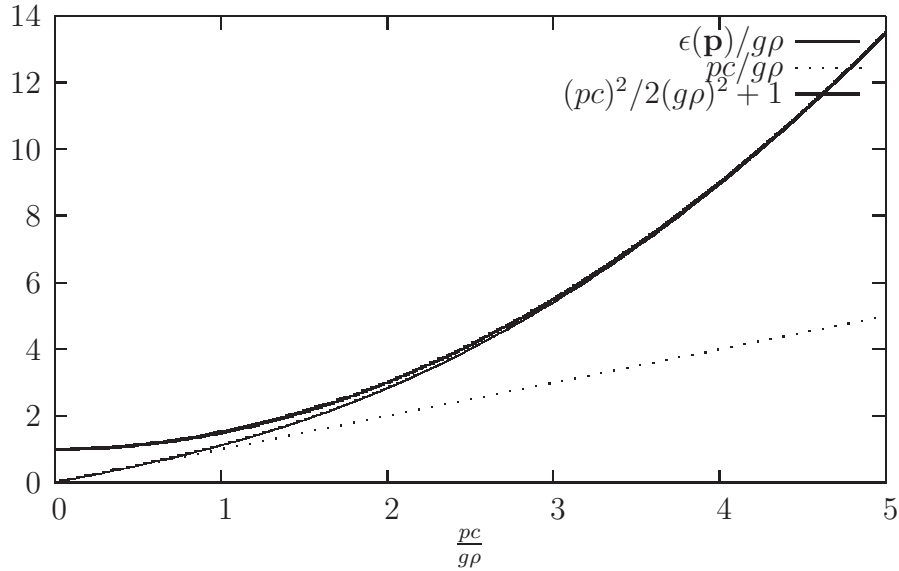


Figure 7.1: The dispersion relation normalized to  $g\rho$  as well as the small - and large- $p$  limits as functions of  $cp/g\rho$ .

The average particle number outside the condensate is given by the integral

$$N' = \sum_{\mathbf{p}} \langle a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} \rangle ,\tag{7.42}$$

where

$$\begin{aligned}\langle a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} \rangle &= \langle \Phi | (u_p b_{\mathbf{p}}^{\dagger} + v_p b_{-\mathbf{p}}) (u_p b_{\mathbf{p}} + v_p b_{-\mathbf{p}}^{\dagger}) | \Phi \rangle \\ &= v_p^2 .\end{aligned}\tag{7.43}$$

This yields

$$\begin{aligned}
\rho' &= \frac{1}{V} \sum_{\mathbf{p}}' v_p^2 \\
&= \int \frac{d^3p}{(2\pi)^3} v_p^2 \\
&= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \frac{p^2/2m + g\rho}{\epsilon(p)} - 1 \right], \tag{7.44}
\end{aligned}$$

where we in the second line has taken the thermodynamic limit, i.e.  $V \rightarrow \infty$ ,  $N \rightarrow \infty$  such that the density of the system is constant. The large- $p$  behavior of  $\epsilon(p)$  is given by

$$\epsilon(p) = \frac{p^2}{2m} + g\rho - \frac{1}{8} \frac{g^2 \rho^2}{p^2/2m} + \dots \tag{7.45}$$

From the large- $p$  behavior of  $\epsilon(p)$ , it follows that the integral in Eq. (7.44) is finite. A straightforward calculation gives

$$\begin{aligned}
\rho' &= \frac{8}{3\sqrt{\pi}} \left( \frac{mg\rho}{4\pi} \right)^{3/2} \\
&= \frac{8}{3} \sqrt{\frac{\rho^3 a^3}{\pi}}. \tag{7.46}
\end{aligned}$$

The total density then becomes

$$\begin{aligned}
\rho &= \rho_0 + \rho' \\
&= \rho_0 \left[ 1 + \frac{8}{3} \sqrt{\frac{\rho_0 a^3}{\pi}} \right]. \tag{7.47}
\end{aligned}$$

Note that we have replaced  $\rho$  by  $\rho_0$  in the second term as the difference is supposed to be small. The fact that not all particles is in the  $\mathbf{p} = 0$  state is a quantum effect. A fraction of the particles are kicked out of the ground state and the effect is referred to as *depletion*. The result was first derived by Bogoliubov in 1947 [14].

The ground state is given by the state with no quasiparticles present. The energy of the ground state is denoted by  $E_0$  and the corresponding energy density by  $\mathcal{E}$ . It follows from the Hamiltonian (7.39) that the latter is given by

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2} g \rho^2 + \frac{1}{2V} \sum_{\mathbf{p}}' \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho \right] \\
&= \frac{1}{2} g \rho^2 + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho \right]. \tag{7.48}
\end{aligned}$$

It follows from Eq. (7.45) that the above integral is ultraviolet divergent. The divergent term is given by

$$\mathcal{E}_{\text{div}} = -\frac{1}{2} m g^2 \rho^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2}. \tag{7.49}$$

Having isolated the divergence, the energy density can be rewritten as

$$\mathcal{E} = \frac{1}{2}g\rho^2 \left[ 1 - mg \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} \right] + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho + \frac{mg^2\rho^2}{p^2} \right]. \quad (7.50)$$

The first integral is the same as the expression for the scattering length in the second Born approximation! We then redefine the coupling constant by replacing the first term by the *physical*, i. e. the measured coupling constant  $g$ <sup>5</sup>. The second integral is finite since we have subtracted the divergent piece. This yields

$$\mathcal{E} = \frac{1}{2}g\rho^2 + \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho + \frac{mg^2\rho^2}{p^2} \right]. \quad (7.51)$$

After having performed the integral, the ground-state energy density can then be written as

$$\mathcal{E} = \frac{2\pi a\rho^2}{m} \left[ 1 + \frac{128}{15} \sqrt{\frac{\rho a^3}{\pi}} \right]. \quad (7.52)$$

The first term is the mean-field energy while the second term is the first quantum correction to the energy density. We notice that the dimensionless expansion parameter for the ground-state energy is the gas parameter  $\sqrt{\rho a^3}$ . The result (7.52) was first obtained by Lee, Huang, and Yang [15, 16]. The weakly interacting Bose gas is then defined by  $\sqrt{\rho a^3} \ll 1$ . In this case, one hopes that the first correction in Eq. (7.52) will be small compared to the mean-field result  $\mathcal{E}_0 = \frac{2\pi a\rho^2}{m}$ . By Legendre transforming, one can compute the free energy density and therefore the pressure since  $\mathcal{P} = -\mathcal{F}$ . By measuring the s-wave scattering length  $a$  in scattering experiments, we have all the parameters in the equation of state. By measuring the pressure for various densities, we should be able to see the effects of the quantum fluctuations, i. e. the deviations from mean-field theory!

### 7.3 Bose-Einstein condensation and spontaneous symmetry breaking

The field operator is written as

$$\psi(\mathbf{x}, t) = \sqrt{\frac{N_0}{V}} + \frac{1}{\sqrt{V}} \sum_{\mathbf{p}}' a_{\mathbf{p}} e^{-i\mathbf{p}\mathbf{x}}. \quad (7.53)$$

What is the expectation value of  $\psi$  in the interacting ground state? In the trivial ground state, i.e. in the vacuum, the expectation value is zero since the annihilation operator  $a_{\mathbf{p}}$  by definition annihilates  $|0\rangle$ . We know that the interacting nontrivial ground state  $|\Phi\rangle$  is

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<sup>5</sup>We define  $g = g_{\text{bare}} + \delta g$ . The counterterm  $\delta g$  then cancels the divergence in the usual way.

a complicated sum of states with different particle numbers (recall the fluctuations of the operator  $N_{\mathbf{p}}$ ). One finds

$$\langle \Phi | \psi | \Phi \rangle = \sqrt{\frac{N_0}{V}} + \frac{1}{\sqrt{V}} \sum'_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}} e^{-ipx} | \Phi \rangle . \quad (7.54)$$

The first term on the right-hand-side is nonzero, while the second can be written as

$$\begin{aligned} \frac{1}{\sqrt{V}} \sum'_{\mathbf{p}} \langle \Phi | a_{\mathbf{p}} e^{-ipx} | \Phi \rangle &= \frac{1}{\sqrt{V}} \sum'_{\mathbf{p}} \langle \Phi | [u_p b_{\mathbf{p}} + v_p b_{-\mathbf{p}}^\dagger] e^{-ipx} | \Phi \rangle \\ &= 0 , \end{aligned} \quad (7.55)$$

since the quasiparticle operator  $b_{\mathbf{p}}$  and  $b_{-\mathbf{p}}^\dagger$  annihilate the states  $|\Phi\rangle$  and  $\langle\Phi|$ , respectively. We can therefore write

$$\begin{aligned} \langle \Phi | \psi | \Phi \rangle &= \sqrt{\frac{N_0}{V}} \\ &\equiv \phi_0 , \end{aligned} \quad (7.56)$$

so  $\phi_0$  is the square root of the condensate density  $\rho_0$ . This yields

$$\psi = \phi_0 + \tilde{\psi} , \quad (7.57)$$

where  $\phi_0$  is a *classical field* and  $\tilde{\psi}$  is a quantum fluctuating field<sup>6</sup>. The expectation value of the fluctuating field is zero. We next insert the expression for the field operator (7.57) into the Kamiltonian density. This yields

$$\mathcal{K} = \frac{1}{2m} \nabla \tilde{\psi}^\dagger \cdot \nabla \tilde{\psi} - \mu [\phi_0^2 + \phi_0 \tilde{\psi} + \phi_0 \tilde{\psi}^\dagger + \tilde{\psi}^\dagger \tilde{\psi}] + \frac{1}{2} g [\phi_0^2 + \phi_0 \tilde{\psi} + \phi_0 \tilde{\psi}^\dagger + \tilde{\psi}^\dagger \tilde{\psi}]^2 . \quad (7.58)$$

Neglecting the quantum fluctuations in the Kamiltonian yields

$$\mathcal{K} = -\mu \phi_0^2 + \frac{1}{2} g \phi_0^4 \quad (7.59)$$

$$K = \int d^3x \left[ -\mu \phi_0^2 + \frac{1}{2} g \phi_0^4 \right] . \quad (7.60)$$

We define the *effective potential*<sup>7</sup> by  $V = [-\mu \phi_0^2 + \frac{1}{2} g \phi_0^4]$ . The value that minimizes the effective potential, i. e. which gives the ground-state energy depend on the sign of  $\mu$ . The minimum is given by

$$\begin{aligned} V' &= -\mu \phi_0 + g \phi_0^3 \\ &= 0 . \end{aligned} \quad (7.61)$$

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<sup>6</sup>It is called a classical field since we are ignoring the commutator  $[a_0, a_0^\dagger]$ .

<sup>7</sup>At this level,  $V$  the effective potential is equal to the classical potential  $V_0$  in analogy to the anharmonic oscillator in quantum mechanics. There are, however, quantum corrections to it.

If  $\mu < 0$ , the minimum is given by  $\phi_0 = 0$ , while for  $\mu > 0$ , one finds that  $\phi_0$  is

$$\phi_0^2 = \mu/g. \quad (7.62)$$

Inserting this value into the effective potential  $V$  and recalling that  $\phi_0^2 = \rho_0$ , we obtain

$$V = -\frac{\mu^2}{2g}. \quad (7.63)$$

In other words  $\langle \mathcal{K} \rangle = -\frac{\mu^2}{2g}$ . This yields

$$\begin{aligned} \langle \mathcal{H} \rangle &= \langle \mathcal{K} \rangle + \mu \rho_0 \\ &= \frac{1}{2} g \rho_0^2, \end{aligned} \quad (7.64)$$

where we have used  $\rho_0 = -\frac{\partial \langle \mathcal{K} \rangle}{\partial \mu} = \mu/g$  and expressed  $\langle \mathcal{H} \rangle$  in terms of  $\rho_0$  instead of  $\mu$  (Legendre transform). So far we have chosen  $\phi_0$  real, but we can equally well consider have chosen it with a nonzero phase (recall  $\psi$  is a complex field). The potential is then written as

$$V = -\mu |\phi_0|^2 + \frac{1}{2} g |\phi_0|^4. \quad (7.65)$$

The potential is invariant under global phase transformations. It has the form of a Mexican hat - see Fig. 7.2.

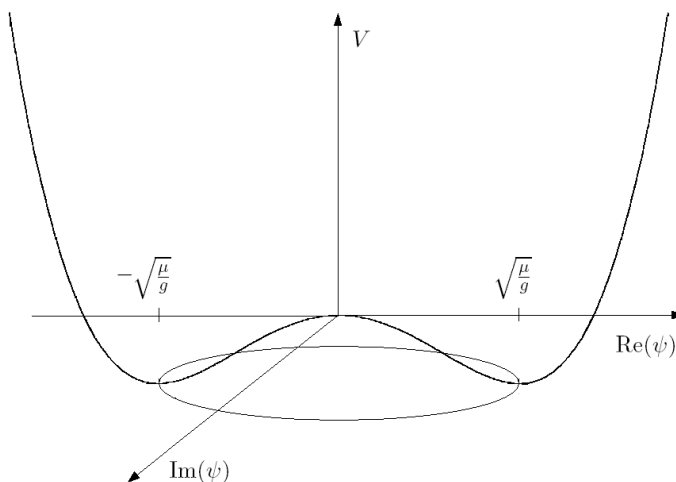


Figure 7.2: Plot of potential  $V$  as a function of magnitude and phase.

The fact that the system picks out a phase, implies that the ground state is no longer invariant under the global phase transformation - it *breaks* the global symmetry<sup>8</sup> This is not as scary as it sounds. Actually, it is analogous to what is happening in a ferromagnet

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<sup>8</sup>This is related to the fact that the ground state is not an eigenstate of the number operator.

as you cool it below a critical temperature and it acquires a nonzero magnetization. The Hamiltonian is invariant under spatial rotations, but the system picks out a direction - the direction of the magnetization. The ground state then breaks the rotational symmetry of the Lagrangian.

## 7.4 Problems

### 7.4.1 Complex scalar field, symmetry breaking, and Goldstone mode

Consider a complex scalar field with a quartic interaction

$$\mathcal{L} = (\partial_\mu \Phi)^* (\partial^\mu \Phi) - m^2 \Phi^* \Phi - \frac{\lambda}{6} (\Phi^* \Phi)^2. \quad (7.66)$$

The Lagrangian is invariant under global phase transformation. The complex field can be written in terms of two real fields  $\phi_1$  and  $\phi_2$

$$\Phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2). \quad (7.67)$$

Assume that  $\phi_1$  has a nonzero vacuum expectation value  $v$  in analogy with the dilute Bose gas<sup>9</sup>. We therefore write

$$\Phi = \frac{1}{\sqrt{2}} (v + \phi_1 + i\phi_2), \quad (7.68)$$

where  $v$  is a classical field and  $\phi_1$  and  $\phi_2$  are quantum fluctuating fields.

- 1) Substitute Eq. (7.68) into the Lagrangian (7.66) and expand it to second order in the fluctuating fields i. e. neglect terms that are third order or higher in  $\phi_1$  and  $\phi_2$ .
- 2) Find the classical potential  $V$  and show that  $v \neq 0$  at its minimum if  $m^2 < 0$ .
- 3) Find the terms that are linear in the fluctuating fields and show that they vanish if they are evaluated at the minimum of  $V$ .
- 4) Find the dispersion relations for the fields and show that there is a Goldstone boson.

### 7.4.2 Weakly interacting Bose gas

Consider the weakly interacting Bose gas.

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<sup>9</sup>In this case, the chemical potential is zero, so we cannot interpret it as a condensate of particles in the  $p = 0$  state. It is more like the spontaneous magnetization in a ferromagnet.

1) Calculate the fluctuations of the particle number  $\langle(\Delta N_{\mathbf{p}})^2\rangle$  in a state with momentum  $\mathbf{p}$ , i.e. calculate

$$\langle N_{\mathbf{p}}^2 \rangle - \langle N_{\mathbf{p}} \rangle^2, \quad (7.69)$$

where  $N_{\mathbf{p}} = a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$ . Take the limits  $p \rightarrow 0$  and  $p \rightarrow \infty$ . Comments? Calculate the integral

$$I = \int \frac{d^3 p}{(2\pi)^3} \langle (\Delta N_{\mathbf{p}})^2 \rangle. \quad (7.70)$$

2) The ground-state energy density of the Bose gas is

$$\mathcal{E} = \frac{2\pi a \rho^2}{m} \left[ 1 + \frac{128}{15} \sqrt{\frac{\rho a^3}{\pi}} \right]. \quad (7.71)$$

Calculate  $\mu$ , which is given by

$$\mu = \frac{d\mathcal{E}}{d\rho}. \quad (7.72)$$

What is the (dimensionless) expansion parameter of  $\mu$ ? Calculate the free energy density  $\mathcal{F} = \mathcal{E} - \mu\rho$ . Recall  $\mathcal{F}$  is a function of  $\mu$  and not  $\rho$ !

3) Define

$$\phi(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_{\mathbf{p}} e^{-ipx}. \quad (7.73)$$

Calculate

$$J(|\mathbf{x} - \mathbf{y}|, t - t') = \langle \Phi | T \left\{ \phi(\mathbf{x}, t) \phi^\dagger(\mathbf{y}, t') \right\} | \Phi \rangle, \quad (7.74)$$

where  $\Phi$  is the ground state of the weakly interacting Bose gas. Find its Fourier transform  $J(p)$ .

### 7.4.3 Dimensional regularization

The energy of a weakly interacting Bose gas is given by

$$\mathcal{E} = \frac{1}{2} g \rho^2 + \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} \left[ \epsilon(p) - \frac{p^2}{2m} - g\rho \right]. \quad (7.75)$$

1) Calculate separately the three integrals in the above expression using dimensional regularization. Comments?

2) Use the results in 1) to rederive Eq. (7.52). Why do you need no counterterms in “dimreg” to render the ground-state energy finite?

# Chapter 8

## Fermi gases

We will now take a closer look at Fermi gases. We first consider an ideal Fermi gas at finite chemical potential. Then we will introduce interactions and compute quantum corrections to the various physical quantities.

### 8.1 Perfect Fermi gas

The field operator  $\psi(x)$  can be expanded as

$$\psi(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}, \lambda} a_{\mathbf{p}, \lambda} \chi(\lambda) e^{-i\mathbf{p}x}, \quad (8.1)$$

where  $\lambda$  denotes other quantum numbers, e. g. that are necessary to completely specify the state, and  $\chi(\lambda)$  is the associated part of the wavefunction. For example, for a spin- $\frac{1}{2}$  particle, we may write

$$\chi(\lambda = \tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi(\lambda = -\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (8.2)$$

The Fermi propagator  $iG(x, y)$  is defined by

$$iG(x, y) = \langle \Phi | T \{ \psi(x) \psi^\dagger(y) \} | \Phi \rangle \quad (8.3)$$

$$= \begin{cases} \langle \Phi | \psi(x) \psi^\dagger(y) | \Phi \rangle, & y^0 < x^0, \\ -\langle \Phi | \psi^\dagger(y) \psi(x) | \Phi \rangle, & x^0 < y^0, \end{cases} \quad (8.4)$$

where  $T$  denotes time-ordering as usual. Note the minus sign when  $x^0 < y^0$ , which is not included in the definition of the bosonic propagator (see chapter six). We first consider the case  $y^0 < x^0$ . Inserting the expansion Eq. (8.1) of the field operator into the first line of Eq. (8.4), we obtain

$$iG(x, y) = \theta(x^0 - y^0) \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \langle \Phi | e^{-i(px - qy)} a_{\mathbf{p}, \lambda} a_{\mathbf{q}, \lambda}^\dagger | \Phi \rangle. \quad (8.5)$$

Note that we have taken the continuum limit and used the orthogonality between the functions,  $\chi(\lambda_1)^* \chi(\lambda_2) = \delta_{\lambda_1, \lambda_2}$ . Using the anticommutator, we can write this as

$$iG(x, y) = \theta(x^0 - y^0) \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} e^{-i(px - qy)} \langle \Phi | (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) - a_{\mathbf{q}, \lambda}^\dagger a_{\mathbf{p}, \lambda} | \Phi \rangle \quad (8.6)$$

In the fermionic ground state, we know that all the states up to Fermi energy  $\epsilon_F$  are occupied and all other states are empty. Thus  $\mathbf{p}$  must be smaller than the Fermi momentum<sup>1</sup>  $p_F$ , when  $a_{\mathbf{p}, \lambda}$  acts on  $|\Phi\rangle$ . To recover the state  $|\Phi\rangle$  when  $a_{\mathbf{q}, \lambda}$  acts afterwards, we must have  $\mathbf{q} = \mathbf{p}$ . This yields

$$\begin{aligned} iG(x, y) &= \theta(x^0 - y^0) \sum_{\lambda} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} e^{-i(px - qy)} \langle \Phi | (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) (1 - \theta(p_F - p)) | \Phi \rangle \\ &= g_s \int \frac{d^3 p}{(2\pi)^3} e^{-ip(x - y)} \theta(x^0 - y^0) \theta(p - p_F), \end{aligned} \quad (8.7)$$

where we in the line have summed over  $\lambda$  giving rise to a degeneracy factor, which is denoted by  $g_s$ . Using the integral representation for the step function (Exercise 5.6.4), we obtain

$$G(x, y) = g_s \int \frac{d^3 p}{(2\pi)^3} \frac{dp_0}{2\pi} \theta(p - p_F) \frac{e^{-i\omega(x^0 - y^0)}}{p_0 - \omega_p + i\eta} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}. \quad (8.8)$$

For  $y^0 > x^0$ , a similar calculation yields

$$G(x, y) = g_s \int \frac{d^3 p}{(2\pi)^3} \frac{dp_0}{2\pi} \theta(p_F - p) \frac{e^{-i\omega(x^0 - y^0)}}{p_0 - \omega_p - i\eta} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}. \quad (8.9)$$

The momentum space propagator then reads

$$G(p) = g_s \left[ \frac{\theta(p - p_F)}{p_0 - \omega_p + i\eta} + \frac{\theta(p_F - p)}{p_0 - \omega_p - i\eta} \right]. \quad (8.10)$$

Note that this reduces to the standard propagator in the vacuum, i.e. when  $p_F = 0$ .

The density can now be written in terms of the propagator:

$$\begin{aligned} \rho &= -i \int \frac{d^4 p}{(2\pi)^4} G(p) e^{i\eta p_0} \\ &= -ig_s \int \frac{d^4 p}{(2\pi)^4} \left[ \frac{\theta(p - p_F)}{\omega - \omega_p + i\eta} + \frac{\theta(p_F - p)}{\omega - \omega_p - i\eta} \right] e^{i\eta p_0}. \end{aligned} \quad (8.11)$$

Since  $\eta > 0$ , we must close the contour in the upper half plane such that the contribution from the semicircle vanishes as  $R \rightarrow \infty$ . We then pick up a contribution from the pole

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<sup>1</sup>The Fermi momentum is defined by  $\epsilon_F = \sqrt{m^2 + p_F^2}$ . For a nonrelativistic particle, this reduces to  $\epsilon_F = p_F^2/2m$ .

$\omega = \omega_p + i\eta$  with residue  $2\pi i$ . The density then reduces to

$$\begin{aligned}\rho &= g_s \int \frac{d^3p}{(2\pi)^3} \theta(p_F - p) \\ &= \frac{g_s}{2\pi^2} \int_0^\infty \theta(p_F - p) p^2 dp \\ &= \frac{g_s}{6\pi^2} p_F^3 .\end{aligned}\tag{8.12}$$

This result is well known from e.g. TFY4230.

Using the same techniques as we did to express the density  $\langle \psi^\dagger \psi \rangle$ , we can express the expectation value of the kinetic energy density  $\mathcal{T} \equiv \langle \psi^\dagger \frac{1}{2m} \nabla^2 \psi \rangle$  in terms of the derivative of the propagator. It is not so difficult to show that

$$\mathcal{T} = \pm \lim_{\mathbf{x}' \rightarrow \mathbf{x}^+} -\frac{1}{2m} \nabla^2 G(\mathbf{x}, t, \mathbf{x}', t^+) ,\tag{8.13}$$

where  $+$  means taking the limit from above and  $t^+$  is infinitesimally larger than  $t$ <sup>2</sup>. The  $\pm$  sign apply to bosons or fermions, respectively. The latter ensures that pick up the contribution from the pole in the upper half plane. For fermions, this yields

$$\mathcal{T} = -ig_s \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{2m} \left[ \frac{\theta(p - p_F)}{\omega - \omega_p + i\eta} + \frac{\theta(p_F - p)}{\omega - \omega_p - i\eta} \right] e^{i\eta p_0} .\tag{8.14}$$

Closing the contour in the upper half plane and picking up the contribution from the pole  $\omega = \omega_p + i\eta$ , we find

$$\begin{aligned}\mathcal{T} &= g_s \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} \theta(p_F - p) \\ &= \frac{g_s}{2\pi^2} \int_0^\infty \theta(p_F - p) \frac{p^4}{2m} dp \\ &= \frac{g_s}{20\pi^2 m} p_F^5 .\end{aligned}\tag{8.15}$$

The energy per particle is given by  $\mathcal{T}/\rho$  and so we obtain

$$\mathcal{E}_0 = \frac{p_F^2}{2m} \frac{3}{5} .\tag{8.16}$$

Thus the average energy of a particle is  $3/5$  times the Fermi energy.

Since we have not included interactions,  $\mathcal{T}$  equals the total energy density  $\mathcal{E}$ <sup>3</sup>. The free energy density is

$$\begin{aligned}\mathcal{F} &= \mathcal{E} - \mu\rho \\ &= \frac{g_s}{20\pi^2 m} p_F^5 - \frac{p_F^2}{2m} \frac{g_s}{6\pi^2} p_F^3 \\ &= -\frac{g_s}{30\pi^2 m} p_F^5 ,\end{aligned}\tag{8.17}$$

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<sup>2</sup>This ensures the correct ordering, i.e.  $\psi^\dagger \psi$  and *not*  $\psi \psi^\dagger$ .

<sup>3</sup>Note that we are using  $\epsilon = p^2/2m$  and have subtracted the rest mass energy  $m$ . Taking it into account yields an additional term in the energy density,  $m\rho$ .

where we have used that  $\mu = p_F^2/2m$ . The pressure  $\mathcal{P}$  is minus the free energy density and is a function of  $\rho$ . Inverting Eq. (8.12) to express  $p_F$  in terms of  $\rho$  and inserting the expression into  $\mathcal{P} = -\mathcal{F}$ , we obtain

$$\mathcal{P} = \frac{1}{5m} \left( \frac{6\pi^2}{g_s} \right)^{2/3} \rho^{5/3}. \quad (8.18)$$

which is the well known equation of state for an ideal Fermi gas at  $T = 0$ .

## 8.2 Interacting Fermi gas

The Lagrangian of a Fermi gas with a local two-body interaction with strength  $g$  is given by

$$\mathcal{L}_{\text{int}} = i\psi^\dagger \partial_0 \psi - \frac{1}{2m} (\nabla \psi^\dagger)(\nabla \psi) + \mu \psi^\dagger \psi - \frac{1}{2} g (\psi^\dagger \psi)^2. \quad (8.19)$$

The perturbation  $H_{\text{int}}$  is given

$$H_{\text{int}} = \frac{1}{2} g \int d^3x (\psi^\dagger \psi)^2. \quad (8.20)$$

The first-order correction to the ground-state energy,  $E_1$ , is given by the expectation of the perturbation in the unperturbed ground state  $|\Phi\rangle$ :

$$E_1 = \langle \Phi | H_{\text{int}} | \Phi \rangle. \quad (8.21)$$

We can write

$$\frac{1}{2} g (\psi^\dagger \psi)^2 = \frac{g}{2V^2} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \lambda_1, \lambda_2} e^{i(p-q+r-s)x} a_{\mathbf{p}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{s}, \lambda_2}, \quad (8.22)$$

where we have used the orthogonality of  $\chi(\lambda_i)$ .

Integrating over  $\mathbf{x}$  gives momentum conservation:

$$\mathbf{p} - \mathbf{q} + \mathbf{r} - \mathbf{s} = 0. \quad (8.23)$$

This yields

$$H_{\text{int}} = \frac{g}{2V} \sum_{\mathbf{q}, \mathbf{r}, \mathbf{s}, \lambda_1, \lambda_2} a_{\mathbf{q}-\mathbf{r}+\mathbf{s}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{s}, \lambda_2} \quad (8.24)$$

We first redefine the momentum,  $\mathbf{q}' = \mathbf{q} - \mathbf{r}$ , then  $\mathbf{s} \rightarrow \mathbf{p}$ , drop the prime on the dummy variable  $\mathbf{q}'$ , and obtain

$$H_{\text{int}} = \frac{g}{2V} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p}+\mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}+\mathbf{r}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p}, \lambda_2} \quad (8.25)$$

We next separate the sum (8.25) into two terms, one with  $\mathbf{q} = 0$  and one with  $\mathbf{q} \neq 0$ . The first term reads

$$H_{\text{int}}^a = \frac{g}{2V} \sum_{\mathbf{p}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p}, \lambda_1}^\dagger a_{\mathbf{r}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p}, \lambda_2} . \quad (8.26)$$

Firstly, we note that we must have  $\mathbf{p} < p_F$  to get a nonzero result. We next distinguish between the cases  $\mathbf{r} < p_F$  and  $\mathbf{r} > p_F$ . In the first case, this implies that  $\mathbf{p} = \mathbf{r}$ . If we denote the corresponding operator by  $H_{\text{int}}^{a1}$ , we obtain

$$H_{\text{int}}^{a1} = \frac{g}{2V} \sum_{\mathbf{p}, \lambda_1 \lambda_2} a_{\mathbf{p}, \lambda_1}^\dagger a_{\mathbf{p}, \lambda_1} a_{\mathbf{p}, \lambda_2}^\dagger a_{\mathbf{p}, \lambda_2} . \quad (8.27)$$

The matrix element of this operator is

$$\begin{aligned} \langle \Phi | H_{\text{int}}^{a1} | \Phi \rangle &= \frac{g}{2V} \sum_{\mathbf{p}, \lambda_1 \lambda_2} \theta(p_F - p) \\ &\rightarrow \frac{1}{2} g g_s^2 \int \frac{d^3 p}{(2\pi^3)} \theta(p_F - p) . \end{aligned}$$

When we calculate the energy per particle, we first divide by the volume  $V$  to get the energy density and then by  $\rho = p_F^3/6\pi^2$ . Hence this term vanishes in the thermodynamic limit.

When  $\mathbf{r} > p_F$ , we must have  $\lambda_1 = \lambda_2$  since  $a_{\mathbf{r}, \lambda_1}$  must annihilate the particle above the Fermi sea created by  $a_{\mathbf{r}, \lambda_2}^\dagger$ . If we denote the corresponding operator by  $H_{\text{int}}^{a2}$ , the matrix element becomes

$$\begin{aligned} \langle \Phi | H_{\text{int}}^{a2} | \Phi \rangle &= \frac{g}{2V} \sum_{\mathbf{p}, \mathbf{r}, \lambda_1} [1 - \theta(p_F - r)] \theta(p_F - p) \\ &\rightarrow \frac{1}{2} g g_s V \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 r}{(2\pi)^3} [1 - \theta(p_F - r)] \theta(p_F - p) \\ &= -\frac{1}{2} g g_s \theta(p_F - p) V \int \frac{d^3 r}{(2\pi)^3} - \frac{1}{2} g g_s V \left( \frac{p_F^3}{6\pi^2} \right)^2 . \end{aligned} \quad (8.28)$$

The term involving is divergent and is related to normal ordering and renormalization of the chemical potential. The second term is finite and gives the following total contribution to the energy per particle from  $H_{\text{int}}^a$ :

$$\mathcal{E}_1^a = -\frac{1}{2} g \left( \frac{p_F^3}{6\pi^2} \right) . \quad (8.29)$$

The second reads

$$H_{\text{int}}^b = \frac{g}{2V} \sum'_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p}+\mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}+\mathbf{r}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p}, \lambda_2} , \quad (8.30)$$

where the prime on the sum is a reminder that the term  $\mathbf{q} = 0$  is omitted from the sum. Again we must have  $\mathbf{p} < p_F$  to get a nonzero contribution. If  $\mathbf{r} > p_F$ , we create a particle

above the Fermi sea and therefore this particle must be annihilated by the operator  $a_{\mathbf{q}+\mathbf{r},\lambda_1}$  (why?). Hence  $\mathbf{q} = 0$  but this is excluded. This implies that  $\mathbf{r} < p_F$  and therefore  $\mathbf{p} = \mathbf{r}$  (why?). first. This yields

$$H_{\text{int}}^b = \frac{g}{2V} \sum_{\mathbf{p}, \mathbf{q}, \lambda_1, \lambda_2}^{\prime} a_{\mathbf{p}+\mathbf{q}, \lambda_1}^{\dagger} a_{\mathbf{p}+\mathbf{q}, \lambda_1} a_{\mathbf{p}, \lambda_2}^{\dagger} a_{\mathbf{p}, \lambda_2} , \quad (8.31)$$

The energy then becomes

$$E_{1b} = \frac{g}{2V} g_s^2 \sum_{\mathbf{p}, \mathbf{q}}^{\prime} \theta(p_F - p) \theta(p_F - |\mathbf{p} + \mathbf{q}|) . \quad (8.32)$$

Taking the continuum limit, we obtain

$$E_{1b} = \frac{1}{2} g V g_s^2 \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \theta(p_F - p) \theta(p_F - |\mathbf{p} + \mathbf{q}|) . \quad (8.33)$$

Note that we have removed the prime from the integral as it only affects the integrand at a single point. If we use dimensional regularization, we can redefine the momenta,  $q' = \mathbf{q} + p$ . The integrals then decouple and we obtain

$$E_{1b} = \frac{1}{2} g V g_s^2 \left( \frac{p_F^3}{6\pi^2} \right)^2 . \quad (8.34)$$

The contribution to the energy per particle is then

$$\mathcal{E}_1^b = \frac{1}{2} g g_s \frac{p_F^3}{6\pi^2} . \quad (8.35)$$

Adding Eqs. (8.29) and (8.35), we obtain the total energy per particle to first order in the interaction

$$\mathcal{E}_1 = \frac{p_F^2}{2m} \left[ \frac{3}{5} + \frac{2}{3\pi} (g_s - 1) p_F a + \dots \right] , \quad (8.36)$$

where we have reinstated  $a = mg/4\pi$ . We notice that the expansion parameter is the dimensionless quantity  $p_F a$ . Note also that a nonzero spin degeneracy is necessary to obtain a nonzero correction. The reason is the Pauli principle: The  $s$ -wave scattering length is zero for two identical fermions, unless they have different spin.

### 8.3 Degenerate electron gas

So far, we have been discussing a degenerate Fermi gas with a local quartic interaction characterized by the  $s$ -wave scattering length  $a$ . We will next consider a nonrelativistic degenerate electron gas in a volume  $V$ . The interaction is therefore the usual static Coulomb interaction between charges. In order to ensure that the system is overall electrically neutral, we introduce a homogeneous background of positive charge. The interactions we will take into account are thus

- Self-interaction of the homogeneous background.
- Electron-background interaction.
- Static electron-electron interaction.

From courses on electromagnetism, we know the expression for the Coulomb potential. However, we now have developed the machinery to *derive* it from QED. This is very instructive and so we consider this first.

In Fig. 8.1, we have shown the  $e^2$ -order Feynman diagram that contributes to electron-electron scattering. The photon which is exchanged between the two electrons is virtual and in order to evaluate the expression given by the Feynman graph, we need to know the photon propagator.

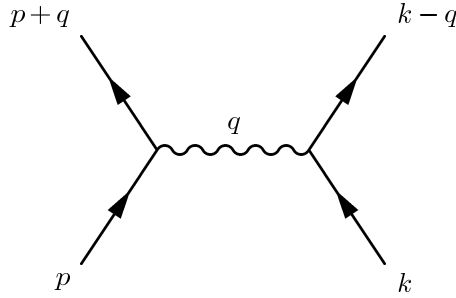


Figure 8.1: Electron-electron scattering

### 8.3.1 Photon propagator and the Coulomb potential

The Lagrangian for free Maxwell theory is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \quad (8.37)$$

The Green's function for the photon field is a  $4 \times 4$  matrix and denoted by  $D_{\nu\beta}(x-y)$ . The equation for the Green's function reads

$$[\partial_\alpha \partial^\alpha g^{\mu\nu} - \partial^\mu \partial^\nu] D_{\nu\beta}(x-y) = i\delta_\beta^\mu \delta^4(x-y). \quad (8.38)$$

In momentum space, this becomes

$$[-k^2 g^{\mu\nu} + k^\mu k^\nu] D_{\nu\beta}(p) = i\delta_\beta^\mu. \quad (8.39)$$

This equation has no solution since the operator  $-k^2 g^{\mu\nu} + k^\mu k^\nu$  is singular, i. e.  $\det(-k^2 g^{\mu\nu} + k^\mu k^\nu) = 0$ . The problem is that we have too many degrees of freedom. Recall that QED is a gauge theory which is invariant under local phase transformations of the fermion field  $\psi$  accompanied by a transformation of the gauge field  $A_\mu$ . Hence there are (infinitely) many

gauge-field configurations that correspond to the same physical fields  $\mathbf{E}$  and  $\mathbf{B}$ <sup>4</sup>. We must therefore *choose* a gauge, i.e. somehow pick out one representative from each class of gauge-equivalent configurations. This is done by adding to the Lagrangian a *gauge-fixing* term<sup>5</sup>

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \quad (8.40)$$

where  $\alpha$  is a so-called gauge parameter. The equation for the momentum-space propagator now becomes<sup>6</sup>

$$\left[ -k^2 g^{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) k^\mu k^\nu \right] D_{\nu\beta}(p) = i\delta_\beta^\mu. \quad (8.41)$$

The solution is

$$D_{\mu\nu}(q) = \frac{ig_{\mu\nu}}{q^2 + i\epsilon} - \frac{(1 - \alpha)q_\mu q_\nu / q^2}{q^2 + i\epsilon}. \quad (8.42)$$

Now it is important to stress that all physical quantities that one computes are independent of the gauge parameter  $\alpha$ , and more generally independent of the gauge-fixing term in the Lagrangian<sup>7</sup>. Two common and convenient choices are  $\alpha = 1$ , which is called the Feynman gauge and  $\alpha = 0$ , which is the Landau gauge.

We now return to the Feynman diagram Fig. 8.1. Up to some prefactors, the expression for it reads

$$V_{ijkl} \sim e^2 \bar{u}^{(i)}(p+q) \gamma^\mu u^{(j)}(p) \left[ \frac{g_{\mu\nu}}{q^2} - \frac{(1 - \alpha)q_\mu q_\nu / q^2}{q^2 + i\epsilon} \right] u^{(k)}(k-q) \gamma^\nu u^{(l)}(k), \quad (8.43)$$

where  $i, j, k$ , and  $l = 1, 2$ , and where the  $\gamma$ -matrices are expressed in terms of the Pauli spin matrices  $\sigma_i$

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (8.44)$$

The spinors are given by

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}, \quad u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}, \quad (8.45)$$

---

<sup>4</sup>In fact, the space of gauge-field configurations can be divided into disjoint sets (equivalence classes) that can be connected by a gauge transformation.

<sup>5</sup>This class of gauges is called covariant gauge. There exists another widely used class, the Coulomb gauge where  $\mathcal{L}_{\text{gf}} = -\frac{1}{2\alpha}(\partial_i A^i)^2$ .

<sup>6</sup>Exercise!

<sup>7</sup>This is highly nontrivial at this stage.

We are interested in the nonrelativistic limit of the expression (8.43). In this limit, the spinors become

$$u^{(1)}(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u^{(2)}(p) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad (8.46)$$

The indices  $i, j, k$ , and  $l$  of  $V_{ijkl}$  then denote the spins of the electrons.

Using the expressions for the  $\gamma$ -matrices, it is easy to show that

$$\bar{u}^{(i)}(p)\gamma^0 u^{(j)}(q) = \delta_{ij} \quad \bar{u}^{(i)}(p)\gamma^k u^{(j)}(q) = 0 \quad (8.47)$$

for all values of  $p$  and  $q$  and  $k = 1, 2, 3$ . This yields

$$V_{ijkl} = e^2 \delta_{ij} \frac{g_{00}}{q^2 + i\epsilon} \left[ 1 - (1 - \alpha) q_0^2 / q^2 \right] \delta_{kl}. \quad (8.48)$$

In the nonrelativistic limit, we have  $p \approx (m, \mathbf{p})$  and  $(p + q) \approx (m, \mathbf{p} + \mathbf{q})$ . This yields  $q \approx (0, \mathbf{q})$  and therefore

$$V_{ijkl} = e^2 \delta_{ij} \frac{1}{\mathbf{q}^2} \delta_{kl}. \quad (8.49)$$

Note that the  $\alpha$ -dependence drops out in this limit. The Fourier transform of this expression is essentially the Coulomb potential. Note that the Coulomb potential is time independent, i.e. it is static. It is an instantaneous interaction, which is reasonable since the nonrelativistic limit is obtained by taking the limit  $c \rightarrow \infty$  and so it takes zero time for any signal or interaction to propagate.

The Hamiltonian for the electron-electron interaction can be written as

$$H_{\text{el-el}} = \frac{1}{2} e^2 \int d^3x \int d^3y \psi_{\lambda_1}^\dagger(\mathbf{x}) \psi_{\lambda_1}(\mathbf{x}) \frac{1}{|\mathbf{x} - \mathbf{y}|} \psi_{\lambda_2}^\dagger(\mathbf{x}) \psi_{\lambda_2}(\mathbf{x}), \quad (8.50)$$

where we have replaced the indices  $i, j, k$ , and  $l$  by the indices spin indices  $\lambda_1$  and  $\lambda_2$  and the field  $\psi_{\lambda_1}^\dagger(\mathbf{x})$  creates an electron with spin  $\lambda_1$  at position  $\mathbf{x}$ .

### 8.3.2 Self-interaction of homogeneous background

The self-interaction energy of the homogeneous background of two infinitesimal regions in space located at  $\mathbf{x}$  and  $\mathbf{y}$  with volumes  $d^3x$  and  $d^3y$  is

$$dV_{\text{bg}} = \frac{1}{2} e^2 \frac{\rho_b(\mathbf{x}) \rho_b(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d^3x d^3y, \quad (8.51)$$

where  $\rho_b(\mathbf{x})$  is the number density of the background at position  $\mathbf{x}$ . The total interaction energy  $V_b$  is found by integrating over  $x$  and  $y$ . Since the density of the background is assumed homogeneous, we obtain

$$V_{\text{bg}} = \frac{1}{2} e^2 \rho_b^2 \int \frac{d^3x d^3y}{|\mathbf{x} - \mathbf{y}|}. \quad (8.52)$$

Changing variables  $\mathbf{x}' = \mathbf{x} - \mathbf{y}$ , we can write this as

$$\begin{aligned} V_{\text{bg}} &= \frac{1}{2} e^2 \rho_b^2 \int \frac{d^3x d^3y}{|\mathbf{x}|} \\ &= \frac{1}{2} e^2 \rho_b^2 V \int \frac{d^3x}{|\mathbf{x}|}, \end{aligned} \quad (8.53)$$

where  $V$  is the volume of the system. The integral turn out to be divergent for large values of  $x$  and so we need to regulate it. We do this by adding a small mass  $\mu$  to the photon <sup>8</sup>. The propagator in momentum space then changes to that of a massive particle

$$\frac{1}{q^2} \rightarrow \frac{1}{q^2 + \mu^2}. \quad (8.54)$$

The Coulomb potential then changes to a Yukawa potential

$$\frac{1}{|\mathbf{x}|} \rightarrow \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|}. \quad (8.55)$$

The final integral in Eq. (8.53) is now easy to calculate and we finally obtain

$$V_{\text{bg}} = \frac{1}{2} e^2 \rho_b^2 V \frac{4\pi}{\mu^2}. \quad (8.56)$$

### 8.3.3 Electron interaction with homogeneous background

The interaction between the homogeneous background and a single electron located at  $\mathbf{y}$  is

$$E_1 = -e^2 \int d^3x \frac{\rho_b}{|\mathbf{x} - \mathbf{y}|} e^{-\mu|\mathbf{x} - \mathbf{y}|}, \quad (8.57)$$

where we have replaced the Coulomb potential with the Yukawa potential. After changing variables, this yields

$$\begin{aligned} E_1 &= -e^2 \rho_b \int d^3x \frac{e^{-\mu|\mathbf{x}|}}{|\mathbf{x}|} \\ &= -e^2 \rho_b \frac{4\pi}{\mu^2}. \end{aligned} \quad (8.58)$$

Note that this result is independent of the coordinate  $\mathbf{y}$  of the electron (why?). The total result is then obtained by multiplying by  $N$  for the electron-background energy is then

$$V_{\text{b-el}} = -e^2 \rho_b^2 V \frac{4\pi}{\mu^2}. \quad (8.59)$$

---

<sup>8</sup>It is important to realize that regulating this divergence has nothing to do with the regularization of divergent integrals arising from Feynman graphs. We shall see that the divergence will cancel against another divergent terms arising in the calculations and so regulating the integrals is only used in the intermediate steps. The final result has no  $\mu$ -dependence.

### 8.3.4 Eletron-electron interaction

We finally consider the eletron-electron interaction term. Inserting the expression for the field operators, this yields

$$H_{\text{el-el}} = \frac{e^2}{2V^2} \int d^3x \int d^3y \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \lambda_1, \lambda_2} a_{\mathbf{p}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} \frac{1}{|\mathbf{x} - \mathbf{y}|} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{s}, \lambda_2} e^{i(p-q)x} e^{i(r-s)y} . \quad (8.60)$$

We next need to evaluate

$$\int d^3x \int d^3y e^{i(p-q)x} \frac{1}{|\mathbf{x} - \mathbf{y}|} e^{i(r-s)y} \rightarrow \int d^3x \int d^3y e^{i(p-q)x} \frac{e^{-\mu|\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} e^{i(r-s)y} \quad (8.61)$$

$$\begin{aligned} &= \int d^3x \int d^3y \frac{e^{-\mu|\mathbf{y}|}}{|\mathbf{y}|} e^{i(r-s)y} e^{i(p-q+r-s)x} \\ &= \frac{4\pi}{(\mathbf{r} - \mathbf{s})^2 + \mu^2} V \delta_{\mathbf{p} - \mathbf{q} + \mathbf{r} - \mathbf{s}} , \end{aligned} \quad (8.62)$$

where we in the second line have changed variables. Inserting this result into Eq. (8.60) and summing over  $\mathbf{s}$ , we obtain

$$\begin{aligned} H_{\text{el-el}} &= \frac{e^2}{2V} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} \frac{4\pi}{(\mathbf{p} - \mathbf{q})^2 + \mu^2} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p} - \mathbf{q} + \mathbf{r}, \lambda_2} \\ &= \frac{e^2}{2V} \sum_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p} + \mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} \frac{4\pi}{\mathbf{p}^2 + \mu^2} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p} + \mathbf{r}, \lambda_2} \end{aligned} \quad (8.63)$$

where we have changed variables,  $\mathbf{p}' = \mathbf{p} - \mathbf{q}$ . We are now ready to calculate the expectation value

$$V_{\text{el-el}} = \langle \Phi | H_{\text{el-el}} | \Phi \rangle \quad (8.64)$$

We first consider the case  $\mathbf{p} = 0$ . This term reads

$$\begin{aligned} H_{\text{el-el}}^{\mathbf{p}=0} &= \frac{e^2}{2V} \frac{4\pi}{\mu^2} \sum_{\mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{r}, \lambda_2} \\ &= \frac{1}{2} e^2 V \frac{4\pi}{\mu^2} \sum_{\lambda_1, \lambda_2} \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3r}{(2\pi)^3} a_{\mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{r}, \lambda_2} , \end{aligned} \quad (8.65)$$

where we in the last line have taken the large-volume limit. This operator is proportional to  $N^2$ . This immediately gives

$$E_{\text{el-el}}^{\mathbf{p}=0} = \frac{1}{2V} e^2 N^2 \frac{4\pi}{\mu^2} . \quad (8.66)$$

The interpretation of this contribution is given by the Feynman diagram to the right in Fig. 8.2.

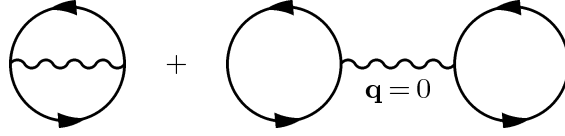


Figure 8.2: Order- $e^2$  vacuum diagrams contributing to the electron-electron interaction energy.

Adding this result to Eqs. (8.56) and (8.58), we notice that the terms that are individually divergent in the limit  $\mu \rightarrow 0$  cancel <sup>9</sup>.

We next consider the  $\mathbf{p} \neq 0$  term in the sum

$$H_{\text{el-el}}^{\mathbf{p} \neq 0} = \frac{e^2}{2V} \sum'_{\mathbf{p}, \mathbf{q}, \mathbf{r}, \lambda_1, \lambda_2} a_{\mathbf{p}+\mathbf{q}, \lambda_1}^\dagger a_{\mathbf{q}, \lambda_1} \frac{4\pi}{\mathbf{p}^2 + \mu^2} a_{\mathbf{r}, \lambda_2}^\dagger a_{\mathbf{p}+\mathbf{r}, \lambda_2} \quad (8.67)$$

where the prime is a reminder that  $\mathbf{p} = 0$  is excluded. The contribution is given by the left Feynman graph in Fig. 8.2. By the usual arguments, we must have  $|\mathbf{p} + \mathbf{r}| < p_F$  and  $|\mathbf{r}| > p_F$ . This implies that  $\mathbf{p} + \mathbf{r} = \mathbf{p} + \mathbf{r}$ , i. e.  $\mathbf{q} = \mathbf{r}$ . This yields

$$\begin{aligned} V_{\text{el-el}} &= \frac{e^2}{2V} \sum'_{\mathbf{p}, \mathbf{q}, \lambda_1} \frac{4\pi}{\mathbf{p}^2} \theta(p_F - |\mathbf{p} + \mathbf{q}|) [1 - \theta(p_F - |\mathbf{q}|)] \\ &= 4\pi e^2 V \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\mathbf{p}^2} \theta(p_F - |\mathbf{p} + \mathbf{q}|) [1 - \theta(p_F - |\mathbf{q}|)] . \end{aligned} \quad (8.68)$$

Notice that the prime is removed from the integral as contributes to a single point only. As usual, the first term is divergent and related to normal ordering. We discard it and are left with <sup>10</sup>

$$V_{\text{el-el}} = -4\pi e^2 V \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 q}{(2\pi)^3} \frac{1}{(\mathbf{p} - \mathbf{q})^2} \theta(p_F - |\mathbf{p}|) \theta(p_F - |\mathbf{q}|) , \quad (8.69)$$

where we have changed variables. We therefore need to calculate the angular average of  $1/|\mathbf{q} - \mathbf{p}|$ . This is

$$\begin{aligned} \left\langle \frac{1}{|\mathbf{q} - \mathbf{p}|} \right\rangle &= 2\pi \int_0^\pi \frac{\sin \theta}{\mathbf{p}^2 + \mathbf{q}^2 + 2|\mathbf{p}||\mathbf{q}| \cos \theta} d\theta \\ &= \frac{\pi}{|\mathbf{p}||\mathbf{q}|} \log \frac{(|\mathbf{p}| + |\mathbf{q}|)^2}{(|\mathbf{p}| - |\mathbf{q}|)^2} , \end{aligned} \quad (8.70)$$

The radial integrals then become

$$\begin{aligned} \frac{1}{16\pi^4} \int_0^{p_F} dp \int_0^{q_F} dq p q \log \frac{(|\mathbf{p}| + |\mathbf{q}|)^2}{(|\mathbf{p}| - |\mathbf{q}|)^2} &= \frac{1}{16\pi^4} \int_0^{p_F} dp \left[ p_F p^2 + \frac{1}{4} p (p_F^2 - p^2) \log \frac{(p_F + p)^2}{(p_F - p)^2} \right] \\ &= \frac{1}{16\pi^4} p_F^4 . \end{aligned} \quad (8.71)$$

<sup>9</sup>The terms with  $\mathbf{p} \neq 0$  from  $H_{\text{el-el}}$  are clearly convergent in the limit  $\mu \rightarrow 0$ .

<sup>10</sup>Note that the way we calculate the remaining integrals is very different from the arguments given in Fetter and Walecka.

This yields

$$V_{\text{el-el}} = -\frac{e^2 V}{4\pi^3} p_F^4 . \quad (8.72)$$

This is the final result for the interaction energy of a degenerate electron plasma and is in agreement with the result of Fetter and Walecka, Eq. (3.36).

# Chapter 9

## Effective potential

In this section, we introduce the concept of the effective potential in the context of the harmonic oscillator. We then apply the formalism to the weakly interacting Bose. In both cases, we rederive well-known results.

### 9.1 Harmonic oscillator

Consider a harmonic oscillator in one dimension with the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2x^2. \quad (9.1)$$

Scaling by setting  $\phi = \sqrt{m}x$ , we can write

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\omega^2\phi^2. \quad (9.2)$$

Recall that quantum mechanics is field theory in  $0+1$  dimensions and we therefore expand the field  $\phi$  as  $\phi_0 + \tilde{\phi}$ . This yields

$$L = \frac{1}{2}\dot{\tilde{\phi}}^2 - \frac{1}{2}\omega^2(\phi_0^2 + 2\phi_0\tilde{\phi} + \tilde{\phi}^2). \quad (9.3)$$

The classical potential is defined by

$$V_0 = \frac{1}{2}\omega^2\phi_0^2, \quad (9.4)$$

and is simply the (scaled) potential energy of the oscillator as a function of  $\phi_0$ . The propagator in momentum space can be computed using standard techniques and reads <sup>1</sup>

$$D(p_0) = \frac{i}{p_0^2 - \omega^2 + i\epsilon}. \quad (9.5)$$

---

<sup>1</sup> $D$  does not depend on any spatial momenta since we are in  $0+1$  dimensions.

Note that the linear term in  $L$  vanishes at the minimum of the classical potential (as usual) i.e. for  $\phi_0 = 0$  and the classical energy of the oscillator vanishes.

One can systematically calculate corrections to the classical potential. The potential is then called the *effective potential* and is the classical potential plus quantum corrections. If the theory is a free field theory, one can calculate the quantum corrections exactly, which we will now do for the harmonic oscillator. The first quantum correction  $V_1$  to the classical potential is a so-called one-loop correction and reads

$$V_1 = -\frac{i}{2} \int \frac{dp_0}{2\pi} \int \frac{d^d p}{(2\pi)^d} \ln[-i \det D^{-1}(p_0, p)] , \quad (9.6)$$

where  $d$  is the number of spatial dimensions and  $D^{-1}(p_0, p)$  is the inverse propagator in momentum space. For a free field theory

$$V = V_0 + V_1 \quad (9.7)$$

is exact. For the oscillator,  $d = 0$  and we obtain

$$V_1 = -\frac{i}{2} \int \frac{dp_0}{2\pi} \ln[\omega^2 - p_0^2 - i\epsilon] .$$

In order to calculate  $V_1$ , we first consider the derivative of  $V_1$  with respect to  $\omega^2$ :

$$\begin{aligned} I &= \frac{i}{2} \int \frac{dp_0}{2\pi} \frac{1}{(p_0^2 - \omega^2 + i\epsilon)} \\ &= \frac{1}{4} \frac{1}{\omega} , \end{aligned} \quad (9.8)$$

where we have used standard contour integration<sup>2</sup>. Integrating with respect to  $\omega^2$  we obtain<sup>3</sup>

$$V_1 = \frac{1}{2} \omega . \quad (9.9)$$

Thus the complete effective potential is

$$V = \frac{1}{2} \omega^2 \phi_0^2 + \frac{1}{2} \omega . \quad (9.10)$$

The ground-state energy  $E_0$  is given by the minimum of the quantum effective potential. The value of  $\phi_0$  which minimizes  $V$  is  $\phi_0 = 0$  and so

$$\begin{aligned} E_0 &= V(\phi_0 = 0) \\ &= \frac{1}{2} \omega . \end{aligned} \quad (9.11)$$

---

<sup>2</sup>Close the contour in either halfplane. The contribution comes from one of the poles  $p_0 = \pm(\omega - i\epsilon)$  and the contribution from the semicircle vanishes.

<sup>3</sup>We set the integration constant to zero!

Reinstating factors of  $\hbar$  one finds

$$E_0 = \frac{1}{2}\hbar\omega. \quad (9.12)$$

Thus the classical energy of a harmonic oscillator is zero (the oscillator is at rest at the origin,  $\phi_0 = 0$ ), while the one-loop correction includes the zero-point fluctuations and gives the exact quantum mechanical ground-state energy.

For interacting field theories, one can write the effective potential in a perturbative series as

$$V = V_0 + V_1 + V_2 + \dots \quad (9.13)$$

$V_2, V_3, \dots$  are corrections to the exactly solvable case with no interactions, i.e. a theory that is quadratic in the fields.

## 9.2 Dilute Bose gas revisited

The Lagrangian of the dilute Bose gas is

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + \mu \psi^\dagger \psi - \frac{1}{2m} \nabla \psi^\dagger \cdot \nabla \psi - \frac{1}{2}g (\psi^\dagger \psi)^2, \quad (9.14)$$

Writing the  $\psi$  as a sum of a classical field and a quantum fluctuating field

$$\psi = \phi_0 + \tilde{\psi}, \quad (9.15)$$

one finds

$$\begin{aligned} \mathcal{L} = & i\tilde{\psi}^\dagger \partial_0 \tilde{\psi} - \frac{1}{2m} \nabla \tilde{\psi}^\dagger \cdot \nabla \tilde{\psi} + \mu [|\phi_0|^2 + \phi_0^* \tilde{\psi} + \phi_0 \tilde{\psi}^\dagger + \tilde{\psi}^\dagger \tilde{\psi}] \\ & - \frac{1}{2}g[|\phi_0|^2 + \phi_0^* \tilde{\psi} + \phi_0 \tilde{\psi}^\dagger + \tilde{\psi}^\dagger \tilde{\psi}]^2, \end{aligned} \quad (9.16)$$

where we have neglected total derivatives and used that  $\phi_0$  is independent of space-time. The complex field  $\tilde{\psi}$  is written in terms of two real fields

$$\tilde{\psi} = \frac{1}{\sqrt{2}}(\tilde{\psi}_1 + i\tilde{\psi}_2). \quad (9.17)$$

Inserting this into the Lagrangian, we obtain

$$\begin{aligned} \mathcal{L} = & \frac{1}{2}(\tilde{\psi}_2 \partial_0 \tilde{\psi}_1 - \tilde{\psi}_1 \partial_0 \tilde{\psi}_2) + \frac{1}{2} \frac{1}{2m} (\tilde{\psi}_1 \nabla^2 \tilde{\psi}_1 + \tilde{\psi}_2 \nabla^2 \tilde{\psi}_2) + \left[ \mu - \frac{1}{2}g|\phi_0|^2 \right] |\phi_0|^2 \\ & + \frac{1}{\sqrt{2}} \left[ \mu - \frac{1}{2}g|\phi_0|^2 \right] \phi_0^* [\tilde{\psi}_1 + i\tilde{\psi}_2] + \frac{1}{\sqrt{2}} \left[ \mu - \frac{1}{2}g|\phi_0|^2 \right] \phi_0 [\tilde{\psi}_1 - i\tilde{\psi}_2] \\ & + \frac{1}{2}(\mu - 3g|\phi_0|^2) \tilde{\psi}_1^2 + \frac{1}{2}(\mu - g|\phi_0|^2) \tilde{\psi}_2^2, \end{aligned} \quad (9.18)$$

where we have integrated by parts and neglected terms that are third and fourth in the fluctuations. The classical potential is

$$V_0 = \left[ \mu - \frac{1}{2}g|\phi_0|^2 \right] |\phi_0|^2. \quad (9.19)$$

Note that the linear terms vanish at the minimum of the classical potential,  $\mu = g|\phi_0|^2$ , and that  $V_0$  reduces to

$$V_0 = \mu^2/2g. \quad (9.20)$$

The terms that are quadratic in the fluctuations can be written in matrix form

$$\mathcal{L}_{\text{quad}} = -\frac{1}{2}(\tilde{\psi}_1, \tilde{\psi}_2) \begin{pmatrix} -\frac{1}{2m}\nabla^2 - \mu + 3g|\phi_0|^2 & -\partial_0 \\ \partial_0 & -\frac{1}{2m}\nabla^2 - \mu + g|\phi_0|^2 \end{pmatrix} \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix}. \quad (9.21)$$

The matrix is denoted by  $G^{-1}$  and defines the inverse propagator. In momentum space,  $G^{-1}$  reads

$$iG^{-1}(p, \omega) = \begin{pmatrix} \frac{p^2}{2m} - \mu + 3g|\phi_0|^2 & -i\omega \\ i\omega & \frac{p^2}{2m} - \mu + g|\phi_0|^2 \end{pmatrix}. \quad (9.22)$$

The determinant is

$$\det iG^{-1} = \left[ \frac{p^2}{2m} - \mu + g|\phi_0|^2 \right] \left[ \frac{p^2}{2m} - \mu + 3g|\phi_0|^2 \right] - \omega^2. \quad (9.23)$$

The zeros of  $\det iG^{-1}$  yields the dispersion relation. This gives

$$\omega^2 = \left[ \frac{p^2}{2m} - \mu + g|\phi_0|^2 \right] \left[ \frac{p^2}{2m} - \mu + 3g|\phi_0|^2 \right]. \quad (9.24)$$

Evaluated at the classical minimum  $\mu = g|\phi_0|^2$ , we find

$$\epsilon^2(p) = \frac{p^2}{2m} \left[ \frac{p^2}{2m} + 2g|\phi_0|^2 \right], \quad (9.25)$$

which is the Bogoliubov dispersion relation derived earlier.

The one-loop effective potential is given by

$$\begin{aligned} V_1 &= -\frac{i}{2} \int \frac{dp_0}{2\pi} \int \frac{d^d p}{(2\pi)^d} \ln \det[-iG^{-1}(p_0, p)] \\ &= \int \frac{dp_0}{2\pi} \int \frac{d^d p}{(2\pi)^d} \ln[\epsilon^2(p) - p_0^2]. \end{aligned} \quad (9.26)$$

Integrating over  $p_0$  yields <sup>4</sup>

$$V_1 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \epsilon(p). \quad (9.27)$$

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<sup>4</sup>Take the derivative of  $V_1$  wrt  $\epsilon^2(p)$  and do the contour integral of  $dV_1/d\epsilon^2(p)$  in the complex  $p_0$  plane. Integrate the resulting expression wrt  $\epsilon^2(p)$  and set the constant of integration to zero.

The expression for  $V_1$  is highly divergent in the UV. In fact, for  $d = 3$  it contains quintic, cubic, and linear divergences. This is seen by expanding the integrand for large  $p$ . After integrating over angles, we obtain

$$V_1 = \frac{1}{4\pi^2} \int_0^\infty dp p^2 \epsilon(p) \quad (9.28)$$

For large- $p$ , the integrand behaves as

$$p^2 \epsilon(p) \simeq p^2 \left[ \frac{p^2}{2m} + \mu - \frac{\mu^2}{p^2/m} + \dots \right] \quad (9.29)$$

We isolate the divergences by adding and subtracting the infinite terms

$$\begin{aligned} V_1 = & \frac{1}{4\pi^2} \int_0^\infty dp p^2 \left[ \epsilon(p) - \frac{p^2}{2m} - \mu + \frac{\mu^2}{p^2/m} \right] \\ & + \frac{1}{4\pi^2} \int_0^\infty dp p^2 \left[ \frac{p^2}{2m} + \mu - \frac{\mu^2}{p^2/m} \right]. \end{aligned} \quad (9.30)$$

The first divergence corresponds the infinite vacuum energy density of an ideal gas and reads

$$\mathcal{E}_{\text{vacuum}} = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m}. \quad (9.31)$$

The counterterm that arises from the classical potential is

$$\Delta V_0 = \Delta \mathcal{E} - \frac{\mu}{g} \Delta \mu + \frac{\mu^2}{2g^2} \Delta g, \quad (9.32)$$

where we have added a vacuum counterterm  $\Delta \mathcal{E}$ . This term in Eq. (9.31) is eliminated by the vacuum counterterm which becomes

$$\Delta \mathcal{E} = -\frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m}. \quad (9.33)$$

This divergence is related to the operator-ordering problem in the formalism<sup>5</sup>. The second term is

$$\frac{1}{2} \mu \int \frac{d^3p}{(2\pi)^3}. \quad (9.34)$$

This term is cancelled by the counterterm

$$-\frac{\mu}{g} \Delta \mu, \quad (9.35)$$

---

<sup>5</sup>Classically the Fourier coefficients  $a$  and  $a^*$  commute and so  $aa^*$  is the same  $a^*a$ . This is no longer the case in the quantum theory where  $aa^\dagger \neq a^\dagger a$ !

and so matching the two terms yields

$$\Delta\mu = \frac{1}{2}g \int \frac{d^3p}{(2\pi)^3} . \quad (9.36)$$

The last divergence is

$$-\frac{1}{2}m\mu^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} . \quad (9.37)$$

This divergence is cancelled by the counterterm

$$\frac{\mu^2}{g^2} \Delta g . \quad (9.38)$$

Matching the two expressions, we find

$$\Delta g = mg^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} . \quad (9.39)$$

Making the subtractions, the renormalized free energy density is

$$\begin{aligned} V_1 &= \frac{1}{2\pi^2} \int_0^\infty dp p^2 \left[ \epsilon(p) - \frac{p^2}{2m} - \mu + \frac{\mu^2}{p^2/2m} \right] \\ &= \frac{2\sqrt{2}}{15\pi} (2m)^{3/2} \mu^{5/2} . \end{aligned} \quad (9.40)$$

The free energy density is the sum of the classical potential  $V_0 = -\mu|\phi_0|^2 + g|\phi_0|^4/2$  and  $V_1$  evaluated at the minimum  $\mu = g|\phi_0|^2$  and expressed as a function of  $\mu$ . This yields

$$\mathcal{F} = -\frac{\mu^2}{2g} \left[ 1 - \frac{4(2m)^{3/2}\sqrt{2\mu g^2}}{15\pi^2} \right] . \quad (9.41)$$

This result is identical to that obtained by first calculating the energy density  $\mathcal{E}$  and then performing a Legendre transform to obtain  $\mathcal{F}$ .

## 9.3 Problems

### 9.3.1

Consider a relativistic scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 . \quad (9.42)$$

1) Write the field as  $\phi = \phi_0 + \tilde{\phi}$  and show that the classical potential is

$$V_0 = \frac{1}{2}m^2\phi_0^2 + \frac{\lambda}{4}\phi_0^4 . \quad (9.43)$$

2) The propagator is

$$D = \frac{i}{p^2 - M^2} , \quad (9.44)$$

where  $M^2 = m^2 + 3\lambda\phi_0^2$ . Show that the one-loop correction to the effective potential is

$$V_1 = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \sqrt{|\mathbf{p}|^2 + M^2} , \quad (9.45)$$

where  $d$  is the number of spatial dimensions.

3) Set  $d = 2$ . The expression for  $V_1$  is divergent. Analyze the divergences and subtract counterterms such that the renormalized effective potential is finite. Hint: you only need to renormalize the vacuum energy density and the mass parameter. The coupling  $\lambda$  needs no renormalization. In other words, write

$$m^2 \rightarrow m^2 + \Delta m^2 , \quad (9.46)$$

and calculate  $\Delta m^2$ .

4) Set  $d = 3$  and repeat using dimensional regularization.

# Chapter 10

## Bose-Einstein condensation in traps

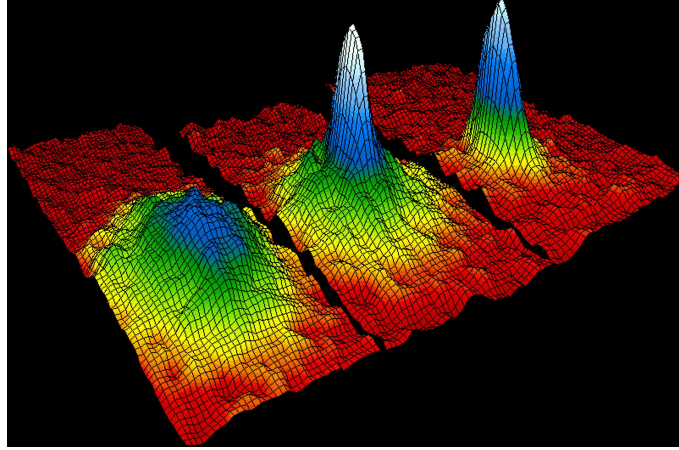


Figure 10.1: Images of the velocity distribution of rubidium atoms in the experiment by Anderson *et al.* (1995). The left frame corresponds to a gas at a temperature just above condensation; the center frame, just after the appearance of the condensate; the right frame, after further evaporation leaves a sample of nearly pure condensate.

Bose-Einstein condensation in harmonic traps is (simply) a many-body problem in an external potential  $V(\mathbf{x})$ . The potential we will be using in the following is a three-dimensional harmonic potential

$$V = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) . \quad (10.1)$$

In most cases, we restrict ourselves to a spherically symmetric potential, i.e.  $\omega_x = \omega_y = \omega_z$ .

The starting point is the action

$$S = \int dt d^3x \left[ i\psi^\dagger(\mathbf{x}, t) \partial_t \psi(\mathbf{x}, t) + (\mu - V(\mathbf{x}))\psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) - \frac{1}{2m} \nabla \psi^\dagger(\mathbf{x}, t) \cdot \nabla \psi(\mathbf{x}, t) - \frac{1}{2}g \left( \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}, t) \right)^2 \right] . \quad (10.2)$$

The new term  $V(\mathbf{x})\psi^\dagger\psi$  comes about since  $\psi^\dagger\psi$  is the density operator and  $V(\mathbf{x})$  is the potential energy of a particle at the position  $\mathbf{x}$ . Hence  $V(\mathbf{x})\psi^\dagger\psi$  is the potential energy density due to the trapping potential.

Mean-field theory is obtained by replacing  $\psi = \phi_0 + \tilde{\psi}$  by  $\phi_0$  and ignoring quantum fluctuations <sup>1</sup>. The classical equation of motion for the field  $\phi_0$  is

$$i\frac{\partial\phi_0}{\partial t} = -\frac{1}{2m}\nabla^2\phi_0 - [\mu - V(\mathbf{x})]\phi_0 + g\rho\phi_0, \quad (10.3)$$

where we have used that  $\rho = \phi_0^2$  in the mean-field approximation. Note that  $[\mu - V(\mathbf{x})]$  acts as a local space-dependent effective chemical potential. This equation is called the *time-dependent Gross-Pitaevskii equation*.

The ground state of the trap is independent of time and satisfies Eq. (10.3) with a vanishing left-hand-side:

$$0 = -\frac{1}{2m}\nabla^2\phi_0 - [\mu - V(\mathbf{x})]\phi_0 + g\rho\phi_0. \quad (10.4)$$

From this equation, it is clear that the ground state is no longer homogeneous in space. In fact, turning off the interactions, we know that the single-particle ground state is a Gaussian (the ground state of a single oscillator). In the many-particle ground state all particles reside in the single-particle ground state and the many-body wavefunction is a product of  $N$  Gaussians.

Assume that the spatial extent of the cloud is  $r_0$ . In the absence of interactions, the kinetic term gives a contribution to the energy per particle  $\sim 1/2mr_0^2$  since a typical momentum is  $p \sim 1/r_0$  which follows from the uncertainty principle. Similarly, the potential term gives a contribution to the energy per particle on the order  $m\omega_0^2 r_0^2/2$ . Balancing these contributions, i.e. minimizing the total energy, gives  $r_0 \sim 1/\sqrt{m\omega_0}$ , which is the length scale  $l$  of the harmonic oscillator. Now consider interactions. If there are  $N$  particles in the trap, the density is  $\rho \sim N/r_0^3$  and the interaction energy per particle is on the order  $g\rho \sim gN/r_0^3$ . The interaction term thus shifts the radius of the cloud towards larger values <sup>2</sup>.

If we assume that interactions are dominant <sup>3</sup>, we can ignore the differential operator  $\nabla^2/2m$  in Eq. (10.4). This approximation is called the *Thomas-Fermi approximation*. We can then find an explicit expression for the density by solving Eq. (10.4) ignoring the first term. For a spherically symmetric trapping potential  $V = m\omega_0^2 r^2/2$ , this yields

$$\rho = \begin{cases} \frac{1}{g}[\mu - V] & , \quad r < r_0 \\ 0 & , \quad r > r_0 \end{cases}, \quad (10.5)$$

where  $r_0$  satisfies

$$\frac{1}{2}m\omega_0^2 r_0^2 = \mu. \quad (10.6)$$

---

<sup>1</sup>In a trap  $\phi_0$  does depend on  $\mathbf{x}$ .

<sup>2</sup>This is correct if  $g > 0$ , i.e. if interactions are repulsive. We assume that  $g > 0$  in the following.

<sup>3</sup>Thus we are considering the strong-coupling limit which is obtained by letting  $gN \rightarrow \infty$ .

The solution is thus an upside-down parabola. Given the trapping potential  $V(\mathbf{x})$ , one can determine the chemical potential  $\mu$  by using the normalization condition

$$\int d^3x \rho = N, \quad (10.7)$$

where  $N$  is the total number of particles in the trap. Integrating over angles, this yields

$$\int_0^{r_0} [\mu - \frac{1}{2}m\omega_0^2 r^2] r^2 dr = \frac{gN}{4\pi}. \quad (10.8)$$

Integrating this expression, we obtain

$$\begin{aligned} \frac{gN}{4\pi} &= \frac{1}{3}\mu r_0^3 - \frac{1}{10}m\omega_0^2 r_0^5 \\ &= \frac{1}{15}m\omega_0^2 r_0^5, \end{aligned} \quad (10.9)$$

where we have substituted  $\mu = m\omega^2 r_0^2/2$ . Solving with respect to  $r_0$ , we find

$$r_0 = \left( \frac{15Na}{l} \right)^{1/5} l, \quad (10.10)$$

where  $l = 1/\sqrt{m\omega_0}$  is the length scale of the oscillator. The chemical potential then becomes

$$\begin{aligned} \mu &= \frac{1}{2}m\omega_0^2 r_0^2 \\ &= \frac{1}{2} \left( \frac{15Na}{l} \right)^{2/5} \omega_0. \end{aligned} \quad (10.11)$$

In Fig. 10.2, we show the density distribution of 80000 sodium atoms in the trap of Hau *et al.* (1998) as a function of the axial co-ordinate. The experimental points correspond to the measured optical density, which is proportional to the column density of the atom cloud along the path of the light beam. The dashed line is the noninteracting ground state, i.e. a Gaussian profile. The solid line is the up-side down parabola, i.e. the Thomas-Fermi approximation. The agreement with the data points is impressive. The figure points out the role of atom-atom interaction in reducing the central density and enlarging the size of the cloud.

## 10.1 Hydrodynamics of BEC

In this section, we briefly discuss the excitations around the ground state for a trapped Bose gas.

The equation of motion for the quantum field  $\psi$  is

$$i\hbar \dot{\psi} = \left[ -\frac{1}{2m} \nabla^2 + V - \mu \right] \psi + g(\psi^\dagger \psi) \psi. \quad (10.12)$$

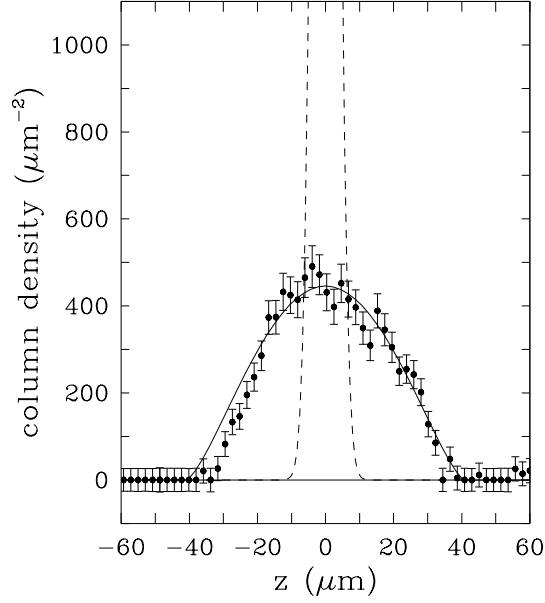


Figure 10.2: Density profile in the  $z$ -direction. See main text for details.

The quantum field is written as the sum of a classical time-dependent field  $\phi_0$  and a quantum fluctuating field  $\tilde{\psi}$  as

$$\psi = \phi_0 + \tilde{\psi} . \quad (10.13)$$

The expectation values of density  $\rho$  and current density  $\mathbf{j}$  are

$$\begin{aligned} \rho &= \langle \psi^\dagger \psi \rangle \\ &= |\phi_0|^2 + \langle \tilde{\psi}^\dagger \tilde{\psi} \rangle , \end{aligned} \quad (10.14)$$

$$\begin{aligned} \mathbf{j} &= -\frac{i}{2m} \langle \psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger \rangle \\ &= -\frac{i}{2m} (\phi_0^* \nabla \phi_0 - \phi_0 \nabla \phi_0^*) - \frac{i}{2m} \langle \tilde{\psi}^\dagger \nabla \tilde{\psi} - \tilde{\psi} \nabla \tilde{\psi}^\dagger \rangle . \end{aligned} \quad (10.15)$$

Taking the expectation value of the equation of motion (10.12) , we obtain

$$i\dot{\phi}_0 = \left[ -\frac{1}{2m} \nabla^2 + V - \mu \right] \phi_0 + g \left[ |\phi_0|^2 \phi_0 + 2\langle \tilde{\psi}^\dagger \tilde{\psi} \rangle \phi_0 + \langle \tilde{\psi} \tilde{\psi} \rangle \phi_0^* + \langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle \right] , \quad (10.16)$$

where we have used that  $\langle \tilde{\psi}^\dagger \rangle = \langle \tilde{\psi} \rangle = 0$ . In the mean-field approximation, the matrix elements are neglected and the equation reduces to the time-dependent Gross-Pitaevskii equation:

$$i\dot{\phi}_0 = \left[ -\frac{1}{2m} \nabla^2 + V - \mu \right] \phi_0 + g |\phi_0|^2 \phi_0 . \quad (10.17)$$

Multiplying Eq. (10.17) by  $\phi_0^*$  and adding its complex conjugate i.e. adding  $\dot{\phi}_0^* \phi_0$ , we obtain

$$i \frac{\partial |\phi_0|^2}{\partial t} + \nabla \cdot \left[ \frac{1}{2m} (\phi_0^* \nabla \phi_0 - \phi_0 \nabla \phi_0^*) \right] = 0 .$$

Since  $\rho = |\phi_0|^2$  in the mean-field approximation, Eq. (10.18) is simply the continuity equation  $\dot{\rho} + \nabla \cdot \mathbf{j}$ , where the current density  $\mathbf{j}$  is given by its mean-field approximation, i.e. by Eq. (10.15) neglecting the matrix elements. If we define the velocity  $\mathbf{v}$  by  $\mathbf{j}/\rho$ , we can write

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 . \quad (10.18)$$

We next write the field  $\phi_0$  in terms of its magnitude  $\sqrt{\rho}$  and its phase  $\phi$  as  $\phi_0 = \sqrt{\rho} e^{i\phi}$ <sup>4</sup>. Inserting this into the expression for  $\mathbf{j}$ , we obtain

$$\begin{aligned} \mathbf{j} &= -\frac{i}{2m} [\rho e^{-i\phi} \nabla e^{i\phi} - \rho e^{i\phi} \nabla e^{-i\phi}] \\ &= \frac{\rho}{m} \nabla \phi . \end{aligned} \quad (10.19)$$

Comparing this with  $\mathbf{j} = \rho \mathbf{v}$ , we find

$$\mathbf{v} = \frac{1}{m} \nabla \phi . \quad (10.20)$$

Inserting  $\phi_0 = \sqrt{\rho} e^{i\phi}$  into the equation of motion and taking the real part of the resulting expression, one finds

$$-\frac{\partial \phi}{\partial t} = -\frac{1}{2m\sqrt{\rho}} \nabla^2 \sqrt{\rho} + \frac{1}{2} m v^2 + g\rho + V - \mu . \quad (10.21)$$

Taking the gradient of this equation and the time derivative of  $\nabla \phi = m\mathbf{v}$  and equating the result we find an expression for  $m\partial\mathbf{v}/\partial t$ .

$$m \frac{\partial \mathbf{v}}{\partial t} = -\nabla \left( \tilde{\mu} + \frac{1}{2} m v^2 \right) , \quad (10.22)$$

where

$$\tilde{\mu} = V + g\rho - \mu - \frac{1}{2m\sqrt{\rho}} \nabla^2 \sqrt{\rho} . \quad (10.23)$$

Eqs. (10.18) and (10.22) constitute the *hydrodynamic equations* for the problem since they involve the hydrodynamic variables density  $\rho$  and velocity  $\mathbf{v}$ .

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<sup>4</sup>This is called the polar representation of the field for obvious reasons. This is in contrast to the cartesian approximation used earlier.

We are now going to study oscillations on top of the ground state in the Thomas-Fermi limit. We can then neglect the last term in Eq. (10.23) to a first approximation. Taking the time derivative of the continuity equation, we obtain

$$\begin{aligned}\ddot{\rho} + \nabla \cdot (\rho \dot{\mathbf{v}} + \dot{\rho} \mathbf{v}) &\approx \ddot{\rho} + \nabla \cdot (\rho \dot{\mathbf{v}}) \\ &= 0 ,\end{aligned}\tag{10.24}$$

where we neglect the term  $\dot{\rho} \mathbf{v}$  since it is higher order in derivatives. Inserting the expression for  $\mathbf{v}$  into Eq. (10.24), we find

$$\ddot{\rho} = \frac{1}{m} \nabla \cdot [\rho \nabla (V - \mu + g\rho)] ,\tag{10.25}$$

where we also neglect the term  $mv^2/2$  treating  $\mathbf{v}$  as a small quantity. The density is now written as a sum of the ground-state density and a time-dependent fluctuation:

$$\rho = \rho_0 + \delta\rho .\tag{10.26}$$

Inserting this into Eq. (10.25), we obtain to first order in the perturbation  $\delta\rho$

$$\ddot{\delta\rho} = \frac{g}{m} \nabla \cdot [\rho_0 \nabla \delta\rho] ,\tag{10.27}$$

where we have used that  $V - \mu = \rho_0$ . Assuming the confining potential is spherically symmetric and the time dependence of  $\delta\rho \propto e^{-i\omega t}$ , this can be written as

$$\omega^2 \delta\rho = \omega_0^2 r \frac{\partial \delta\rho}{\partial r} - \frac{1}{2} \omega_0^2 (r_0^2 - r^2) \nabla^2 \delta\rho ,\tag{10.28}$$

where  $\omega_0$  denotes the oscillator frequency and where we have used that  $\rho_0 = [\mu - V]/g$ . The spherical symmetry implies that the solutions can be written as

$$\delta\rho = D(r) Y_{lm}(\theta, \phi) .\tag{10.29}$$

A simple class of solutions is

$$\delta\rho = C r^l Y_{lm}(\theta, \phi) ,\tag{10.30}$$

where  $C$  is a constant. The radial part of second term on the right-hand-side of Eq. (10.28) is

$$\left[ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right] r^l = 0 .\tag{10.31}$$

This implies that

$$\omega^2 = \omega_0^2 l .\tag{10.32}$$

The solution  $l = 0$  corresponds to a constant change of  $\rho$  everywhere and therefore a constant change in the chemical potential  $\mu$ . The frequency is of course zero in this case. The solutions  $l = 1$  correspond to a translation of the cloud with no change in internal structure. Consider for example  $l = 1$  and  $m = 0$ , i.e.

$$\begin{aligned}\delta\rho &= CrY_{10} \\ &\propto z .\end{aligned}\tag{10.33}$$

The ground-state density  $\rho_0$  of the cloud goes like  $(1 - r^2/r_0^2)$  and so a small translation in the  $z$ -direction is therefore proportional to  $\delta\rho \propto \frac{\partial\rho_0}{\partial z} \propto z$ . Higher-order solutions are more complicated.

# Appendix A

## Sum vs integral

If we use a large quantization volume  $V = L^3$  and periodic boundary conditions, the solutions to the KG-equations are

$$\phi_{\mathbf{k}}(\mathbf{x}, t) \sim e^{-ikx}, \quad (\text{A.1})$$

where

$$k_x = \frac{2\pi}{L}n, \quad (\text{A.2})$$

where  $n = 0, 1, 2, \dots$  with similar expressions for  $k_y$  and  $k_z$ . The normalized solutions are

$$\phi_{\mathbf{k}}(\mathbf{x}, t) = \frac{1}{\sqrt{V}}e^{ikx}. \quad (\text{A.3})$$

The inner product is

$$\begin{aligned} I &= \int d^3x \phi_{\mathbf{k}}^*(\mathbf{x}, t) \phi_{\mathbf{p}}(\mathbf{x}, t) \\ &= \frac{1}{V} e^{i(k_0 - p_0)t} \int e^{-i(\mathbf{k} - \mathbf{p}) \cdot \mathbf{x}} d^3x. \end{aligned} \quad (\text{A.4})$$

If  $\mathbf{p} = \mathbf{k}$ , clearly the integral gives a factor of  $V$ . If  $\mathbf{p} \neq \mathbf{k}$ , the integral vanishes due to the periodic boundary conditions. This yields

$$\begin{aligned} I &= e^{i(k_0 - p_0)t} \delta_{\mathbf{p}, \mathbf{k}} \\ &= \delta_{\mathbf{p}, \mathbf{k}}, \end{aligned} \quad (\text{A.5})$$

where we have used  $k_0 = p_0$  whenever  $\mathbf{p} = \mathbf{k}$ . The integral is now being replaced by a sum using the prescription

$$\int \frac{d^3p}{(2\pi)^3} \rightarrow \frac{1}{V} \sum_{\mathbf{p}}. \quad (\text{A.6})$$

Let us check that this is consistent. The equal-time commutator can be written as

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= \frac{1}{V} \sum_{\mathbf{p}, \mathbf{q}} \left( -\frac{i}{2} \right) \left[ [a_{\mathbf{p}}, a_{\mathbf{q}}] e^{-i(px+qy)} - [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] e^{i(px+qy)} \right. \\ &\quad \left. + [a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}] e^{i(px-xy)} - [a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] e^{-i(px-xy)} \right]. \end{aligned} \quad (\text{A.7})$$

We next *assume* the commutation relations are as for a single oscillator, i.e.

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = \delta_{\mathbf{p}, \mathbf{q}} \quad (\text{A.8})$$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}] = 0, \quad (\text{A.9})$$

$$[a_{\mathbf{p}}^\dagger, a_{\mathbf{q}}^\dagger] = 0. \quad (\text{A.10})$$

This yields

$$\begin{aligned} [\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] &= \frac{i}{V} \sum_{\mathbf{p}, \mathbf{q}} \sqrt{\frac{E_{\mathbf{q}}}{E_{\mathbf{p}}}} \delta_{\mathbf{p}, \mathbf{q}} e^{-i(px-xy)} \\ &= \frac{i}{V} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{y})}. \end{aligned} \quad (\text{A.11})$$

Eq. (A.11) is simply the representation of  $\delta^3(\mathbf{x} - \mathbf{y})$  for discrete values of  $\mathbf{p}$ . Let us check this. We define

$$f(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} \quad (\text{A.12})$$

Integrating over  $\mathbf{x}$ , yields

$$\begin{aligned} \int d^3x f(\mathbf{x}) &= \frac{1}{V} \int d^3x \sum_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{x}} \\ &= \sum_{\mathbf{p}} \delta_{\mathbf{p}, 0} e^{i\mathbf{p} \cdot \mathbf{x}} \\ &= 1, \end{aligned}$$

where we again have used the periodic boundary conditions in the second line. The text book of Fetter and Walecka is using discrete sums instead of integrals and this little note gives the recipe to switch.

# Appendix B

## Interatomic potentials and scattering processes

So far we have been considering noninteracting systems, with a few exceptions such as a Dirac fermion in an external Coulomb potential. We are now going to discuss interacting systems. An important application is the dilute Bose gas and therefore we need to discuss interatomic potentials. A general discussion on scattering can be found Ref. [9] and a discussion on atomic potentials and the dilute Bose gas can be found in Ref. [13].

The action is

$$S = \int dt \left\{ \int d^3x \psi^\dagger(\mathbf{x}, t) \left[ i\partial_t + \frac{1}{2m} \nabla^2 + \mu \right] \psi(\mathbf{x}, t) - \frac{1}{2} \int d^3x \int d^3x' \psi^\dagger(\mathbf{x}, t) \psi^\dagger(\mathbf{x}', t) V_0(\mathbf{x} - \mathbf{x}') \psi(\mathbf{x}, t) \psi(\mathbf{x}', t) + \dots \right\}. \quad (\text{B.1})$$

Here  $\psi^\dagger(\mathbf{x}, t)$  is a quantum field that creates a boson at the position  $\mathbf{x}$  and time  $t$  and  $\psi(\mathbf{x}, t)$  is a quantum field that annihilates an atom at the position  $\mathbf{x}$  and time  $t$ .  $\mu$  is the chemical potential which can be used to tune the average particle density<sup>1</sup>, and  $V_0(\mathbf{x})$  is the two-body potential. Note that the two-body potential is static, i.e. independent of time. This implies we treat the interactions as being instantaneous. Finally, the ellipsis indicate that we are neglecting three-body potentials etc.

The interatomic potential can be divided into a central part  $V_0^c(x)$  and a remainder. The central part of the potential depends only on the separation of the atoms and their electronic spins. It conserves separately the total orbital and the total electronic spin of the atoms. The noncentral part conserves the total angular momentum but does not separately conserve the total orbital or the total electronic spin of the atoms. An example of a noncentral part of the interaction is the dipole-dipole interaction:

$$V_{\text{dipole-dipole}} = \frac{1}{4\pi\epsilon_0 x^3} [\mathbf{d}_1 \cdot \mathbf{d}_2 - 3(\mathbf{d}_1 \cdot \hat{\mathbf{x}})(\mathbf{d}_2 \cdot \hat{\mathbf{x}})] , \quad (\text{B.2})$$

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<sup>1</sup>Recall from statistical mechanics that the  $\mu$  is introduced in the grand canonical ensemble where the particle number is not fixed.

where  $\mathbf{d}_i$  ( $i = 1, 2$ ) are the dipole moments and  $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ . The central part of the potential consists of a short-range part with range  $x_0$  and a long-range Van der Waals tail. The latter goes as  $1/x^6$  as  $x \rightarrow \infty$ <sup>2</sup>.

A typical two-body potential is shown in Fig. B.1.

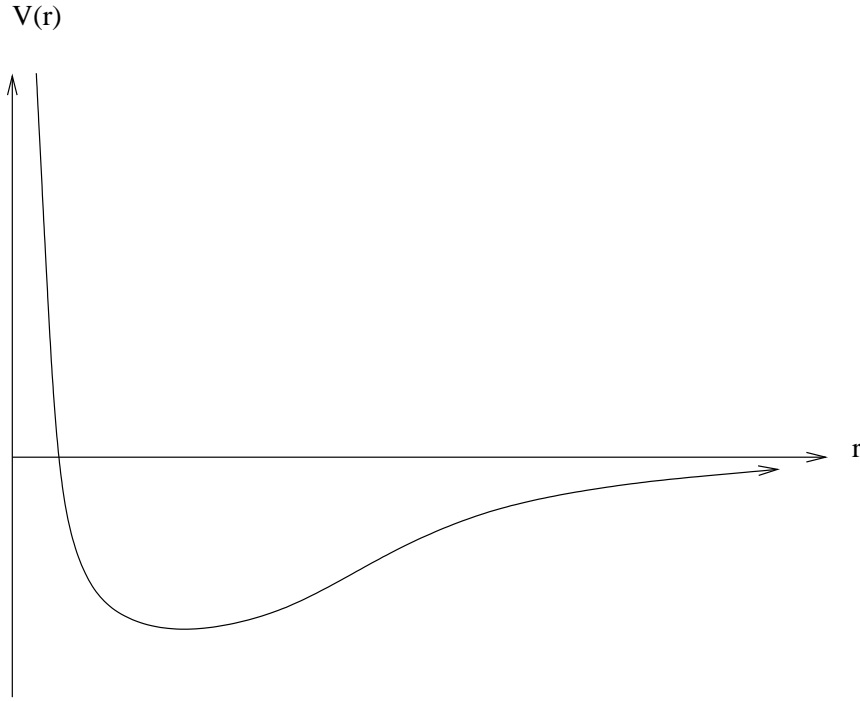


Figure B.1: Typical central part of an interatomic potential.

Now a real interatomic is very difficult to handle in practice. For example, in order to find the bound state of a potential  $V_0(\mathbf{x} - \mathbf{x}')$ , we would have to solve the Schrödinger equation that follows from the action (B.8):

$$\left[ -\frac{1}{2m} \nabla^2 + \int d^3x' \psi^\dagger(\mathbf{x}', t) V_0(\mathbf{x} - \mathbf{x}') \psi(\mathbf{x}', t) \right] \psi(\mathbf{x}, t) = E \psi(\mathbf{x}, t), \quad (\text{B.3})$$

where  $E$  is the energy of the bound state and we have set  $\mu = 0$ . This is an integral equation and is nonlocal<sup>3</sup> and therefore numerically very difficult to solve in practice. It is therefore common to simplify the problem by introducing some model potentials that mimics the physics of a real potential. The model potentials typically have one or more parameters that can be adjusted to reproduce the physics of a real potential. We will return to this point below.

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<sup>2</sup>If retardation effects are taken into account, it goes as  $1/x^7$ .

<sup>3</sup>Nonlocal means that the value of the  $\psi(\mathbf{x}, t)$  depends on the values of not only in  $\mathbf{x}$  but on other values as well.

A model potential of this kind is the sum of a hard-core potential with range  $x_0$  and a van-der-Waals potential:

$$V_0^c(x) = \begin{cases} +\infty, & x < x_0 \\ -\frac{C_6}{x^6}, & x > x_0 \end{cases}, \quad (\text{B.4})$$

where  $C_6$  is a constant. Another example is the hard-core square-well potential

$$V_0^c(x) = \begin{cases} +\infty, & x < x_c \\ -V_0, & x_c < x < x_0 \\ -\frac{C_6}{x^6}, & x_0 < x \end{cases}, \quad (\text{B.5})$$

where  $x_c$  is a constant.

In Fig. B.2, we have shown the Fourier transform  $V(\mathbf{k})$  of a typical short-range potential with range  $x_0$ . Note in particular that the Fourier transform vanishes in the limit  $\mathbf{k} \rightarrow \infty$ .

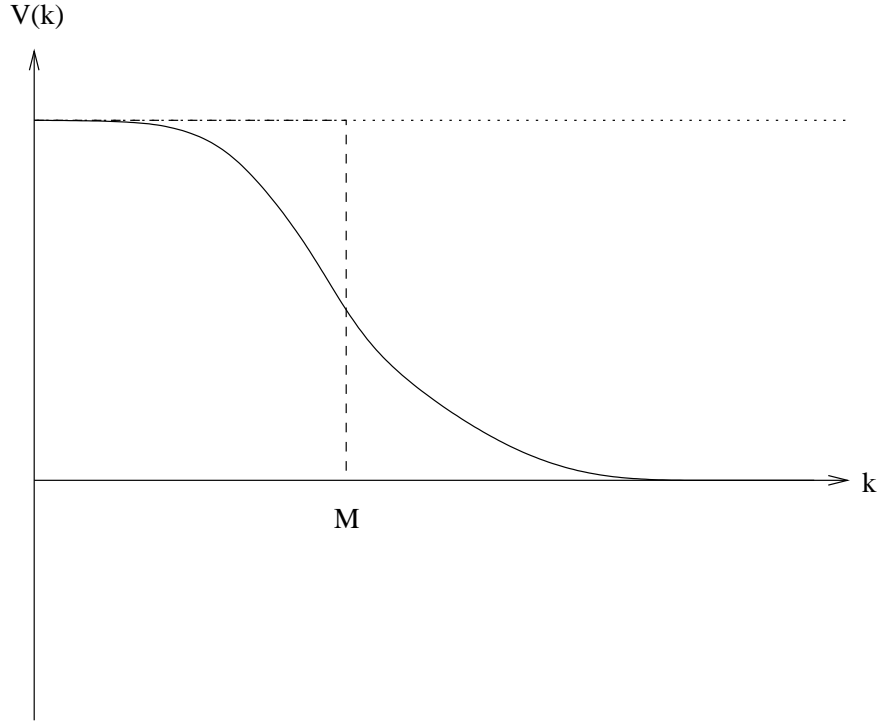


Figure B.2: Fourier transform  $V(\mathbf{k})$  interatomic potential.

The simplest model potential is a modification of the potential (B.5), where we neglect the Van de Waals tail and set  $x_0 = 0$ . The potential is then a contact potential in real space of strength  $g$  and can be written as

$$V_0(\mathbf{x} - \mathbf{x}') = g\delta(\mathbf{x} - \mathbf{x}') . \quad (\text{B.6})$$

The Fourier transform is

$$\begin{aligned} V_0(\mathbf{q}) &= \int d^3x e^{i\mathbf{q}\cdot\mathbf{x}} V_0(\mathbf{x} - \mathbf{x}') \\ &= \int d^3x g \delta(\mathbf{x} - \mathbf{x}') \\ &= g, \end{aligned} \tag{B.7}$$

i.e. it is constant in momentum space. This is shown by the dotted line in Fig. (B.2). The contact potential is singular in coordinate space and is constant in momentum space. This leads to some technical problems with diverging integrals and the need to renormalize the theory<sup>4</sup>. This is related to the fact that  $V(\mathbf{k})$  does not vanish for  $\mathbf{k} \rightarrow \infty$ .

Inserting the potential (B.6) into the action, neglecting three-body potentials, and integrating over  $\mathbf{x}'$ , we obtain

$$S = \int dt \left\{ \int d^3x \psi^*(\mathbf{x}, t) \left[ i\partial_t + \frac{1}{2m} \nabla^2 + \mu \right] \psi(\mathbf{x}, t) - \frac{1}{2} g \left( \psi^\dagger(\mathbf{x}, t) \psi(\mathbf{x}, t) \right)^2 \right\}.$$

This action will be the starting point for our treatment of the weakly interacting Bose gas.

## B.1 Scattering and the Born approximation

In this section, we are very briefly going to discuss scattering of an atom off of a time-independent potential. This is equivalent to two-particle scattering processes where the particles interact via a two-body potential  $V_0(\mathbf{x})$ . The reduced mass of the particle is  $m^* = m/2$ . For a detailed account on scattering, consult Hemmer or any other standard textbook on quantum mechanics.

The scattering process is described by the time-independent Schrödinger equation:

$$\left[ -\frac{1}{2m^*} \nabla^2 + V_0(\mathbf{x}) \right] \psi(\mathbf{x}) = E \psi(\mathbf{x}), \tag{B.8}$$

where  $E$  is the energy of the particle. The boundary conditions for this problem must be such that it describes an incoming ray of particles and outgoing scattered particles. For large  $|\mathbf{x}|$ , the wavefunction is therefore a sum of an incoming plane wave and an outgoing spherical wave

$$\psi \sim e^{i\mathbf{k}\cdot\mathbf{x}} + f(\theta, \phi) \frac{e^{ikr}}{r}, \tag{B.9}$$

where  $f(\theta, \phi)$  is the scattering amplitude.

We next introduce  $E = k^2/2m^*$  and  $U(\mathbf{x}) = 2m^*V_0(\mathbf{x})$ . This yields

$$\left[ \nabla^2 + k^2 \right] \psi(\mathbf{x}) = U(\mathbf{x}) \psi(\mathbf{x}). \tag{B.10}$$

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<sup>4</sup>We have already seen that the ground-state energy of a quantum field theory is divergent, but we neglected the divergence. In interacting field theories, more divergences appear and we tame them by redefining the coupling constant  $g$ , as we shall see.

The differential equation (B.10) is equivalent to the integral equation

$$\psi(\mathbf{x}) = \psi_0(\mathbf{x}) + \int d^3x' G^0(\mathbf{x} - \mathbf{x}') U(\mathbf{x}') \psi(\mathbf{x}') , \quad (\text{B.11})$$

where  $\psi_0(\mathbf{x})$  is a solution to the homogeneous equation  $(\nabla^2 + k^2)\psi_0(\mathbf{x}) = 0$  and the free-particle Green's functions  $G^0(\mathbf{x} - \mathbf{x}')$  satisfies

$$[\nabla^2 + k^2] G^0(\mathbf{x} - \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') . \quad (\text{B.12})$$

The equivalence of these equations is easy to check by acting on equation (B.10) with  $\nabla^2 + k^2$ , using that  $(\nabla^2 + k^2)\psi_0(\mathbf{x}) = 0$  and that  $G^0(\mathbf{x} - \mathbf{x}')$  satisfies Eq. (B.12). The solution to Eq. (B.12) is given by

$$G^0(\mathbf{x} - \mathbf{x}') = -\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|} , \quad (\text{B.13})$$

and in momentum space the free propagator is

$$G^0(p) = \frac{1}{k^2 - p^2 + i\eta} , \quad (\text{B.14})$$

Collecting the various pieces, the integral equation (B.11) can be written as

$$\psi(\mathbf{x}) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{4\pi} \int d^3x' \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} U(\mathbf{x}') \psi(\mathbf{x}') . \quad (\text{B.15})$$

For large values of  $\mathbf{x}$ , we can expand the exponent and write

$$k|\mathbf{x} - \mathbf{x}'| \approx kx - \mathbf{k}' \cdot \mathbf{x}' , \quad (\text{B.16})$$

where  $\mathbf{k}' \equiv k\mathbf{x}/x$ . This implies that the asymptotic wavefunction can be written as

$$\psi(\mathbf{x}) \approx e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{1}{4\pi} \frac{e^{ikx}}{x} \int d^3x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} U(\mathbf{x}') \psi(\mathbf{x}') . \quad (\text{B.17})$$

The scattering amplitude is therefore

$$f(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi} \int d^3x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} U(\mathbf{x}') \psi(\mathbf{x}') . \quad (\text{B.18})$$

The Fourier transform of  $G^0(\mathbf{x} - \mathbf{x}')$ ,  $\psi(\mathbf{x})$ , and  $U(\mathbf{x})$  are given implicitly by

$$G^0(\mathbf{x} - \mathbf{x}') = \int \frac{d^3p}{(2\pi)^3} e^{-i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})} \frac{1}{k^2 - p^2 + i\eta} , \quad (\text{B.19})$$

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \psi(\mathbf{p}) , \quad (\text{B.20})$$

$$U(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} U(\mathbf{p}) , \quad (\text{B.21})$$

By going to momentum space, Eqs. (B.11) and (B.18) can be written as

$$\begin{aligned}\psi(\mathbf{p}) &= (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) + \frac{1}{k^2 - p^2 + i\eta} \int \frac{d^3q}{(2\pi)^3} U(\mathbf{q}) \psi(\mathbf{p} - \mathbf{q}), \\ f(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi} \int \frac{d^3q}{(2\pi)^3} U(\mathbf{q}) \psi(\mathbf{k}' - \mathbf{q}).\end{aligned}\tag{B.22}$$

By combining these equations, we can write

$$\psi(\mathbf{p}) = (2\pi)^3 \delta(\mathbf{p} - \mathbf{k}) - \frac{4\pi f(\mathbf{p}, \mathbf{k})}{k^2 - p^2 + i\eta}.\tag{B.23}$$

Multiplying Eq. (B.23) by  $-U(\mathbf{k}' - \mathbf{p})$  and integrating with respect to  $\mathbf{p}$ , we obtain the following integral equation for the scattering amplitude

$$4\pi f(\mathbf{k}', \mathbf{k}) = -U(\mathbf{k}' - \mathbf{k}) + 4\pi \int \frac{d^3p}{(2\pi)^3} \frac{U(\mathbf{k}' - \mathbf{p})}{k^2 - p^2 + i\eta} f(\mathbf{p}, \mathbf{k}).\tag{B.24}$$

This equation can be solved by iteration. The first approximation to  $f(\mathbf{k}', \mathbf{k})$  is found by ignoring the second term on the right-hand-side of (B.24). This yields

$$f(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi} U(\mathbf{k}' - \mathbf{k}).\tag{B.25}$$

This approximation is called the first Born approximation.

If we take the limit  $k \rightarrow 0$ , the partial wave method yields

$$f(0, 0) = -a,\tag{B.26}$$

where  $a$  is the scattering length. The scattering length is an observable that one can measure in experiments. The idea of using an effective potential or a model potential is to tune the parameters in the potential such that one reproduces the correct observed scattering length. This procedure is called *matching*. Using the delta-function potential, which in momentum space is  $U(\mathbf{k} - \mathbf{k}') = 2m^*g$ , we find the in the Born approximation

$$f(\mathbf{k}', \mathbf{k}) = -\frac{m^*g}{2\pi},\tag{B.27}$$

and therefore

$$a = \frac{m}{4\pi}g,\tag{B.28}$$

where we have used  $m^* = m/2$ . Using the first-order solution  $f(\mathbf{k}', \mathbf{k}) = -mg/4\pi$  in the integral equation and iterating, we find the second approximation

$$4\pi f(\mathbf{k}', \mathbf{k}) = -mg - mg^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{k^2 - p^2 + i\eta}.\tag{B.29}$$

The scattering length then becomes

$$\begin{aligned}
 a &= \frac{mg}{4\pi} \left[ 1 - mg \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} \right] \\
 &= \frac{mg}{4\pi} \left[ 1 - \frac{mg}{2\pi^2} \int_0^\infty dp \right] .
 \end{aligned} \tag{B.30}$$

We have taken the limit  $\eta \rightarrow 0$ . The integral is divergence and this is because we have replaced the true potential by a singular delta-function potential. Note that true potential always goes to zero in the limit  $\mathbf{p} \rightarrow \infty$ , see Fig. B.2, and so the integral always converges. This is the price we have to pay by using a simple potential. However, we have systematic ways of dealing with these divergences and the procedure is called *renormalization*.

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