Example: Maxwell stress tensor for two charged cylinders

Two cylinders of radius $R$ are positioned as shown in the figure below. Both extend indefinitely along the $z$ axis. We assume that each cylinder has a uniform charge density on the cylinder surface. Let the surface charge density be $\sigma$ ($-\sigma$) on the “+” (“-“) cylinder centered at $x = +a$ ($x = -a$).

The electric field $E_{\pm}$ produced by the ±-cylinder can, due to its high symmetry, most easily be found from Gauss’s law. Picking the Gaussian surface to be a cylinder of length $l$ and radius $s (> R)$ centered at the cylinder, there is no contribution from the two “end faces” (with normal vector $\hat{n} = \pm \hat{z}$) since $E_{\pm \hat{n}} = 0$ by symmetry. Symmetry also dictates that $E_{\pm}$ points radially from each cylinder center and that its magnitude only depends on $r$. Gauss’s law thus gives (where $E_{\pm}$ is the radial component)

$$\oint S E_{\pm} \cdot ds = \frac{Q_{\text{inside}}}{\varepsilon_0} \Rightarrow E_{\pm} 2\pi s l = \frac{\sigma_{\pm} 2\pi R l}{\varepsilon_0}$$
\[ E_{\pm} = \frac{\sigma_{\pm} R}{\epsilon_0} \frac{a}{\sqrt{(x - a)^2 + y^2}} \] (same sign as \( \sigma_{\pm} \))

(valid for \( x > R \))

Let \( \vec{r}_{\pm} = \vec{r} - \vec{r}_{\pm} \), where \( \vec{r}_{\pm} = (\pm a, 0) \) is the position vector of the \( \pm \) cylinder. Then

\[ E_{\pm, x} = E_{\pm} \cos \Theta_{\pm}, \quad E_{\pm, y} = E_{\pm} \sin \Theta_{\pm} \]

where \( \cos \Theta_{\pm} = \frac{\vec{r} - \vec{r}_{\pm}}{|\vec{r} - \vec{r}_{\pm}|} \cdot \hat{x} = \frac{x - a}{(x - a)^2 + y^2} \)

and \( \sin \Theta_{\pm} = \frac{y}{\sqrt{(x - a)^2 + y^2}} \)

\[ E_{\pm, x} = \frac{\sigma_{\pm} R}{\epsilon_0} \frac{x + a}{(x - a)^2 + y^2} \] (valid for regions outside \( \pm \) cylinder, respectively (inside, \( E_{\pm} = 0 \))

\[ E_{\pm, y} = \frac{\sigma_{\pm} R}{\epsilon_0} \frac{y}{(x - a)^2 + y^2} \]

The components of the total electric field \( \vec{E} = E_+ + E_- \) are then

\[ E_x = E_{+, x} + E_{-, x} \]
\[ E_y = E_{+, y} + E_{-, y} \]

The total force on a cylinder can be calculated from

\[ \vec{F} = \oint \vec{T} \cdot d\vec{a} \]

where \( \vec{T} \) is Maxwell's stress tensor for the problem and \( dS \) is a surface.
enclosing the volume \( \Omega \) which includes the cylinder of interest (and none of the other cylinders). Since \( E \) is in the \( xy \) plane, only \( F_x \) and \( F_y \) need to be considered. (In fact, \( F_y = 0 \) by symmetry, which we will also verify explicitly.) So let us consider

\[
F_x = \oint (T_{xx} \, dx + T_{xy} \, dy + T_{xz} \, dz) \, dz
\]

\[
F_y = \oint (T_{yx} \, dx + T_{yy} \, dy + T_{yz} \, dz) \, dz
\]

We have, for this problem,

\[
T_{xx} = -T_{yy} = \frac{\varepsilon_0}{2} (E_x^2 - E_y^2)
\]

\[
T_{xy} = T_{yx} = \varepsilon_0 E_x E_y
\]

\[
T_{yz} = T_{xz} = 0
\]

Vector field plots of \((T_{xx}, T_{xy})\) (relevant for the calculation of \( F_x \)) and \((T_{yx}, T_{yy})\) (relevant for the calculation of \( F_y \)) are shown below for the two cases

\[
\sigma_\pm = \sigma_0 \pm (\text{cylinders with opposite charges}) \quad \text{and} \quad \sigma_\pm = \sigma_0 \pm (\text{same charges})
\]

(Note that only the relative sign of the charges matters for \( T \), not which cylinder has which sign of the charge, since \( T \) is invariant under \( E \to -E \).)
Opposite charges

\[ (T_{xx}, T_{xy}) \]

Same charges

\[ (T_{xx}, T_{xy}) \]

\[ (T_{yx}, T_{yy}) \]
For concreteness let us calculate $F_x$ and $F_y$ on the left ("-"") cylinder. The volume $\Omega$ can be any volume enclosing this cylinder and none of the other. For symmetry reasons a good choice is to pick $\Omega$ to be the region $x < 0$, as then we only need to evaluate $T_{ij}$ on the high-symmetry face $x = 0$ (the other parts of $\Omega$ are "at infinity" and thus give no contribution since $T_{ij} \to 0$ there).

For $F_x$ we get

$$F_x = \int_{x=0}^{\infty} (T_{xx}, T_{xy}) \cdot (dx, dy)$$

plane

$$= (dx, 0) \text{ since } \hat{n} = \hat{e}_x$$

$$= (dx, 0) \text{ since } \hat{n} \text{ points out of the volume } x < 0$$

$$= \int_{-\infty}^{\infty} T_{xx}(x=0, y) \ dy \ L \text{ where } L \text{ is the cylinder length}$$

Since $L = \infty$, $F_x$ obviously diverges. Let us therefore calculate the force per unit length given by

$$\left( \lim_{L \to \infty} \frac{F_x}{L} \right) = \int_{-\infty}^{\infty} dy \ T_{xx}(x=0, y)$$

$$= \frac{\mu_0}{2} \int_{-\infty}^{\infty} dy \left[ E_x^2(x=0, y) - E_y^2(x=0, y) \right]$$
\[ - \frac{e_0}{2} \left( \frac{R^2}{e_0} \right) \int_{-\infty}^{\infty} dy \frac{1}{(a^2 + y^2)^2} \left\{ \left( \sigma_+ (\sigma_+) + \sigma_- (\sigma_-) \right)^2 - \left( \sigma_+ (\sigma_+) + \sigma_- (\sigma_-) \right)^2 \right\} \]

\[ = \frac{R^2}{2e_0} \int_{-\infty}^{\infty} dy \frac{1}{(a^2 + y^2)^2} \left\{ a^2 (\sigma_+ - \sigma_-)^2 - \gamma^2 (\sigma_+ + \sigma_-)^2 \right\} \]

\[ = \frac{R^2}{2e_0} \left\{ \frac{\pi}{2} \frac{1}{a^2} \cdot a^2 (\sigma_+ - \sigma_-)^2 - \frac{\pi}{2a} (\sigma_+ + \sigma_-)^2 \right\} \]

\[ = \frac{\pi R^2}{4e_0 a} \left\{ (\sigma_+ - \sigma_-)^2 - (\sigma_+ + \sigma_-)^2 \right\} \]

\[ \text{For opposite charges } (\sigma_+ = -\sigma_- \equiv \sigma) \text{ this gives} \]

\[ F_x = \frac{\pi R^2 \sigma^2}{4e_0 a} \quad (>0, \text{ so attracted by right cyl.)} \]

\[ \text{while for same charges } (\sigma_+ = \sigma_- \equiv \sigma) \]

\[ F_x = -\frac{\pi R^2 \sigma^2}{4e_0 a} \quad (<0, \text{ so repelled by right cyl.)} \]

Thus as expected, \( F_x \) has the same magnitude in both cases but opposite sign. (This result can be shown to be identical to the force/length experienced between two infinite line charges a distance \( 2a \) apart, if the charge per unit length is the same as for the cylinders, a reasonable result given that the cylinder charge is assumed from the outset to be uniform.)
Finally,

\[ F_y = \int_{x=0}^{x=\infty} T_{yx} \, dx = L \, \varepsilon_0 \int_{-\infty}^{\infty} dy \, E_x(x=0,y) E_y(x=0,y) \]

\( \Rightarrow \quad \frac{F_y}{L} = \varepsilon_0 \left( \frac{\varepsilon}{\varepsilon_0} \right)^2 \int_{-\infty}^{\infty} dy \, \frac{\alpha}{(a^2 + y^2)^2} \left( -\sigma_+ + \sigma_- \right) (\sigma_+ + \sigma_-) \]

\( = 0 \)

We see that \( F_y = 0 \) not only for the two cases of identical and opposite charges, but also more generally due to the integrand being odd in \( y \).