Exam TFY 4210 Quantum theory of many-particle systems, spring 2012

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Examination support:

Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae

The exam has 3 problems, with subproblems (a), (b), ... All subproblems have the same weight. There are 6 pages in total. Some useful formulas are given on the last page.
Problem 1

(a) The Dirac equation reads (with $\hbar = c = 1$)

$$i \frac{\partial \psi}{\partial t} = H \psi \quad \text{where} \quad H = \vec{\alpha} \cdot \vec{p} + \beta m.$$  

Briefly describe why Dirac sought an equation of this form.

(b) It turns out that an equation of this form also arises in the low-energy description of some 1-dimensional condensed matter systems. In the rest of this problem we therefore consider the Dirac equation in 1 spatial dimension. There is then only one $\alpha$ matrix, $\alpha_1$. Use the same kind of reasoning as for the 3-dimensional case to show that in the 1-dimensional case one gets the conditions

$$\alpha_1^2 = \beta^2 = 1, \quad \alpha_1 \beta + \beta \alpha_1 = 0.$$  

(c) A valid representation for $\beta$ and $\alpha_1$ that satisfies these equations is $\beta = \sigma_1$ and $\alpha_1 = \sigma_3$. Using this Pauli matrix representation, show that the eigenvalues of $H$ are given by

$$E = \pm \sqrt{p^2 + m^2}$$  

where $p$ is the momentum eigenvalue.

(d) In terms of $\gamma$ matrices ($\gamma^0 \equiv \beta$ and $\gamma^1 \equiv \beta \alpha_1$) the Dirac equation reads

$$(i \gamma^\mu \partial_\mu - m)\psi = 0.$$  

where $\mu$ runs over 0 and 1. Derive this equation from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(i \gamma^\mu \partial_\mu - m)\psi$$  

where $\bar{\psi} = \psi^\dagger \gamma^0$.

(e) With our chosen representations for $\beta$ and $\alpha_1$, the $\gamma$ matrices become $\gamma^0 = \sigma_1$ and $\gamma^1 = -i\sigma_2$. Consider the matrix $\gamma^5 \equiv \gamma^0 \gamma^1$, which is used to define a chiral transformation as

$$\psi \to e^{i\theta \gamma^5} \psi$$  

where $\theta$ is an angular parameter. Show that under this transformation, $\bar{\psi}$ transforms as

$$\bar{\psi} \to \bar{\psi} e^{i\theta \gamma^5},$$
and show furthermore that the two-component vector
\[
\begin{pmatrix}
\bar{\psi}\psi \\
i\bar{\psi}\gamma^5\psi
\end{pmatrix}
\]
transforms as a rotation,
\[
\begin{pmatrix}
\bar{\psi}\psi \\
i\bar{\psi}\gamma^5\psi
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\bar{\psi}\psi \\
i\bar{\psi}\gamma^5\psi
\end{pmatrix}
\]
where the rotation angle $\phi = \phi(\theta)$. Find the values of $\theta$ that leave this two-component vector invariant. (These results have a natural interpretation in the condensed matter context that we mentioned in the introduction, but we don’t go into that here.)

**Problem 2**

In this problem we consider $\varphi^4$ quantum field theory. Subproblems (a) and (b) are about (position-space) Feynman diagrams for the 2-point function $\langle \Omega | T\{\varphi(x)\varphi(y)\} | \Omega \rangle \equiv D_F(x - y)_{\text{int}}$ in $\varphi^4$ theory. Subproblem (c) involves (momentum-space) Feynman diagrams for the Fourier transform $\tilde{D}_F(p)_{\text{int}}$ of the 2-point function.

(a) Using the Feynman rules for position-space Feynman diagrams, write down the expression for the two Feynman diagrams (i)-(ii) below (you can leave the symmetry factor $S$ unspecified).

(b) After some simplifications, the perturbation expansion for the 2-point function can be written schematically as a sum over Feynman diagrams, i.e. $D_F(x - y)_{\text{int}} = \sum_i A_i$, where $A_i$ represents a Feynman diagram appearing in this expansion. Among the 4 diagrams (i)-(iv) below, at least one of them is not of the valid type $A_i$. Identify the invalid diagram(s), and if a diagram is invalid, briefly state why.
(c) Consider the following approximation for $\tilde{D}(p)_{\text{int}}$:

Using the momentum-space Feynman rules, find an expression for the diagram with $n$ loops in this series. [Hint: It may be helpful to start by finding expressions for the diagrams with 0, 1, and 2 loops, and then if necessary look at diagrams with more loops until you see a pattern. Note that the symmetry factor for the diagram with $n$ loops is $2^n$.] Use this to find an expression for $\tilde{D}_F(p)_{\text{int}}$ in this approximation. (Don’t try to evaluate nontrivial integrals.)
Problem 3

Consider a tight-binding model of noninteracting electrons in a one-dimensional crystal with \( N \) sites and periodic boundary conditions. The Hamiltonian is

\[
H = -t \sum_{j,\sigma} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + \text{h.c.}) + t' \sum_{j,\sigma} (c_{j,\sigma}^\dagger c_{j+2,\sigma} + \text{h.c.}).
\]

Here \( c_{j,\sigma}^\dagger \) (\( c_{j,\sigma} \)) creates (annihilates) an electron with spin projection \( \sigma \) (= \( \pm 1/2 \)) on site \( j \). The first (second) term in \( H \) describes hopping between nearest-neighbour (next-nearest-neighbour) sites. These terms have hopping amplitudes \(-t\) and \( t'\), respectively.

(a) Show that \( H \) can be written on diagonal form as

\[
H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma}
\]

where \( c_{k,\sigma}^\dagger \) (\( c_{k,\sigma} \)) creates (annihilates) an electron with wavevector \( k \) and spin projection \( \sigma \), the \( k \) sum is over the 1st Brillouin zone \([-\pi, \pi]\) and

\[
\varepsilon_k = -2t \cos k + 2t' \cos 2k
\]

(the wavevectors are dimensionless as we have set the lattice spacing to 1).

From now on, assume that \( t \) is positive and that the system is half-filled, i.e. the number of electrons \( N_e \) equals the number of sites \( N \). We will consider the ground state of the Hamiltonian for different nonnegative values of \( t' \). To be precise we define here a Fermi wavevector of a one-dimensional system as a wavevector that separates a region of occupied wavevectors from a region of unoccupied wavevectors in the ground state of the system.

(b) First consider the case \( t' = 0 \). Sketch \( \varepsilon_k \). What are the values of the Fermi wavevectors and the occupied wavevectors?

(c) Next consider \( t' \) to be positive and define the ratio \( r = t'/t \) (\( > 0 \)). Show that there is a critical value \( r_c \) such that for \( r < r_c \) the system has two Fermi wavevectors while for \( r > r_c \) the system has four Fermi wavevectors. Derive the value of \( r_c \) and find the Fermi energy at \( r = r_c \).
Formulas

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

\[ \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{i,j} \quad (i, j = 1, 2, 3) \]

\[ \hat{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \]

\[ \frac{1}{N} \sum_j e^{i(k-k')j} = \delta_{k,k'} \]