Exercises for week 6

In these exercises we will use classical field theory to study some properties of the Dirac equation. Thus we must consider a Lagrangian density that gives the Dirac equation

\[(i\gamma^\mu \partial_\mu - m)\psi = 0\]  \hfill (1)

as the equation of motion. Because the \(\gamma\)-matrices are \(4 \times 4\) matrices, the field \(\psi\) has 4 components:

\[
\psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4
\end{pmatrix}.
\]  \hfill (2)

Thus

\[
\psi^\dagger = \begin{pmatrix}
\psi_1^* \\
\psi_2^* \\
\psi_3^* \\
\psi_4^*
\end{pmatrix}.
\]  \hfill (3)

Now define

\[
\bar{\psi} = \psi^\dagger \gamma^0.
\]  \hfill (4)

The Lagrangian density ("Dirac Lagrangian") can then be written

\[
\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi.
\]  \hfill (5)

Exercise 1

(a) Treating \(\psi\) and \(\bar{\psi}\) as if they were independent fields, show that the Euler-Lagrange equation for \(\bar{\psi}\) gives the Dirac equation (1).

(b) Find the conjugate momentum densities \(\pi_\psi\) and \(\pi_{\bar{\psi}}\) and use these to construct the Hamiltonian density \(\mathcal{H}\). Show that \(\mathcal{H}\) can be written

\[
\mathcal{H} = \psi^\dagger (i\alpha \cdot \nabla + \beta m) \psi
\]  \hfill (6)

where the \(\alpha\) and \(\beta\) matrices were defined in the lectures.

(c) Show that \(\mathcal{L}\) is invariant under the global phase transformation

\[
\psi \to \psi e^{-i\lambda},
\]

\[
\bar{\psi} \to \bar{\psi} e^{i\lambda},
\]  \hfill (7)

and identify the conserved current \(j^\mu\).

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\(^1\)Thus the Dirac equation is a matrix equation. However, it will not be necessary to write the equation in component form.
Exercise 2

Consider the following global transformation, known as a *chiral* transformation:

\[ \psi \rightarrow e^{i\gamma^5 \lambda} \psi, \]  
\[ \bar{\psi} \rightarrow \bar{\psi} e^{-i(\gamma^5)^\dagger \lambda}, \]  

(note that both exponents have the same sign) where \( \gamma^5 \) is a matrix defined as

\[ \gamma^5 = i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \]  

(a) Show that

\[ (\gamma^5)^\dagger = \gamma^5, \]  
\[ (\gamma^5)^2 = 1, \]  
\[ \{\gamma^5, \gamma^\mu\} = 0. \]  

(In the last expression we have used the standard definition \( \{A, B\} \equiv AB + BA \), called the *anticommutator* of \( A \) and \( B \).)

(b) Show that

\[ \gamma^\mu e^{i\gamma^5 \lambda} = e^{-i\gamma^5 \lambda} \gamma^\mu. \]  

(c) For what value(s) of \( m \) is the chiral transformation a symmetry of the Dirac Lagrangian (5)? Find the corresponding conserved current \( j^\mu \).