Exercises for week 13

Exercise 1

(a) Consider an arbitrary Hermitian single-particle operator $\hat{O}$. In the "1st quantization" formalism, $\hat{O}$ takes the form

$$\hat{O} = \sum_{i=1}^{N} \hat{o}_i$$

(1)

where $N$ is the number of particles in the system. Let $\phi_\nu(x)$ and $\epsilon_\nu$ be the eigenfunctions and associated eigenvalues of $\hat{o}$, i.e.

$$\hat{o} \phi_\nu(x) = \epsilon_\nu \phi_\nu(x).$$

(2)

Here $\nu$ is a set of quantum numbers which completely characterize the single-particle eigenfunctions. As discussed in the lectures, we can use these to construct a basis set for many-particle wavefunctions. The (normalized) basis functions can be written

$$\Phi_{\nu_1,\nu_2,\ldots,\nu_N}(x_1, x_2, \ldots, x_N) = \frac{1}{\sqrt{N!} \prod_{\nu} n_\nu!} \sum_{P \in S_N} \xi^{t_P} \cdot P \phi_{\nu_1}(x_1) \phi_{\nu_2}(x_2) \ldots \phi_{\nu_N}(x_N)$$

(3)

where $\xi = \pm 1$ for bosons/fermions and $t_P$ is the number of transpositions (2-particle permutations) associated with the permutation $P$. $S_N$ is the set of all $N!$ permutations. Furthermore, $n_\nu$ is the number of particles in the single-particle state $\phi_\nu$ in the many-particle state $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ (for fermions this can only be 0 or 1, hence $\sqrt{\prod_{\nu} n_\nu!} = 1$ in the fermionic case and can therefore be omitted).

(i) Write down an example of a basis function for a system of 3 fermions where all single-particle states $\nu_1, \nu_2, \nu_3$ are different (write the state out explicitly, i.e. all $3! = 6$ terms).

(ii) Demonstrate that the function changes sign if the coordinates of two particles are interchanged, e.g. $x_1$ and $x_2$. Furthermore, demonstrate that if two or more of the single-particle states are chosen to be the same, the basis function vanishes. These properties are in accordance with the Pauli exclusion principle.

(iii) Show that $\Phi_{\nu_1,\nu_2,\ldots,\nu_N}$ as given in (3) is an eigenfunction of $\hat{O}$ with eigenvalue $\sum_\nu o_\nu n_\nu$.

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As discussed in the lectures, $t_P$ is not unique, but its evenness/oddness is, so that the sign $\xi^{t_P}$ is well-defined.
Exercise 2

In the "2nd quantization" formalism, many-particle basis states take the form

\[ |n_1, n_2, n_3, \ldots, n_N \rangle = (a_1^{\dagger})^{n_1}(a_2^{\dagger})^{n_2}(a_3^{\dagger})^{n_3} \ldots |0\rangle, \tag{4} \]

where \( |0\rangle \) is the "vacuum" state with no particles, and \( a_\nu^{\dagger} \) creates a particle in the single-particle state with wavefunction \( \phi_\nu(x) \).\(^2\)

(i) Write down the basis state in 2nd quantization that corresponds to the 1st quantization basis state discussed in Exercise 1.(i).

(ii) An explicit connection between the basis states \( |n_1, n_2, \ldots, n_N \rangle \equiv |n\rangle \) in 2nd quantization and the basis functions \( \Phi_n(x_1, x_2, \ldots, x_N) \) in 1st quantization can be established. Let

\[ |x_1, x_2, \ldots, x_N \rangle \equiv \frac{1}{\sqrt{N!}} \psi^{\dagger}(x_1)\psi^{\dagger}(x_2)\ldots\psi^{\dagger}(x_N)|0\rangle. \tag{5} \]

Then the (correctly normalized) basis wavefunctions are given by

\[ \Phi_n(x_1, \ldots, x_N) = \langle x_1, \ldots, x_N | n \rangle. \tag{6} \]

As an example, take \( |n\rangle \) to be the 3-particle fermionic state considered in Exercise 2.(i). Calculate the rhs of Eq. (6) to show that the wavefunction is indeed that considered in Exercise 1.(i).

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\(^2\)In Eq. (4) we have for simplicity just replaced the quantum numbers \( \nu \) labeling a single-particle state by an integer label, i.e. the different states in the single-particle basis are labeled 1, 2, 3, \ldots.