Exercises for week 16

Exercise 1

Consider the following Hamiltonian describing electrons hopping between nearest-neighbour sites on a two-dimensional square lattice:

\[ H = -t \sum_{\langle i,j \rangle, \sigma} (c_{j,\sigma}^\dagger c_{i,\sigma} + \text{h.c.}). \] (1)

Here \( t > 0 \) is the hopping amplitude, \( i \) and \( j \) are labels for the sites of the square lattice, and \( \sigma = \pm 1/2 \) labels the electron spin projection. The leftmost sum is over all pairs of nearest-neighbour sites (each such pair being counted once).

(a) Show that the Hamiltonian can be written on the diagonalized form

\[ H = \sum_{k,\sigma} \varepsilon_k c_{k,\sigma}^\dagger c_{k,\sigma}, \] (2)

where the sum over \( k \) runs over the 1st Brillouin zone of the square lattice. Give the dispersion relation \( \varepsilon_k \).

(b) Consider the density parameter \( n = N_e/N \), where \( N_e \) is the number of electrons in the system and \( N \) is the number of sites. Show that in the ground state of the system (for a given number \( N_e \) of electrons), \( n \) is proportional to the \( k \)-space area enclosed by the Fermi surface and find the proportionality constant.

(c) Sketch the Fermi surface for (i) \( n \ll 1 \), (ii) \( n = 1 \), and (iii) \( n = 2 \).

Exercise 2

(a) Show that an alternative and equivalent form of the spin commutation relations

\[ [S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \] (3)

(where we have set \( \hbar = 1 \)) is given by

\[ [S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm, \] (4)

where \( S^\pm = S^x \pm iS^y \) are the spin raising and lowering operators.
(b) The Holstein-Primakoff (HP) representation is given by

\[ S^+ = \sqrt{2S - \hat{n}} a, \]
\[ S^- = a^\dagger \sqrt{2S - \hat{n}}, \]
\[ S^z = S - \hat{n}, \]

where \( a \) and \( a^\dagger \) are canonical boson operators, and \( \hat{n} = a^\dagger a \). Show that the HP representation satisfies the correct spin commutation relations and the relation \( S \cdot S = S(S + 1) \).