Solution to week 7 exercise

(a) We use the Pauli matrix identity
\[ \sigma_i \sigma_j = \delta_{ij} + i \epsilon_{ijk} \sigma_k \] (1)
where \( \epsilon_{ijk} \) is the totally antisymmetric Levi-Civita tensor with \( \epsilon_{123} = +1 \) and there is a sum over \( k = 1, 2, 3 \) in the last term. This gives
\[ \{ \sigma_i, \sigma_j \} = 2 \delta_{ij} + i (\epsilon_{ijk} + \epsilon_{jik}) \sigma_k = 2 \delta_{ij}. \] (2)

From this one easily sees that
\[ \{ \gamma^\mu, \gamma^\nu \} = \begin{cases} 2 & \text{if } \mu = \nu = 0 \\ -2 & \text{if } \mu = \nu = 1, 2 \\ 0 & \text{if } \mu \neq \nu \end{cases} = 2 \eta^{\mu \nu}. \] (3)

(b) The electric field vanishes for all three configurations since \( A^0 = 0 \) and \( \partial \vec{A} / \partial t = 0 \). Moreover,
\[ \vec{B} = \nabla \times \vec{A}. \]
In the first case, we obtain
\[ \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ 0 & Bx & 0 \end{vmatrix} = B \vec{k}. \]
The other gauge fields yield the same constant magnetic field pointing along the z-axis.

(c) With the given gauge field the Dirac equation becomes
\[ (i \gamma^0 \partial_t + i \gamma^1 \partial_x + i \gamma^2 \partial_y - q \gamma^2 A_2 - m) \psi = 0. \] (4)
Using \( A_2 = -A^2 = -Bx \) and the expressions for the \( \gamma \) matrices we get
\[ (i \sigma_3 \partial_t - \sigma_2 \partial_x + \sigma_1 \partial_y - iq Bx \sigma_1 - m) \psi = 0. \] (5)
Inserting the Pauli matrices, this becomes

\[
\begin{pmatrix}
i \partial_t & 0 & 0 & i \partial_t \\
0 & -i \partial_x & 0 & 0 \\
i \partial_x & 0 & -i \partial_x & 0 \\
0 & \partial_y & 0 & 0 \\
i \partial_x - m & i \partial_x + \partial_y - iqBx & -i \partial_x - m & 0 \\
-i \partial_x + \partial_y - iqBx & -i \partial_x - m & 0 & 0
\end{pmatrix}\psi
\]

\[
= \begin{pmatrix}
i \partial_t - m \\
-i \partial_x - m \\
i \partial_x - m & \partial_y - iqBx \\
0 & 0
\end{pmatrix}\psi = 0. \tag{6}
\]

(d) Since \(\partial_y\) appears in this matrix but \(y\) does not, the \(y\)-dependence of the eigenstates will have a plane-wave form, i.e. \(\propto e^{ip_y y}\).

(e) Inserting the expression for \(\psi\) into the Dirac equation and carrying out the differentiations, we obtain

\[
\begin{pmatrix}
E - m & -\xi_+ \\
\xi_- & -E - m
\end{pmatrix}
\begin{pmatrix}
f(x) \\
g(x)
\end{pmatrix}
= 0. \tag{7}
\]

(f) The set of equations can be written as

\[
(E - m)f(x) - \xi_+ g(x) = 0, \tag{8}
\]

\[
\xi_- f(x) - (E + m)g(x) = 0. \tag{9}
\]

The second equation gives

\[
g(x) = \frac{1}{E + m} \xi_- f(x). \tag{10}
\]

Inserting this into the first equation, we obtain

\[
(E^2 - m^2)f(x) = \xi_+ \xi_- f(x). \tag{11}
\]

(g) Multiplying out, simplifying, rearranging, and dividing by \(2m\) for convenience, we find

\[
\frac{1}{2m} \left[ -\partial_x^2 + (qB)^2 \left( x - \frac{p_y}{qB} \right)^2 \right] f(x) = \frac{E^2 - m^2 + qB}{2m} f(x). \tag{12}
\]

This takes the form of the eigenvalue equation \(H\psi_n = \Omega_n \psi_n\) for a harmonic oscillator centered at \(x = p_y/qB\). We identify \(m\omega^2/2 = (qB)^2/2m\) (so \(\omega = qB/m\)), \(\Omega_n = (E^2 - m^2 + qB)/2m\), and \(\psi_n(x - p_y/qB) = f(x)\). From \(\Omega_n = \omega(1/2 + n)\) we obtain

\[
E^2 = m^2 + 2qBn \quad (n = 0, 1, 2, \ldots). \tag{13}
\]

(h) The function \(g(x)\) is found using Eq. (10).