Problem 1

(a) \( b_\uparrow b_\uparrow^\dagger |n_\uparrow, n_\downarrow\rangle = \sqrt{n_\uparrow + 1} b_\uparrow |n_\uparrow + 1, n_\downarrow\rangle \)
\[ = \sqrt{n_\uparrow + 1} \sqrt{n_\downarrow + 1} |(n_\uparrow + 1) - 1, n_\downarrow\rangle \]
\[ = (n_\uparrow + 1) |n_\uparrow, n_\downarrow\rangle \]

\( b_\uparrow^\dagger b_\uparrow |n_\uparrow, n_\downarrow\rangle = \sqrt{n_\downarrow} b_\uparrow^\dagger |n_\downarrow - 1, n_\downarrow\rangle \)
\[ = \sqrt{n_\downarrow} \sqrt{(n_\downarrow - 1) + 1} |(n_\downarrow - 1) + 1, n_\downarrow\rangle \]
\[ = n_\downarrow |n_\uparrow, n_\downarrow\rangle \]

\( \Rightarrow (b_\uparrow b_\uparrow^\dagger - b_\uparrow^\dagger b_\uparrow) |n_\uparrow, n_\downarrow\rangle \)
\[ = (n_\uparrow + 1 - n_\uparrow) |n_\uparrow, n_\downarrow\rangle = |n_\uparrow, n_\downarrow\rangle \]

i.e. \( [b_\uparrow, b_\uparrow^\dagger] |n_\uparrow, n_\downarrow\rangle = 1 |n_\uparrow, n_\downarrow\rangle \).

As this result is valid for an arbitrary basis state \( |n_\uparrow, n_\downarrow\rangle \), we obtain the operator identity \( [b_\uparrow, b_\uparrow^\dagger] = 1 \quad \text{QED} \).

More generally, the commutation relations that follow from (1) can be written \((\alpha, \beta = \uparrow, \downarrow)\)
\[ [b_\alpha, b_\beta^\dagger] = \delta_{\alpha, \beta} \quad [b_\alpha, b_\beta] = [b_\beta^\dagger, b_\alpha^\dagger] = 0 \]
\[ (b) \quad [\hat{S}_+^\dagger, \hat{S}_-] = b_\uparrow^\dagger b_\downarrow b_\uparrow b_\downarrow - b_\uparrow b_\downarrow b_\uparrow^\dagger b_\downarrow \]

\[ = b_\uparrow^\dagger (1 + \hat{n}_\psi) b_\uparrow - b_\uparrow^\dagger (1 + \hat{n}_\uparrow) b_\downarrow \quad \text{(using bosonic commutation relation, including the \(\uparrow\) and \(\downarrow\) operators that commute)} \]

\[ = \hat{n}_\uparrow (1 + \hat{n}_\psi) - \hat{n}_\psi (1 + \hat{n}_\uparrow) \]

\[ = \hat{n}_\uparrow + \hat{n}_\uparrow \hat{n}_\psi - \hat{n}_\psi - \hat{n}_\uparrow \hat{n}_\uparrow \]

\[ = \hat{n}_\uparrow - \hat{n}_\psi = 2 \hat{S}_z \]

\[ [\hat{S}_z, \hat{S}_+^\dagger] = \left[ \frac{1}{2} \left( \hat{n}_\uparrow - \hat{n}_\downarrow \right), b_\uparrow^\dagger b_\downarrow \right] \]

\[ = \frac{1}{2} \left[ \hat{n}_\uparrow, b_\uparrow^\dagger b_\downarrow \right] - \frac{1}{2} \left[ \hat{n}_\downarrow, b_\uparrow^\dagger b_\downarrow \right] \]

\[ = \frac{1}{2} \left[ \hat{n}_\uparrow, b_\uparrow^\dagger \right] b_\downarrow - \frac{1}{2} b_\uparrow^\dagger \left[ \hat{n}_\downarrow, b_\downarrow \right] \quad \text{(using Eqs. (23), (24))} \]

\[ = b_\uparrow^\dagger \leftarrow \quad = - b_\downarrow \]

\[ = b_\uparrow^\dagger b_\downarrow = \hat{S}_+ \]

\[ [\hat{S}_z, \hat{S}_-^\dagger] = - \hat{S}_- \quad \text{can be proven in the same way.} \]

Alternatively, one can note that since the Schwinger boson expressions for \(\hat{S}_+\) and \(\hat{S}_-\) are h.c.'s of each other,

\[ [\hat{S}_z, \hat{S}_-] = \hat{S}_z \hat{S}_- - \hat{S}_- \hat{S}_z = (\hat{S}_+ \hat{S}_z^\dagger) - (\hat{S}_z \hat{S}_-^\dagger) \]

\[ = [\hat{S}_+^\dagger, \hat{S}_z^\dagger] \quad \text{by h.c.'s} \]

\[ = (- \hat{S}_-^\dagger)^+ = - \hat{S}_- \]
(c) \[ \hat{S}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 \]

\[ = (\frac{1}{2}) (\hat{S}^+ + \hat{S}^-)(\hat{S}^+ - \hat{S}^-) + \frac{1}{4i} (\hat{S}^+ - \hat{S}^-)(\hat{S}^- - \hat{S}^+) + \hat{S}_z^2 \hat{S}_z^2 \]

\[ = \frac{1}{2} (\hat{S}^+ \hat{S}^- + \hat{S}^- \hat{S}^+) + \hat{S}_z^2 \hat{S}_z^2 \]

\[ = \frac{1}{2} (b^+_\uparrow b^+ \downarrow b^+_\downarrow b^+ \downarrow + b^+_\psi b^+ \uparrow b^+_\uparrow b^+ \psi) + \frac{1}{2i} (\hat{\eta}_\uparrow - \hat{\eta}_\psi)(\hat{\eta}_\psi - \hat{\eta}_\uparrow) \]

\[ = \frac{1}{2} \left[ \hat{\eta}_\uparrow (1 + \hat{\eta}_\psi) + \hat{\eta}_\psi (1 + \hat{\eta}_\uparrow) \right] + \frac{1}{4} (\hat{\eta}_\uparrow - \hat{\eta}_\psi)^2 \]

\[ = \frac{1}{2} \left[ \hat{\eta}_\uparrow + \hat{\eta}_\psi + 2 \hat{\eta}_\uparrow \hat{\eta}_\psi \right] + \frac{1}{4} \left[ \hat{\eta}_\uparrow^2 - 2 \hat{\eta}_\uparrow \hat{\eta}_\psi + \hat{\eta}_\psi^2 \right] \]

\[ = \frac{1}{2} (\hat{\eta}_\uparrow + \hat{\eta}_\psi) + \frac{1}{4} (\hat{\eta}_\uparrow + \hat{\eta}_\psi)^2 \]

\[ = \frac{\hat{\eta}_\uparrow + \hat{\eta}_\psi}{2} \left[ \frac{\hat{\eta}_\uparrow + \hat{\eta}_\psi}{2} + 1 \right] \]

\[ \Rightarrow \quad \hat{S}^2 \left| \eta_\uparrow, \eta_\psi \right\rangle \]

\[ = \frac{\hat{\eta}_\uparrow + \hat{\eta}_\psi}{2} \left[ \frac{\hat{\eta}_\uparrow + \hat{\eta}_\psi}{2} + 1 \right] \left| \eta_\uparrow, \eta_\psi \right\rangle \]

\[ = \frac{\eta_\uparrow + \eta_\psi}{2} \left[ \frac{\eta_\uparrow + \eta_\psi}{2} + 1 \right] \left| \eta_\uparrow, \eta_\psi \right\rangle \]

\[ = \hat{S} (\hat{S} + 1) \left| \eta_\uparrow, \eta_\psi \right\rangle \quad \text{if} \quad \eta_\uparrow + \eta_\psi = 2S \]
(2) \[ U(\Theta) = e^{-i\frac{\Theta^2}{2} \hat{N}} \]

Setting \( \Theta = 1 \) as usual and using the Schwinger boson representation gives:

\[ U(\Theta) = \exp\left[-i\frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)\Theta\right] \]

\[ \Rightarrow U(\Theta) \hat{b}_\alpha U^+(\Theta) = e^{-i\frac{\Theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)} \hat{b}_\alpha e^{i\frac{\Theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)} \]

To calculate this we use the Baker-Hausdorff theorem (25):

\[ e^{A} e^{B} = A + [A, B] + \frac{1}{2!} [A, [A, B]] + \ldots \]

with \( A = \hat{b}_\alpha \) and \( B = i\frac{\Theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow) \)

For \( \alpha = \uparrow \), \[ [A, B] = [\hat{b}_\uparrow, i\frac{\Theta}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)] \]

\[ = i\frac{\Theta}{2} [\hat{b}_\uparrow, \hat{n}_\uparrow] = -i\frac{\Theta}{2} \hat{b}_\uparrow \]

where we used (24). As the result is just a constant \(-i\frac{\Theta}{2}\) times \( \hat{b}_\uparrow \), it follows that:

\[ [A, B]_n = (-i\frac{\Theta}{2})^n \hat{b}_\uparrow \]

Therefore:

\[ U(\Theta) \hat{b}_\uparrow U^+(\Theta) = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\frac{\Theta}{2})^n \hat{b}_\uparrow = e^{-i\frac{\Theta}{2} \hat{b}_\uparrow} \]

For \( \alpha = \downarrow \), \[ [A, B] = i\frac{\Theta}{2} [\hat{b}_\downarrow, -\hat{n}_\downarrow] = i\frac{\Theta}{2} \hat{b}_\downarrow \]

Thus giving:

\[ U(\Theta) \hat{b}_\downarrow U^+(\Theta) = e^{i\frac{\Theta}{2} \hat{b}_\downarrow} \]
Use that \( |S, m\rangle = |n_\uparrow, n_\downarrow\rangle \) with \( n_\uparrow = S + m \)
and \( \langle n_\uparrow, n_\downarrow| \propto (b^+_\uparrow)^{n_\uparrow} (b^+_\downarrow)^{n_\downarrow} 10, 0\rangle \)

\[ \Rightarrow U(2\pi) |S, m\rangle \propto U(2\pi) (b^+_\uparrow)^{S + m} (b^+_\downarrow)^{S - m} 10, 0\rangle \]

\[ \Rightarrow U(2\pi) \underbrace{b^+_\uparrow b^+_\uparrow \cdots b^+_\uparrow}_{S + m \text{ factors}} \underbrace{b^+_\downarrow b^+_\downarrow \cdots b^+_\downarrow}_{S - m \text{ factors}} 10, 0\rangle \]

Now insert \( I = U^+(2\pi) U(2\pi) \) between adjacent \( b^+_\alpha \)
operators and after the last \( b^+_\downarrow \)

\[ \Rightarrow (U(2\pi) b^+_\uparrow U^+(2\pi))^{S + m} (U(2\pi) b^+_\downarrow U^+(2\pi))^{S - m} U(2\pi) 10, 0\rangle \]

Use \( U(2\pi) b^+_\uparrow U^+(2\pi) = e^{-i \frac{2\pi}{2}} b^+_\uparrow = e^{-i \pi} b^+_\uparrow = -b^+_\uparrow \)
\[ U(2\pi) b^+_\downarrow U^+(2\pi) = e^{i \frac{2\pi}{2}} b^+_\downarrow = e^{i \pi} b^+_\downarrow = -b^+_\downarrow \]

\[ (U(2\pi) 10, 0\rangle = e^{-i \frac{2\pi}{2} \langle \hat{n}_\uparrow - \hat{n}_\downarrow \rangle} = \sum_{m=0}^{\infty} \frac{(-i \pi)^m}{m!} (\hat{n}_\uparrow - \hat{n}_\downarrow)^m 10, 0\rangle \]

\[ = 10, 0\rangle \quad \text{since only the } m=0 \text{ term gives a non-zero contribution (since } \langle \hat{n}_\uparrow 10, 0\rangle = \hat{n}_\downarrow 10, 0\rangle = 0) \]

\[ \Rightarrow U(2\pi) |S, m\rangle \propto (-b^+_\uparrow)^{S + m} (-b^+_\downarrow)^{S - m} 10, 0\rangle \]

\[ = (-1)^{S + m + S - m} (b^+_\uparrow)^{S + m} (b^+_\downarrow)^{S - m} 10, 0\rangle \]

\[ = (-1)^{2S} (b^+_\uparrow)^{S + m} (b^+_\downarrow)^{S - m} 10, 0\rangle \]

i.e. \( U(2\pi) |S, m\rangle = (-1)^{2S} |S, m\rangle \)

which is what we wanted to verify using the Schrödinger boson representation.
Remarks:

- Depending on at what stage one inserts the value $\theta = 2\pi$, one may end up with the factor $(-1)^{2m}$ instead of $(-1)^{2S}$. But these have the same value, since if $2S$ is an even (odd) integer, then $2m$ is also always even (odd).

- The result $U(2\pi) |S,m\rangle = (-1)^{2S} |S,m\rangle$ is trivial to verify from the general relations:

$$U(2\pi) |S,m\rangle = \exp (-i \cdot 2\pi S_z) |S,m\rangle$$

$$= \exp (-i \cdot 2\pi m) |S,m\rangle = (-1)^{2m} |S,m\rangle = (-1)^{2S} |S,m\rangle$$
Problem 2.

(a) \[ C_{k+2\pi m} = \frac{1}{\sqrt{N}} \sum_j e^{-i(k+2\pi m)j} C_j \]

\[ = \frac{1}{\sqrt{N}} \sum_j e^{-ikj} C_j e^{-i2\pi mj} = C_k \quad (\star) \]

Periodic BC's gives \( k = 2\pi m/N \) (m integer). Pick k's to lie in region of length \( 2\pi \) (cf. \( \star \)) i.e. we choose the standard region \(-\pi \leq k \leq \pi\) (1B2)

Let us define \( H = H_t + H_D \)

\[ H_t = -t \sum_j (C_j^+ C_{j+1} + h.c.) \]

\[ = -t \sum_j \left( \frac{1}{\sqrt{N}} \right)^2 \sum_{k,k'} e^{-ikj} e^{ik'(j+1)} C_{k'}^+ C_k + h.c. \]

\[ = -t \sum_{k,k'} C_{k'}^+ C_k e^{ik} \frac{1}{\sqrt{N}} \sum_j e^{-i(k-k')j} + h.c. \]

\[ = \sum_{k,k'} \delta_{kk'} \quad \text{from using (28) with} \]

\[ f(k-k') = k-k' \]

\[ = -t \sum_k \left( e^{ik} + e^{-ik} \right) C_{k'}^+ C_k = \sum_k \varepsilon_k C_{k'}^+ C_k \]

where \( \varepsilon_k = -2t \cos k \)

We have \( \varepsilon_{k+\pi} = -2t \cos (k+\pi) \)

\[ = -2t \left[ \cos k \cos \pi + \sin k \sin \pi \right] = -\varepsilon_k \]
\[ H_\Delta = \Delta \sum_j (-1)^j c_j^+ c_j \]  
\[ = \Delta \sum_j e^{i\pi j} \left( \frac{1}{N} \right)^2 \sum_{k,k'} e^{-ik'j} e^{ikj} c_{k'}^+ c_k \]  
\[ = \Delta \sum_{kk'} c_{k'}^+ c_k \frac{1}{N} \sum_j e^{ij(k-k'+\pi)} \]

To calculate the sum we use (28) with

\[ F(k-k') = k - k' + \pi. \]  
The sum is thus 0 unless \( k - k' + \pi = 2\pi m \) (\( m \) integer)

\[ \Rightarrow k - k' = (2m-1)\pi \]

Given that both \( k \) and \( k' \) are in \( 1BZ \),

solutions are only possible for \( m = 0,1 \), i.e.

for \( k - k' = \pm \pi \). Thus we may write

\[ H_\Delta = \Delta \sum_{kk'} c_{k'}^+ c_k \sum_{p=\pm 1} S_{k,k+p\pi} \]

For kets with \( k \in [\pi,0) \) we only get a contribution from \( k' = k + \pi \) \( (p = +1) \)

\[ \Rightarrow \Delta c_{k+\pi}^+ c_k \]

For kets with \( k \in (0,\pi) \) we only get a contribution from \( k' = k - \pi \) \( (p = -1) \)

\[ \Rightarrow \Delta c_{k-\pi}^+ c_k = \Delta c_{k+\pi}^+ c_k \]

↑ using \( c_{k+2\pi} = c_k \) which follows from \( c_{k+2\pi} = c_k \)

\[ \Rightarrow H_\Delta = \Delta \sum_k c_{k+\pi}^+ c_k \]

Thus we have shown that

\[ H = \sum_{k \in 1BZ} \left[ E_k c_k^+ c_k + \Delta c_{k+\pi}^+ c_k \right] \]
(b) It is convenient to divide the MBZ into 4 regions I, II, III, IV as shown below:

\[ + \quad \Pi \quad 0 \quad \Pi \quad - \quad \Pi \]

\[ \Rightarrow H = \left( \sum_{k \in I} \Sigma + \sum_{k \in II} \Sigma + \sum_{k \in III} \Sigma + \sum_{k \in IV} \Sigma \right) \left( \varepsilon_k c_k^+ c_k + \Delta c_k^{+\Pi} c_k \right) \]

Regions II + III make up the MBZ, and we see that their contribution gives the first and fourth terms in (16). The contribution from region I is

\[ \sum_{k \in I} \left( \varepsilon_k c_k^+ c_k + \Delta c_k^{+\Pi} c_k \right) \]

\[ = \sum_{k \in III} \left[ \frac{\varepsilon_{k-\Pi}}{C_k^{+\Pi}} \frac{C_k^{+\Pi}}{C_k^+} + \frac{\Delta c_k^{+\Pi}}{C_k^{+\Pi}} \frac{C_k^{+\Pi}}{C_k^+} \right] \]

\[ = \sum_{k \in III} \left[ - \varepsilon_k c_k^+ c_k^{+\Pi} + \Delta c_k^+ c_k^{+\Pi} \right] \]

Similarly, the contribution from region IV is

\[ \sum_{k \in IV} \left( \varepsilon_k c_k^+ c_k + \Delta c_k^{+\Pi} c_k \right) \]

\[ = \sum_{k \in II} \left( \varepsilon_{k+\Pi} c_k^{+\Pi} c_k^{+\Pi} + \Delta c_k^{+\Pi+\Pi} c_k^{+\Pi} \right) \]

\[ = \sum_{k \in II} \left[ - \varepsilon_k c_k^{+\Pi} c_k^{+\Pi} + \Delta c_k^+ c_k^{+\Pi} \right] \]

Thus the contributions from regions I + IV give the second and third terms in (16). In conclusion,

\[ H = \sum_{k \in MBZ} \left[ \varepsilon_k (c_k^+ c_k - c_k^{+\Pi} c_k^{+\Pi}) + \Delta (c_k^+ c_k^{+\Pi} + c_k^{+\Pi} c_k) \right] \]
(c) Each term in the k-sum can now be diagonalized separately. To save some writing we drop the k-index in the following and also define \( c_k \equiv a_k \), \( c_{k+1} \equiv b_k \). Thus we will diagonalize the operator

\[
h = \varepsilon (a^+a - b^+b) + \Delta (a^+b + b^+a)
\]

\[
a^+a = (u^\alpha + v^\beta)^+(u^\alpha - v^\beta)
= u^2 \alpha^+\alpha + v^2 \beta^+\beta - u\varepsilon (\alpha^+\beta + \beta^+\alpha)
\]

\[
b^+b = (v^\alpha + u^\beta)^+(v^\alpha + u^\beta)
= v^2 \alpha^+\alpha + u^2 \beta^+\beta + u\varepsilon (\alpha^+\beta + \beta^+\alpha)
\]

\[
a^+b = (u^\alpha + v^\beta)^+(v^\alpha + u^\beta)
= u\varepsilon (\alpha^+\beta - \beta^+\alpha) + u^2 \alpha^+\beta - v^2 \beta^+\alpha
\]

\[
b^+a = (a+b)^+
= u\varepsilon (\alpha^+\beta - \beta^+\alpha) - v^2 \alpha^+\beta + u^2 \beta^+\alpha
\]

\[
\Rightarrow h = F (\alpha^+\alpha - \beta^+\beta) + G (\alpha^+\beta + \beta^+\alpha)
\]

where \( F = \varepsilon (u^2 - v^2) + 2\Delta \varepsilon \)

\( G = -2\varepsilon \varepsilon \Delta \varepsilon + \Delta (u^2 - v^2) \)

The off-diagonal terms (i.e. the kms prop. to \( (\alpha^+\beta - \beta^+\alpha) \))

vanish if \( G = 0 \) \( \Rightarrow \Delta (u^2 - v^2) = \varepsilon \cdot 2\varepsilon \varepsilon \Delta \varepsilon \)

Using the parametrization \( u = \cos \theta \), \( v = \sin \theta \), and

\((29)^c(30)\), the condition \( G = 0 \) becomes \( \Delta \cos 2\theta = \varepsilon \sin 2\theta \)

\[
\Rightarrow \tan 2\theta = \frac{\Delta}{\varepsilon}
\]

This gives furthermore

\[
F = \varepsilon \cos 2\theta + \Delta \sin 2\theta = \cos 2\theta [\varepsilon + \Delta \tan 2\theta]
\]

Using \((31)\) gives \( \cos 2\theta = \pm \frac{1}{\sqrt{1 + \tan^2 2\theta}} = \pm \frac{1}{\sqrt{1 + (\Delta/\varepsilon)^2}} \)
Let us choose the positive sign (choosing the negative sign would however also be allowed and would not affect the physical results).

\[
F = \frac{1}{\sqrt{1 + \left(\frac{\Delta}{\varepsilon}\right)^2}} \left[ \varepsilon + \Delta \frac{\Delta}{\varepsilon} \right] = \frac{\varepsilon^2 + \Delta^2}{\varepsilon \sqrt{1 + \left(\frac{\Delta}{\varepsilon}\right)^2}}
\]

\[
= -\sqrt{\varepsilon^2 + \Delta^2} \quad \text{(where we used that } \varepsilon = -|E| \text{ for } k \in \text{MBZ)}
\]

This gives

\[
H = \sum_{k \in \text{MBZ}} \left[ E^{(\alpha)}_k \alpha^+_k \alpha_k + E^{(\beta)}_k \beta^+_k \beta_k \right]
\]

where \( E^{(\beta)}_k = -E^{(\alpha)}_k = \sqrt{E_k^2 + \Delta^2} \)

Here \( E^{(\alpha)}_k \) (\( E^{(\beta)}_k \)) is the energy of the \( \alpha \)-mode \( \alpha^+_k |0\rangle \) (\( \beta \)-mode \( \beta^+_k |0\rangle \)) with wavevector \( k \) (here \( |0\rangle \) is the vacuum state with no fermions).

(d) As there are \( N \) wavevectors in IBZ, there are \( N/2 \) wavevectors in MBZ. Thus there are \( N/2 \) \( \alpha \)-modes and \( N/2 \) \( \beta \)-modes. Furthermore, for the generic case \( \varepsilon > 0 \) and \( \Delta \neq 0 \), \( \sqrt{E_k^2 + \Delta^2} > 0 \) for all \( k \in \text{MBZ} \). Thus the energy of any \( \beta \)-mode is higher than that of any \( \alpha \)-mode.

Invoking also the Pauli principle, it follows that in a system with \( N/2 \) fermions, the ground state \( |G\rangle \) is characterized by each \( \alpha \)-mode being occupied by one fermion while all \( \beta \)-modes are unoccupied. The ground state energy is therefore

\[
E_G = \sum_{k \in \text{MBZ}} E^{(\alpha)}_k = -\sum_{k \in \text{MBZ}} \sqrt{E_k^2 + \Delta^2}
\]
The ground state $|G\rangle$ can be written $|G\rangle = \left( \frac{1}{\Delta} \alpha_k^+ \right)_{10}$.

\[(e) \quad \hat{N}_{\text{even}} - \hat{N}_{\text{odd}} = \sum (-1)^j c_j^+ c_j = \frac{1}{\Delta} H_{\Delta}\]

\[= \sum_{k \in MBZ} \left[ c_k^+ c_{k+\pi} + c_{k+\pi}^+ c_k \right]
\]

\[= \sum_{k \in MBZ} \left[ 2 u_k s_k (\alpha_k^+ \alpha_k - \beta_k^+ \beta_k) + (u_k^2 - v_k^2)(\alpha_k^+ \beta_k + \text{h.c.}) \right].\]

Given the form of the ground state as discussed in (a), the ground state expectation values are

\[\langle \alpha_k^+ \alpha_k \rangle = 1, \quad \langle \beta_k^+ \beta_k \rangle = \langle \alpha_k^+ \beta_k \rangle = \langle \beta_k^+ \alpha_k \rangle = 0\]

\[\Rightarrow \langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle = \sum_{k \in MBZ} 2 u_k s_k = \sum_{k \in MBZ} \sin 2\theta_k\]

\[= \sum_{k \in MBZ} \cos 2\theta_k + \tan 2\theta_k = \sum_{k \in MBZ} \frac{1}{\sqrt{1 + (\Delta/\varepsilon_k)^2}} \frac{\Delta}{\varepsilon_k}\]

\[= -\sum_{k \in MBZ} \frac{\Delta}{\sqrt{\varepsilon_k^2 + \Delta^2}}\]

(where we again used that $\varepsilon_k = -|\varepsilon_k|$ for $k \in MBZ$)

(i) $|\Delta| < |t| \Rightarrow |\langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle|$ small ($\propto \frac{|\Delta|}{t}$)

This is reasonable as in this case there is no appreciable energy difference between even and odd sites.

(ii) $|\Delta| \gg |t| \Rightarrow |\langle (\hat{N}_{\text{even}} - \hat{N}_{\text{odd}}) \rangle| \approx \frac{N}{2}

Reasonable if now strong preference for those $N/2$ sites with smallest energy cost, leaving the other $N/2$ sites empty.

(ii) The sign equals $-\text{sign}(\Delta)$: reasonable, as it means odd sites preferred for $\Delta > 0$, even sites for $\Delta < 0$. 

Problem 3

(a) Left diagram:

\[ N^2 \left[ \phi^{(0)}(\overline{t}) \right]^2 \sum_{\overline{k}_1 \overline{k}_2} U(\overline{t}_1 - \overline{t}) \overline{\xi}^{(0)}(\overline{k}_1) U(\overline{t} - \overline{t}_1) U(\overline{k}_2 - \overline{t}) \overline{\xi}^{(0)}(\overline{k}_2) U(\overline{t}_2 - \overline{t}) \]

The diagram is reducible, as it falls apart by cutting the middle electron line (the cut is indicated by the red line).

Right diagram:

\[ N^2 \left[ \phi^{(0)}(\overline{t}) \right]^2 \sum_{\overline{k}_1 \overline{k}_2} U(\overline{k}_1 - \overline{t}) \overline{\xi}^{(0)}(\overline{k}_1) U(\overline{k}_2 - \overline{t}_1) \overline{\xi}^{(0)}(\overline{k}_2) U(\overline{k}_1 - \overline{t}_2) \]

\[ - \overline{\xi}^{(0)}(\overline{k}_1) U(\overline{t} - \overline{k}_1) \]

The diagram is irreducible, as it does not fall apart by cutting a single internal electron line. The corresponding self-energy diagram is
(b) 1. Any term in $\mathcal{G}$ can be written on the form

$$g^{(a)} \Sigma^{(a)} \Sigma^{(b)} \Sigma^{(c)} \ldots \Sigma^{(m)} \Sigma^{(a)} g^{(a)}$$

where $\Sigma^{(i)}$ is a self-energy diagram, and $m$ is the number of self-energy diagrams in the term. The exact $\mathcal{G}$ is obtained by summing over $m$ ($m = 0, \ldots, \infty$) and for each $m$ summing each factor $\Sigma^{(i)}$ over all self-energy diagrams. Typically, one however restricts to some approximate self-energy, corresponding to only summing each $\Sigma^{(i)}$ over a certain subset of all self-energy diagrams. Even if that subset is finite, one still generates an infinite subset of Feynman diagrams from it, as there is no limit to the number $m$ of self-energy diagrams in a term.

2. In the case of the 1st Born approximation, the only self-energy diagrams taken into account are $\Sigma^{(1)}$ and $\Sigma^{(2)}$.

The left Feynman diagram is included in this approximation, as it contains two self-energy diagrams of type $\Sigma^{(1)}$, which is included in $\Sigma^{(1)}$.

In contrast, the right Feynman diagram is not included, as it contains the self-energy diagram $\Sigma^{(2)}$ which is not included in $\Sigma^{(2)}$. 
(c) The 1st Born approximation for \( G \) becomes

\[
G^\text{R}(\mathbf{k}, \omega) = \frac{1}{ip_m - \mathbf{k}^2 - \Delta + \frac{i}{2\tau} \text{sgn}(p_m)}
\]

\( G^\text{R}(\mathbf{k}, \omega) \) is obtained from analytic continuation:

\[
G^\text{R}(\mathbf{k}, \omega) = G(\mathbf{k}, ip_m) \bigg|_{ip_m \to \omega + i\eta}
\]

Thus the term \( ip_m \) in \( G \) is replaced by \( \omega + i\eta \), while \( \text{sgn}(p_m) = \text{sgn}(\text{Im}(ip_m)) \) is replaced by \( \text{sgn}(\text{Im}(\omega + i\eta)) = \text{sgn}(\eta) = \pm 1 \). This gives

\[
G^\text{R}(\mathbf{k}, \omega) = \frac{1}{\omega + i\eta - \mathbf{k}^2 - \Delta + \frac{i}{2\tau}} \xrightarrow{\text{finite}} \frac{1}{\omega - (\mathbf{k}^2 + \Delta) + \frac{i}{2\tau}}
\]

when in the last transition we could neglect the infinitesimal \( \eta \) since \( \frac{i}{2\tau} \) is finite.

\[
A(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} G^\text{R}(\mathbf{k}, \omega) = -\frac{1}{\pi} \text{Im} \frac{\omega - (\mathbf{k}^2 + \Delta) - \frac{i}{2\tau}}{(\omega - (\mathbf{k}^2 + \Delta))^2 + (\frac{1}{2\tau})^2}
\]

\[
= \frac{1}{\pi} \frac{1/2\tau}{(\omega - (\mathbf{k}^2 + \Delta))^2 + (1/2\tau)^2}
\]

(d) - peak at \( \mathbf{k}^2 + \Delta \)

- width \( \propto \frac{1}{\tau^2} \)

- height \( \propto \tau \)

- In the absence of scattering:

\( \Delta = 0, \tau = \infty \) \( \Rightarrow \) Dirac \& -function peaked

- (Lorentzian)

\( A(\mathbf{k}, \omega) \uparrow \)

- \( \omega \rightarrow \mathbf{k}^2 + \Delta \)

- \( \mathbf{k}^2 \)