1. Explicit connection between first and second quantization.

An explicit connection between the basis states $|n⟩ ≡ |n_1, n_2, ...⟩$ with $\sum_\nu n_\nu = N$ in second quantization and the basis wavefunctions $\Phi_n(x_1, ..., x_N)$ in first quantization can be established. Let

$$|x_1, x_2, ..., x_N⟩ \equiv \frac{1}{\sqrt{N!}} \hat{\psi}^\dagger(x_1) ... \hat{\psi}^\dagger(x_N)|0⟩. \quad (1)$$

Then the (correctly normalized) basis wavefunctions are given by

$$\Phi_n(x_1, ..., x_N) = \langle x_1, ..., x_N|n⟩. \quad (2)$$

As an example, consider a fermionic 2-particle basis state $|...|1_\mu,1_\nu,...⟩$ in which the single-particle states $\mu$ and $\nu$ are occupied and all others are empty. Evaluate the rhs of (2) to show that the wavefunction is indeed the correct Slater determinant,

$$\Phi(x_1, x_2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \phi_\mu(x_1) & \phi_\mu(x_2) \\ \phi_\nu(x_1) & \phi_\nu(x_2) \end{vmatrix}. \quad (3)$$

2. Density operators.

The density operator $\hat{\rho}(x)$ is in first quantization given by $\hat{\rho}(x) = \sum_{i=1}^{N} \delta(x - x_i)$.

(a) Show that in second quantization,

$$\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x). \quad (4)$$

(b) Using second quantization, show that

$$\int dx \, \hat{\rho}(x) = \hat{N} \quad (5)$$

where $\hat{N}$ is the total number operator.

In first quantization a two-particle operator $\hat{H}_I$ takes the form

$$\hat{H}_I = \frac{1}{2} \sum_{\substack{i,j=1 \atop i \neq j}}^N v(x_i, x_j). \quad \text{(6)}$$

(a) Show that this can be rewritten (with $\hat{\rho}(x)$ given by its first-quantization expression)

$$\hat{H}_I = \frac{1}{2} \left[ \int dx \int dx' v(x, x') \hat{\rho}(x) \hat{\rho}(x') - \int dx v(x, x) \hat{\rho}(x) \right]. \quad \text{(7)}$$

(b) Show that in second quantization $\hat{H}_I$ can be written (for both fermionic and bosonic systems)

$$\hat{H}_I = \frac{1}{2} \int dx \int dx' v(x, x') \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(x') \hat{\psi}(x') \hat{\psi}(x). \quad \text{(8)}$$

(c) Use this to show that the second-quantized representation of $\hat{H}_I$, expressed using the arbitrary basis $\{ | \alpha \rangle \}$ for single-particle states, is given by

$$\hat{H}_I = \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} \left( \int dx \int dx' \phi_\alpha^* (x) \phi_\beta^* (x') v(x, x') \phi_\delta (x') \phi_\gamma (x) \right) \hat{c}_\alpha^\dagger \hat{c}_\beta^\dagger \hat{c}_\delta \hat{c}_\gamma, \quad \text{(9)}$$

which is exactly of the form claimed in the lectures (the expression enclosed in parentheses is the matrix element denoted $\langle \alpha \beta | v | \gamma \delta \rangle$ in the lectures).