1. The Holstein-Primakoff representation.

(a) Show that an alternative and equivalent form of the spin commutation relations

\[ [S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \]  

is given by

\[ [S^+, S^-] = 2S^z, \quad [S^z, S^\pm] = \pm S^\pm. \]  

(b) The Holstein-Primakoff (HP) representation is given by

\[ S^+ = \sqrt{2} S^- a^\dagger a, \]  
\[ S^- = a^\dagger \sqrt{2} S^- a^\dagger a, \]  
\[ S^z = S - a^\dagger a, \]  

where \( a \) and \( a^\dagger \) are canonical boson operators. Show that the HP representation satisfies the correct spin commutation relations and the relation \( \mathbf{S} \cdot \mathbf{S} = S(S + 1) \). NB! For this you should use the exact HP expressions; do NOT expand the square roots. Hint: Use that \([f(\hat{O}), g(\hat{O})] = 0\) for functions \( f \) and \( g \) of an operator \( \hat{O} \).

2. Ferromagnetic Heisenberg model with a spin anisotropy.

Consider spins on a two-dimensional square lattice with Hamiltonian

\[ H = J \sum_{(i,j)} \mathbf{S}_i \cdot \mathbf{S}_j - D \sum_i (S^z_i)^2. \]  

The sum in the first term is over all pairs of nearest-neighbour sites, and the sum in the second term is over all sites. The exchange constant \( J = -|J| < 0 \).

(a) Assuming that the parameter \( D > 0 \), argue that the spins will order along the +z or −z direction.

(b) Use spin-wave theory to calculate the ground state energy \( E_0 \) and the magnon dispersion \( \omega_k \).

(c) The energy gap is defined as \( \Delta \equiv E_1 - E_0 \) where \( E_1 \) is the energy of the lowest excited state. Give an expression for \( \Delta \).

(d) What do you predict about the ordering direction if \( D < 0 \), and why?
3. Physical picture of ferromagnetic spin waves

Consider a Heisenberg ferromagnet which has magnetic order with the magnetization vector pointing in the $z$ direction. Let the state $|\mathbf{k}\rangle \equiv a^\dagger_{\mathbf{k}}|0\rangle$ where $|0\rangle$ is the ferromagnetic ground state, i.e. the state $|\mathbf{k}\rangle$ contains one magnon with wavevector $\mathbf{k}$ ($\neq 0$). Let us define the transverse correlation function in the state $|\mathbf{k}\rangle$ as

$$
\langle \mathbf{k}| \mathbf{S}_\perp^i \cdot \mathbf{S}_\perp^j |\mathbf{k}\rangle,
$$

(7)

where the transverse spin operator $\mathbf{S}_\perp^i$ is the projection of the spin operator onto the $xy$ plane:

$$
\mathbf{S}_\perp^i = S_i^x \hat{x} + S_i^y \hat{y}.
$$

(8)

Use the truncated HP representation to show that

$$
\langle \mathbf{k}| \mathbf{S}_\perp^i \cdot \mathbf{S}_\perp^j |\mathbf{k}\rangle = \frac{2S}{N} \cos[\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)].
$$

(9)

Thus on average each spin has a small transverse component (i.e. perpendicular to the direction of the magnetization) and the orientations of the transverse components of two spins $i$ and $j$ differ by an angle $\mathbf{k} \cdot (\mathbf{r}_i - \mathbf{r}_j)$. This should hopefully make the origin of the terminology spin wave clearer.