TFY4210, Quantum theory of many-particle systems, 2016: Tutorial 6

1. The Holstein-Primakoff representation.

(a) Show that an alternative and equivalent form of the spin commutation relations
\[
[S^x, S^y] = iS^z, \quad [S^y, S^z] = iS^x, \quad [S^z, S^x] = iS^y \tag{1}
\]
is given by
\[
[S^+, S^-] = 2S^z, \quad [S^z, S^±] = ±S^±. \tag{2}
\]

(b) The Holstein-Primakoff (HP) representation is given by
\[
S^+ = \sqrt{2S - a^\dagger a} a, \quad (3)
\]
\[
S^- = a^\dagger \sqrt{2S - a^\dagger a}, \quad (4)
\]
\[
S^z = S - a^\dagger a, \quad (5)
\]
where \(a\) and \(a^\dagger\) are canonical boson operators. Show that the HP representation satisfies the correct spin commutation relations and the relation \(S \cdot S = S(S + 1)\). NB! For this you should use the exact HP expressions; do NOT expand the square roots. Hint: Use that \([f(\hat{O}), g(\hat{O})] = 0\) for functions \(f\) and \(g\) of an operator \(\hat{O}\).

2. Ferromagnetic Heisenberg model with a spin anisotropy.

Consider spins on a two-dimensional square lattice with Hamiltonian
\[
H = J \sum_{(i,j)} S_i \cdot S_j - D \sum_i (S^z_i)^2. \tag{6}
\]
The sum in the first term is over all pairs of nearest-neighbour sites, and the sum in the second term is over all sites. The exchange constant \(J = -|J| < 0\).

(a) Assuming that the parameter \(D > 0\), argue that the spins will order along the +z or −z direction.

(b) Use spin-wave theory to calculate the ground state energy \(E_0\) and the magnon dispersion \(\omega_k\).

(c) The energy gap is defined as \(\Delta \equiv E_1 - E_0\) where \(E_1\) is the energy of the lowest excited state. Give an expression for \(\Delta\).

(d) What do you predict about the ordering direction if \(D < 0\), and why?
3. Physical picture of ferromagnetic spin waves

Consider a Heisenberg ferromagnet which has magnetic order with the magnetization vector pointing in the \( z \) direction. Let the state \( |k\rangle \equiv a_k^\dagger |0\rangle \) where \( |0\rangle \) is the ferromagnetic ground state, i.e. the state \( |k\rangle \) contains one magnon with wavevector \( k \) \( (\neq 0) \). Let us define the transverse correlation function in the state \( |k\rangle \) as

\[
\langle k| S_i^\perp \cdot S_j^\perp |k\rangle, \tag{7}
\]

where the transverse spin operator \( S_i^\perp \) is the projection of the spin operator onto the \( xy \) plane:

\[
S_i^\perp = S_i^x \hat{x} + S_i^y \hat{y}. \tag{8}
\]

Use the truncated HP representation to show that

\[
\langle k| S_i^\perp \cdot S_j^\perp |k\rangle = \frac{2S}{N} \cos[k \cdot (r_i - r_j)]. \tag{9}
\]

Thus on average each spin has a small transverse component (i.e. perpendicular to the direction of the magnetization) and the orientations of the transverse components of two spins \( i \) and \( j \) differ by an angle \( k \cdot (r_i - r_j) \). This should hopefully make the origin of the terminology spin wave clearer.