Consider the Hamiltonian
\[ H = \varepsilon(a_1^{\dagger}a_1 + a_2^{\dagger}a_2) + \Delta(a_1a_2 + \text{h.c.}) \] (1)
where \( a_1 \) and \( a_2 \) are bosonic operators satisfying canonical commutation relations \([a_i, a_j^{\dagger}] = \delta_{ij}\) etc. \((i, j = 1, 2)\). Assume that \( \varepsilon \) and \( \Delta \) are positive numbers with \( \varepsilon > \Delta \).

In order to write the Hamiltonian in diagonal form we transform to a new set \( b_1, b_2 \) of bosonic operators. The transformation reads (here \( u \) and \( v \) are real numbers)
\[ a_1 = ub_1 - vb_2^{\dagger}, \] (2)
\[ a_2 = ub_2 - vb_1^{\dagger}. \] (3)

(a) Use the requirement that the \( b \)-operators should also satisfy canonical commutation relations to show (e.g. by just calculating one selected commutator) that
\[ u^2 - v^2 = 1. \] (4)
This result can be used to write \( u = \cosh \eta \), \( v = \sinh \eta \).

(b) Show that \( H \) becomes diagonal in terms of the \( b \)-operators provided that \( \eta \) is chosen to satisfy
\[ \tanh 2\eta = \frac{\Delta}{\varepsilon}. \] (5)
Show that with this choice, \( H \) can be written
\[ H = F(b_1^{\dagger}b_1 + b_2^{\dagger}b_2) + G \] (6)
and give expressions for \( F \) and \( G \) in terms of \( \varepsilon \) and \( \Delta \).

(c) Argue from this result that the ground state \( |\Psi_0\rangle \) can be defined in terms of an equation expressing what happens when an annihilation operator \( b_i \) acts on \( |\Psi_0\rangle \). What is the ground state energy? Explain your reasoning.

(d) What is the energy of the lowest excited state(s)? Explain your reasoning.

(e) Express the annihilation operator \( b_1 \) in terms of \( a \)-operators.

(f) The ground state \( |\Psi_0\rangle \) is given by
\[ |\Psi_0\rangle = C \ \exp(-\tanh \eta \ a_1^{\dagger}a_2^{\dagger})|0\rangle \] (7)
where $|0\rangle$ is the vacuum state of the $a$-operators, i.e. $a_1|0\rangle = a_2|0\rangle = 0$. The constant $C$ is just a normalization factor and can be neglected in the following.

Verify Eq. (7) for the ground state by showing that it satisfies the defining equation for $|\Psi_0\rangle$ discussed in (c) (it’s sufficient that you only consider the equation that involves $b_1$). [Hint: The Baker-Hausdorff theorem

$$e^{-Q}P e^Q = P + [P,Q] + \frac{1}{2!}[[P,Q],Q] + \frac{1}{3!}[[[P,Q],Q],Q] + \ldots$$

may be useful.]