1. The sublattice magnetization correction for the Heisenberg antiferromagnet at nonzero temperature.

(a) We have
\[
\gamma_k = \frac{2}{z} \sum_{\delta} \cos(k \cdot \delta) = \frac{1}{d} \sum_{\delta} \cos(k \cdot \delta). \tag{1}
\]
Using \(\cos x \approx 1 - x^2/2\) for \(x \to 0\) we get, for small \(k\),
\[
\gamma_k \approx \frac{1}{d} \sum_{\delta} \left[1 - \frac{1}{2} (k \cdot \delta)^2\right] = \frac{1}{d} \left[d - \frac{1}{2} k^2\right] = 1 - \frac{k^2}{2d} \tag{2}
\]
where \(k = |k|\). Here we used that \(\delta\) runs over the \(d\) orthogonal unit vectors in \(d\) dimensions (e.g. in 3 dimensions, \(\delta\) runs over \(\hat{x}, \hat{y},\) and \(\hat{z}\)). This gives, for small \(k\),
\[
\omega_k = Jsz \sqrt{1 - \gamma_k^2} \approx 2JSd \sqrt{1 - \left(1 - \frac{k^2}{2d}\right)^2} = 2JSd \sqrt{\frac{k^2}{d} - \frac{k^4}{4d^2}} \approx 2JS\sqrt{d} k. \tag{3}
\]

(b) Converting the \(k\)-sum to an integral and using that \(n_k\) is given by the Bose-Einstein distribution function, the temperature-dependent part of the sublattice magnetization correction can be written
\[
\frac{2}{N} \sum_{k \in \text{MBZ}} \frac{n_k}{\sqrt{1 - \gamma_k^2}} \propto \int_{\text{MBZ}} d^d k \frac{1}{e^{\beta \omega_k} - 1} \frac{1}{\sqrt{1 - \gamma_k^2}} \tag{4}
\]
where MBZ denotes the magnetic Brillouin zone. Since we here want to look at the contribution from the vicinity of \(k = 0\) we use the results in (a) and also (since \(\omega_k \to 0\) as \(k \to 0\)) \(e^{\beta \omega_k} \approx 1 + \beta \omega_k\) to get
\[
\frac{1}{e^{\beta \omega_k} - 1} \frac{1}{\sqrt{1 - \gamma_k^2}} \propto \frac{1}{k^2} \text{ for } k \to 0. \tag{5}
\]
Also taking into account the factor \(k^{d-1}\) from the integration measure, the \(k\)-dependence of the radial part of the \(k\)-integral then becomes proportional to \(k^{d-1} \frac{1}{k^2} = k^{d-3}\) at small \(k\). Therefore the contribution from the lower limit \(k = 0\) to the radial integral becomes
\[
\int_0^1 dk \ k^{d-3} = \begin{cases} 
\int_0^1 dk \ k^{-2} = -\frac{1}{k} \big|_0^1 = +\infty & d = 1, \\
\int_0^1 dk \ k^{-1} = \ln k \big|_0^1 = +\infty & d = 2.
\end{cases} \tag{6}
\]
Thus both in \(d = 1\) and \(d = 2\) there is a divergence coming from the lower integration limit.
(c) By putting \( d = 3 \) in the integral on the lhs of (6) one sees that there is no divergence from the \( k = 0 \) limit in this case. To find the leading \( T \)-dependence of the finite-temperature correction \( (T \to 0) \) to the sublattice magnetization, we note that due to the factor \( e^{βωk} \) the contributions to the integral will decrease very rapidly with \( k \) when \( β \to ∞ \). We therefore approximate the integral by using the small-\( k \) approximations derived in (a) for all \( k \) and also replacing the integral over the MBZ by an integral over all \( k \). Thus (4) becomes

\[
\int_{\text{all } k} d^3k \frac{1}{e^{2βJS\sqrt{3}k} - 1} = \sqrt{3} \cdot 4π \int_0^∞ dk \frac{k}{e^{2βJS\sqrt{3}k} - 1} = \frac{4π \sqrt{3}}{(2\sqrt{3}βJS)^2} \int_0^∞ dx \frac{x}{e^x - 1}
\]

where a change of integration variable led to the last expression in which the remaining integral is a dimensionless number. The \( T^2 \) temperature dependence is now evident from the factor \( β^{-2} \).

2. 0th and 1st order perturbation theory for the interacting electron gas.

(a) We have

\[
\frac{1}{n} = \frac{Ω}{N} = \frac{4π}{3} r_0^3 = \frac{4π}{3} (r_s a_B)^3.
\]

Using \( n = k_F^3/(3π^2) \) (derived in the lectures) then gives

\[
k_F a_B = (3π^2 n)^{1/3} a_B = \left(3π^2 \cdot \frac{3}{4π(r_s a_B)^3}\right)^{1/3} = \left(\frac{9π}{4}\right)^{1/3} \frac{1}{r_s}.
\]

In the lectures we also showed that \( E^{(0)}/N = (3/5)ε_F \), where the Fermi energy \( ε_F = \hbar^2 k_F^2/(2m) \). Thus

\[
\frac{E^{(0)}}{N} = \frac{3 \hbar^2 k_F^2}{2m} = \frac{3}{5} (k_F a_B)^2 \text{Ry} = \frac{3}{5} \left(\frac{9π}{4}\right)^{2/3} \frac{1}{r_s^2} \text{Ry} \approx \frac{2.210}{r_s^2} \text{Ry}.
\]

(b) From 1st order perturbation theory, \( E^{(1)} = ⟨FS|H_I|FS⟩ \), where \( H_I \) is the Coulomb interaction term in \( H \). Thus

\[
\frac{E^{(1)}}{N} = \frac{1}{2ΩN} \sum_{q \neq 0} \sum_{k,k'} \sum_{σ,σ'} \frac{e^2}{ε_q^2} ⟨FS|c_{k+q,σ}^† c_{k'-q,σ'}^† c_{k',σ'} c_{k,σ} |FS⟩.
\]

Let us consider the matrix element in this expression. The annihilation operators acting on \( |FS⟩ \) will give zero unless their wavevectors \( k \) and \( k' \) are occupied in \( |FS⟩ \), giving the requirement \( |k| ≤ k_F \) and \( |k'| ≤ k_F \). To get a nonzero matrix element, the two creation
operators must then bring the state back to $|FS\rangle$, which requires $q = 0$ or $k' = k + q$, $\sigma' = \sigma$. But $q = 0$ is excluded from the $q$-sum in $H_I$. Therefore

$$
\langle FS| c^\dagger_{k+q,\sigma} c^\dagger_{k',-q,\sigma'} c_{k',\sigma'} c_{k,\sigma}|FS\rangle = \delta_{k',k+q} \delta_{\sigma',\sigma} \langle FS| n_{k+q,\sigma} n_{k,\sigma}|FS\rangle
$$

$$
= -\delta_{k',k+q} \delta_{\sigma',\sigma} \theta(k_F - |k + q|) \theta(k_F - |k|),
$$

where we again used that $q \neq 0$ when anticommuting the two middle operators. Inserting this in (11) and doing the spin summations (which give a factor of 2) and the summation over $k'$ gives

$$
\frac{E^{(1)}}{N} = -\frac{e^2}{\Omega N \varepsilon_0} \sum_{q \neq 0} \frac{1}{q^2} \sum_k \theta(k_F - |k + q|) \theta(k_F - |k|).
$$

Converting the two sums to integrals ($\sum_k \rightarrow \frac{\Omega}{(2\pi)^3} \int d^3k$) and using spherical coordinates $(\phi_q, \theta_q, q)$ and $(\phi_k, \theta_k, k)$ gives

$$
\frac{E^{(1)}}{N} = -\frac{e^2}{N \varepsilon_0 (2\pi)^6} \int_0^{2\pi} d\phi_q \int_{-1}^1 d(\cos \theta_q) \int_0^\infty d q q^2 \frac{1}{q^2}
$$

$$
\times \int_0^{2\pi} d\phi_k \int_{-1}^1 d(\cos \theta_k) \int_0^\infty dk k^2 \theta(k_F - |k + q|) \theta(k_F - |k|).
$$

Consider the $k$-integral (second line here) for a fixed $q$. The two step functions are equivalent to the requirements $|k| < k_F$ and $|k - (-q)| < k_F$. Thus the $k$-integral is the volume of the intersection of two spheres of radius $k_F$, one centered at $k = 0$ and the other centered at $k = -q$ (see Fig. 1). Although the latter sphere moves as the direction of $q$ is changed, the volume of the intersection of the two spheres is independent of the direction of $q$. Thus the angular part of the $q$-integral can be done trivially, giving $\int_0^{2\pi} d\phi_q \int_{-1}^1 d(\cos \theta_q) = 4\pi$. 

![Figure 1](image1.png)

Figure 1: Left: For a given $q$, the $k$-integral is the volume of the intersection of two spheres of radius $k_F$ centered at $k = 0$ and $k = -q$ respectively. Right: Blowup of the intersection.
Furthermore, one sees from the figure that a nonzero intersection requires the magnitude of $q$ to satisfy $q < 2k_F$. Taking these things into account gives

$$E^{(1)} = N = -\frac{e^2}{N \varepsilon_0} \frac{\Omega}{16\pi^3} \int_0^{2k_F} dq \ Vol(q)$$

where Vol($q$) is the intersection volume, which by symmetry is twice the volume of the right half of the intersection. To calculate this half-volume, we take the $k_z$ axis to point in the direction of $-q$. Then the angle $\theta_k$ is as shown in Fig. 1. For a given $\theta_k$, $k$ is integrated from $k_{\min}$ to $k_{\max}$. While $k_{\max} = k_F$ independently of $\theta_k$, $k_{\min}$ is given by (see Fig. 1):

$$k_{\min} = k_F \cos \theta_k.$$  \hspace{1cm} (16)

As $\theta_k$ is increased, $k_{\min}$ grows. The maximum value of $\theta_k$ corresponds to $k_{\min} = k_{\max}$, giving

$$(\cos \theta_k)_{\min} = \frac{q}{2k_F}. \hspace{1cm} (17)$$

There are no constraints on the variable $\phi_k$, so the $\phi_k$-integral just gives $2\pi$. Thus

$$Vol(q) = 2 \cdot 2\pi \int_{q/(2k_F)}^1 d(cos \theta_k) \int_{q/(2\cos \theta_k)}^{k_F} dk k^2$$

$$= 4\pi \int_{q/(2k_F)}^1 d(cos \theta_k) \cdot \frac{1}{3} \left[ k_F^3 - \left( \frac{q}{2\cos \theta_k} \right)^3 \right]$$

$$= \frac{4\pi}{3} \left\{ k_F^3 \left( 1 - \frac{q}{2k_F} \right) - \left( \frac{q}{2} \right)^3 \cdot \left( \frac{1}{-2} \right) \left[ 1^{-2} - \left( \frac{q}{2k_F} \right)^{-2} \right] \right\}$$

$$= \frac{4\pi}{3} \left( k_F^3 - \frac{3}{4} k_F^2 q + \frac{1}{16} q^3 \right). \hspace{1cm} (18)$$

(As a check of the correctness of this result, note that for $q = 0$ it becomes $4\pi k_F^3/3$ (i.e. the volume of a sphere of radius $k_F$) and for $q = 2k_F$ it becomes 0; both cases are as expected.) Thus

$$\frac{E^{(1)}}{N} = -\frac{e^2}{N \varepsilon_0} \frac{\Omega}{16\pi^3} \int_0^{2k_F} dq \ \left( k_F^3 - \frac{3}{4} k_F^2 q + \frac{1}{16} q^3 \right)$$

$$= -\frac{e^2}{N \varepsilon_0} \frac{\Omega}{12\pi^4} \left( k_F^3 \cdot 2k_F - \frac{3}{4} k_F^2 \cdot \frac{1}{2} (2k_F)^2 + \frac{1}{16} \cdot \frac{1}{4} (2k_F)^4 \right)$$

$$= -\frac{e^2}{N \varepsilon_0} \frac{\Omega}{16\pi^4} k_F^4. \hspace{1cm} (19)$$

Using $\Omega/N = 1/n$ and $k_F^3 = 3\pi^2 n$ gives

$$\frac{E^{(1)}}{N} = -\frac{e^2}{\varepsilon_0} \frac{3}{16\pi^2} k_F. \hspace{1cm} (19)$$

Expressing this in Rydberg units, it can be rewritten as follows:

$$\frac{E^{(1)}}{N} = -\frac{e^2}{\varepsilon_0} \frac{3}{16\pi^2} k_F \frac{2m^2 a_F^2}{\hbar^2} \text{ Ry}$$
\[
= -\frac{e^2}{\varepsilon_0} \frac{3}{16\pi^2} k_F \frac{2m}{\hbar^2} \cdot \frac{4\pi \varepsilon_0 \hbar^2}{m e^2} \cdot a_B \text{ Ry}
= -\frac{3}{2\pi} (k_F a_B) \text{ Ry}
= -\frac{3}{2\pi} \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s} \text{ Ry} \approx -0.916 \frac{1}{r_s} \text{ Ry}. \tag{20}
\]


Given the Hamiltonian for noninteracting bosons,

\[
H_0 = \sum_\nu \xi_\nu c_\nu^\dagger c_\nu, \tag{21}
\]
we consider the single-particle retarded Green function

\[
G_0^R(\nu, t) = -i\theta(t) \langle [c_\nu(t), c_\nu^\dagger(0)] \rangle. \tag{22}
\]
Here

\[
c_\nu(t) = e^{iH_0 t} c_\nu e^{-iH_0 t} = e^{-i\xi_\nu t} c_\nu, \tag{23}
\]
where the last expression follows in exactly the same way as for the fermionic case discussed in the lecture notes (Sec. 2.4). Thus

\[
G_0^R(\nu, t) = -i\theta(t)e^{-i\xi_\nu t} \langle [c_\nu, c_\nu^\dagger] \rangle = -i\theta(t)e^{-i\xi_\nu t} \tag{24}
\]
where we used the equal-time commutation relation \([c_\nu, c_\nu^\dagger] = 1\). The result (24) takes exactly the same form as for fermions. It follows that the Fourier transform also takes exactly the same form as for fermions:

\[
G_0^R(\nu, \omega) = \frac{1}{\omega - \xi_\nu + i\eta}. \tag{25}
\]
Note however that if we had wished to find the Green functions \(G^>\) or \(G^<\), the expressions would have contained the expectation value \(\langle c_\nu^\dagger c_\nu \rangle\), which for noninteracting bosons is given by the Bose-Einstein distribution function \((e^{\beta \xi_\nu} - 1)^{-1}\) and not the Fermi-Dirac distribution function \((e^{\beta \xi_\nu} + 1)^{-1}\) appearing in the fermionic case.

4. The basis invariance of the trace.

Let us consider two arbitrary basis sets, denoted \(\{|\alpha\rangle\}\) and \(\{|\tilde{\alpha}\rangle\}\). Let us define \(\text{Tr } O\) as the sum of the diagonal matrix elements of \(O\) in the basis \(\{|\alpha\rangle\}\):

\[
\text{Tr } O \equiv \sum_\alpha \langle \alpha | O | \alpha \rangle. \tag{26}
\]
Using the resolution of the identity operator $I$ in terms of the basis $\{|\tilde{\alpha}\rangle\}$, i.e. $I = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle\langle\tilde{\alpha}|$, the two basis sets can be related as

$$ |\alpha\rangle = \sum_{\tilde{\alpha}} |\tilde{\alpha}\rangle\langle\tilde{\alpha}|\alpha\rangle = \sum_{\tilde{\alpha}} \langle\tilde{\alpha}|\alpha\rangle|\tilde{\alpha}\rangle. \quad (27) $$

Thus

$$ \text{Tr } O = \sum_{\alpha} \langle\alpha|O|\alpha\rangle = \sum_{\alpha} \sum_{\tilde{\alpha}} \sum_{\tilde{\beta}} \langle\tilde{\alpha}|O|\tilde{\beta}\rangle\langle\tilde{\alpha}|\alpha\rangle\langle\tilde{\beta}|\alpha\rangle^*$$

$$ = \sum_{\tilde{\alpha}} \sum_{\tilde{\beta}} \langle\tilde{\alpha}|O|\tilde{\beta}\rangle \langle\tilde{\beta}|\alpha\rangle \langle\tilde{\alpha}|\alpha\rangle = \sum_{\tilde{\alpha}} \sum_{\tilde{\beta}} \langle\tilde{\alpha}|O|\tilde{\beta}\rangle \langle\tilde{\beta}|\alpha\rangle \delta_{\tilde{\alpha}\tilde{\beta}}$$

$$ = \sum_{\tilde{\alpha}} \langle\tilde{\alpha}|O|\tilde{\alpha}\rangle, \quad (28) $$

which shows that $\text{Tr } O$ is also equal to the sum of the diagonal elements of $O$ in the basis $\{|\tilde{\alpha}\rangle\}$. As the basis sets used here are arbitrary we conclude that the sum of the diagonal elements is independent of the basis.

In the above proof we expressed both the ket $|\alpha\rangle$ and the bra $\langle\alpha|$ in terms of the new basis, thus using the resolution of the identity twice. Actually, a simpler proof can be given by just using it once, e.g. for the ket only, as follows:

$$ \text{Tr } O = \sum_{\alpha} \langle\alpha|O|\alpha\rangle = \sum_{\alpha,\tilde{\alpha}} \langle\alpha|O|\tilde{\alpha}\rangle\langle\tilde{\alpha}|\alpha\rangle = \sum_{\alpha,\tilde{\alpha}} \langle\tilde{\alpha}|\alpha\rangle\langle\alpha|O|\tilde{\alpha}\rangle = \sum_{\tilde{\alpha}} \langle\tilde{\alpha}|O|\tilde{\alpha}\rangle. \quad (29) $$