1. The sublattice magnetization correction for the Heisenberg antiferromagnet at nonzero temperature.

(a) The dispersion relation for antiferromagnetic magnons is \( \omega_k = JSz\sqrt{1 - \gamma_k^2} \) where \( \gamma_k = \frac{2}{z} \sum \delta \cos(k \cdot \delta) \) and \( z = 2d \) is the number of nearest neighbour sites. Show that for small \( |k| \),

\[
\gamma_k \approx 1 - \frac{k^2}{2d},
\]

\( \omega_k \approx 2JS\sqrt{d|k|}. \)

In the lectures we found that the temperature-dependent part of the sublattice magnetization correction \( \Delta M_A \) is given by

\[
\frac{2}{N} \sum_k n_k \frac{1}{\sqrt{1 - \gamma_k^2}}
\]

where \( n_k = \frac{1}{e^{\omega_k/k_BT} - 1} \).

(b) Show that in \( d = 1 \) and \( d = 2 \) the contribution from small \( k \) makes (3) diverge. Hence there is no magnetic order in the Heisenberg antiferromagnet at finite temperatures in one and two dimensions.

(c) In contrast, show that in \( d = 3 \) the expression (3) is finite (i.e. there is no divergence at small \( k \)) and scales as \( T^2 \).

2. 0th and 1st order perturbation theory for the interacting electron gas.

Consider the 3-dimensional interacting electron gas (more precisely the so-called jellium model introduced in Problem 2 in Tutorial 3) with Hamiltonian

\[
H = \sum_{k,\sigma} \frac{\hbar^2 k^2}{2m} c_{k\sigma}^\dagger c_{k\sigma} + \frac{1}{2\Omega} \sum_{q \neq 0} \sum_{k,\sigma} \sum_{k',\sigma'} \frac{e^2}{\epsilon_0 q^2} c_{k+q,\sigma}^\dagger c_{k'-q,\sigma'}^\dagger c_{k',\sigma'} c_{k,\sigma}.
\]

It will be convenient to introduce the length scale defined by the Bohr radius \( a_B = \frac{4\pi\epsilon_0\hbar^2}{me^2} \) and the energy scale \( \text{Ry} = \frac{\hbar^2}{2ma_B^2} \) (the Rydberg). Let \( r_0 \) be a measure of the average distance between electrons (defined as the radius of a sphere whose volume equals the volume per electron) and define the dimensionless quantity \( r_s \equiv r_0/a_B \).
(a) First consider the noninteracting electron gas, whose Hamiltonian is given by the kinetic energy term only. Its ground state is the filled Fermi sphere $|\text{FS}\rangle$ with radius $k_F$. Show that

$$k_F a_B = \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{r_s}$$

and that the ground state energy per particle is given by

$$\frac{E^{(0)}}{N} = 3 \left( k_F a_B \right)^2 \text{Ry} \approx \frac{2.21}{r_s^2} \text{Ry}. \quad (6)$$

(Here you may make use of results already derived in the lectures for $E^{(0)}/N$ and the relation between $k_F$ and the electron density.)

(b) Next consider the interaction term in (4) as a perturbation on the kinetic energy term. Show that the 1st order correction to the ground state energy per particle is given by

$$\frac{E^{(1)}}{N} = -3 \frac{\pi}{2} (k_F a_B) \text{Ry} \approx -0.916 \frac{r_s}{r_s} \text{Ry}. \quad (7)$$

[A few hints: Note that $\mathbf{q} \neq 0$ in the interaction term and show that therefore

$$\langle \text{FS}| c^{\dagger}_{\mathbf{k}+\mathbf{q},\sigma}\mathbf{c}^{\dagger}_{\mathbf{k}'} c_{\mathbf{k}',\sigma'} c_{\mathbf{k},\sigma}|\text{FS}\rangle = -\delta_{\mathbf{k}',\mathbf{k}+\mathbf{q}} \delta_{\sigma,\sigma'} \theta(k_F - |\mathbf{k} + \mathbf{q}|) \theta(k_F - |\mathbf{k}|)$$

where $\theta(x)$ is the (Heaviside) step function. Convert the sums over $\mathbf{k}$ and $\mathbf{q}$ to integrals over spherical coordinates. Observe that for a fixed $\mathbf{q}$ the $\mathbf{k}$-integration amounts to finding the volume of the intersection of two spheres of radius $k_F$ displaced from each other by a vector $\mathbf{q}$.]


Calculate the Fourier transform of the single-particle retarded Green function, $G_{0R}(\nu,\omega)$, for noninteracting bosons with Hamiltonian

$$H_0 = \sum_{\nu} \xi_{\nu} c^\dagger_{\nu} c_{\nu}. \quad (8)$$

4. The basis invariance of the trace.

Show that the trace of an operator is independent of the basis chosen to evaluate it. [Hint: First define $\text{Tr } O$ as the sum of the diagonal elements of $O$ in some particular, but arbitrarily chosen basis. Then do a transformation to an arbitrary different basis and show that $\text{Tr } O$ can be rewritten as the sum of the diagonal elements of $O$ in the new basis.]