In this tutorial set, all problems concern fermions.

1. Relationship between $G^<(\nu,\omega)$ and the spectral function $A(\nu,\omega)$.

(a) Starting from the definition of the "lesser" Green’s function, show that it can be written
\[ G^<(\nu; t, t') = \frac{i}{Z} \sum_{n,m} e^{-\beta E_m} e^{i(E_n - E_m)(t-t')} |\langle m | c^\dagger_\nu | n \rangle|^2. \] (1)

and that its Fourier transform is
\[ G^<(\nu,\omega) = \frac{2\pi i}{Z} \sum_{n,m} e^{-\beta E_m} |\langle m | c^\dagger_\nu | n \rangle|^2 \delta(\omega + E_n - E_m). \] (2)

(b) Starting from Eq. (53) in the lecture notes, show that $A(\nu,\omega)$ can be rewritten as
\[ A(\nu,\omega) = (1 + e^{\beta\omega}) \frac{1}{Z} \sum_{n,m} |\langle m | c^\dagger_\nu | n \rangle|^2 e^{-\beta E_m} \delta(\omega + E_n - E_m). \] (3)

(c) Show that
\[ -iG^<(\nu,\omega) = 2\pi A(\nu,\omega)n_F(\omega) \] (4)
which is Eq. (56) in the lecture notes. (Eq. (55) can be proved in a similar way.)

2. An alternative form of the Lehmann representation.

Show that [Eq. (54) in the lecture notes]
\[ G^R(\nu,\omega) = \int_{-\infty}^{\infty} d\omega' \frac{A(\nu,\omega')}{\omega - \omega' + i\eta}. \] (5)

3. Calculating $G^R(\nu, t)$ from $G^R(\nu, \omega)$ by contour integration.

In the lectures we calculated $G^R(\nu, \omega)$ from a knowledge of $G^R(\nu, t)$. But suppose that you instead know $G^R(\nu, \omega)$ and want to calculate $G^R(\nu, t)$ from it. Starting from the Lehmann representation for $G^R(\nu, \omega)$ [Eq. (50) in the lecture notes], calculate
\[ G^R(\nu, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-i\omega t} G^R(\nu, \omega) \] (6)
by using **contour integration**: Consider the cases \( t > 0 \) and \( t < 0 \) separately. For each case, identify which half-plane the contour must be closed in. By using the residue theorem to evaluate the resulting contour integrals, show that

\[
G^R(\nu, t) = -i\theta(t) \frac{1}{Z} \sum_{n,m} (e^{-\beta E_n} + e^{-\beta E_m}) e^{i(E_n - E_m)t} |\langle m|c^\dagger_\nu|n\rangle|^2
\]

which is Eq. (48) in the lecture notes.

4. Fermi liquids.

Consider the following simplified model for the spectral function of a Fermi liquid for wavevectors \( k \) in the vicinity of the Fermi surface:

\[
A(k\sigma, \omega) = Z\delta(\omega - \xi_k) + (1 - Z) \frac{\theta(W - |\omega|)}{2W}.
\]

Here \( Z \) and \( W \) are constants (real and positive, with \( Z \leq 1 \)), and \( \theta(x) \) is the Heaviside (step) function.

(a) Briefly compare and contrast this expression with the spectral function of a Fermi liquid as discussed in the lecture notes [Eq. (60)].

(b) Show that (8) satisfies the sum rule

\[
\int_{-\infty}^{\infty} d\omega \ A(k\sigma, \omega) = 1.
\]

(c) Show that at zero temperature the momentum distribution function \( \langle c^\dagger_{k\sigma} c_{k\sigma} \rangle \) of the Fermi liquid described by (8) has a jump of magnitude \( Z \) as \( k = |k| \) crosses the Fermi surface.