1. Relationship between $G^<(v,w)$ and the spectral function $A(v,w)$

(a) $G^<(v,t-t') = i \langle c^+_v(t') c_v(t) \rangle$

(for fermions and $v=\nu$)

$$= i \frac{1}{Z} \text{Tr} \left( e^{-\beta H} e^{i H t'} c^+_v e^{-i H t'} e^{i H t} c_v e^{-i H t} \right)$$

Evaluate trace using eigenstates of $H$ as basis.

(inserting $I = \sum_m |m\rangle \langle m|$ twice)

$$G^<(v,t-t') = \frac{i}{Z} \sum_{m,n} \langle m | e^{-\beta H} e^{i H t'} c^+_v e^{-i H t'} | n \rangle \langle n | e^{i H t} c_v e^{-i H t} | m \rangle$$

$$= \frac{i}{Z} \sum_{m,n} e^{\beta E_m} e^{i E_m t'} e^{-i E_n t'} \langle m | c^+_v | n \rangle e^{i E_n t} e^{-i E_m t} \langle n | c_v | m \rangle$$

$$= \frac{i}{Z} \sum_{m,n} e^{\beta E_m} e^{(E_n - E_m)(t-t')} | \langle m | c^+_v | n \rangle |^2$$
The Fourier transform is

\[ G^<(v,w) = \int_{-\infty}^{\infty} dt \, e^{iwt} \, G^<(v,t) \]

\[ = \frac{i}{Z} \sum_{\text{nm}} e^{-\beta E_m} |\langle m | c_v^+ | n \rangle|^2 \cdot \int_{-\infty}^{\infty} dt \, e^{i(\omega + E_n - E_m)t} \frac{1}{2\pi} \delta(\omega + E_n - E_m) \]

\[ = \frac{2\pi i}{Z} \sum_{\text{nm}} e^{-\beta E_m} |\langle m | c_v^+ | n \rangle|^2 \delta(\omega + E_n - E_m) \]

(b) \( A(v,w) \) is given by [Eq. (53) in notes]

\[ A(v,w) = \frac{1}{Z} \sum_{\text{nm}} |\langle m | c_v^+ | n \rangle|^2 \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \cdot \delta(\omega + E_n - E_m) \]

The \( \delta \)-function gives \( E_n = E_m - \omega \)

\[ e^{-\beta E_n} + e^{-\beta E_m} = e^{-\beta E_m} (1 + e^{\beta \omega}) \]

Because \( (1 + e^{\beta \omega}) \) is independent of \( m,n \) it can be taken outside the \( \text{m,n} \) sums.
\[
A(v, w) = (1 + e^{i w}) \frac{1}{2} \sum \frac{1}{n} \left| \langle m \mid c_v^+ l_n \rangle \right|^2 e^{-\beta E_m} \delta(w + E_n - E_m)
\]

(c) This gives, by comparing the expressions for \( G^<(v, w) \) and \( A(v, w) \), that

\[
+ i \cdot 2\pi \cdot (1 + e^{i w})^{-1} \frac{A(v, w)}{\eta_F(w)} = G^<(v, w)
\]
2. An alternative form of the Lehmann representation.

Consider the spectral function as given by Eq. (53) in the notes,

\[ A(v,w) = \frac{i}{2} \sum_{n,m} \langle m| c_v^+ |l_n \rangle^2 \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \delta(w + E_n - E_m) \]

This gives

\[ \int_{-\infty}^{\infty} \frac{A(v,w')}{w - w' + i\eta} \]

\[ = \frac{i}{2} \sum_{n,m} \langle m| c_v^+ |l_n \rangle^2 \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \]

\[ \cdot \int_{-\infty}^{\infty} \frac{\delta(w' + E_n - E_m)}{w - w' + i\eta} \]

\[ \cdot \frac{1}{w - (E_m - E_n) + i\eta} \]

\[ = \frac{1}{2} \sum_{n,m} \frac{\langle m| c_v^+ |l_n \rangle^2}{w + E_n - E_m + i\eta} \left( e^{-\beta E_n} + e^{-\beta E_m} \right), \]

which we recognize as the Lehmann representation for \( G^R(v,w) \), Eq. (50) in the notes. QED.
4. **Calculating $G^R(v, w)$ from $G^R(v, jw)$ by contour integration.**

We have

$$G^R(v, jw) = \frac{1}{Z} \sum_{n,m} \frac{|\langle m | c^+_n | n \rangle|^2}{w + E_n - E_m + j\eta} \left( e^{-\beta E_n} + e^{-\beta E_m} \right)$$

Want to calculate

$$G^R(v, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \, e^{-iwt} \, G^R(v, jw)$$
\[
\frac{1}{Z} \sum_n \left| \langle m | c^+ | n \rangle \right|^2 \left( e^{-\beta E_n} + e^{-\beta E_m} \right)
\]
\[
\cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} dw \frac{e^{-iwt}}{\omega + E_n - E_m + i\eta}
\]

We will do the \( \omega \)-integral using contour integration. The contour will be taken to be a semicircle in the upper or lower half plane, together with the integral over the real axis. For complex \( \omega \) we have

\[
e^{-iwt} = e^{-i(Re\omega + iJm\omega)t}
\]
\[
= e^{-itRe\omega} e^{tJm\omega}
\]

For \( t > 0 \), \( e^{tJm\omega} \) will lead to decay if \( Jm \omega < 0 \). Therefore, for \( t > 0 \), the semicircle must be in the lower half plane for the integral of the integral to vanish as radius of semicircle goes to \( \infty \):

![Diagram](image-url)
Calculate integral using residue theorem.

The integrand has a pole at

\[ w = E_m - E_n - iy \]

i.e. in the lower half plane, inside the integration contour \( C \).

Thus we get, for \( t > 0 \),

\[
\int_{-\infty}^{\infty} dw \frac{e^{-iwt}}{w + En - Em + iy} = \int_{C} dw \frac{e^{-iwt}}{w + En - Em + iy}
\]

\[
= 2\pi i \cdot (-1) \cdot \frac{\text{Res}}{w = Em - En - iy} \left( \frac{e^{-iwt}}{w + En - Em - iy} \right)
\]

because contour is clockwise

\[
= -2\pi i \cdot e^{-i(Em - En - iy)t}
\]

\[
\lim_{\eta \to 0} - 2\pi i \cdot e^{i(Em - En)t}
\]
On the other hand, for \( t < 0 \) the contour must be closed in the upper half plane \( \text{Im} w > 0 \) since only then does \( e^{t \text{Im} w} \) cause the necessary convergence of the semicircle part of the integral:

\[
\int_{C} \psi (\omega) \frac{1}{2 \pi i} \frac{1}{w - \omega} \text{d} \omega \quad (t < 0)
\]

Integration contour for \( t < 0 \) (radius \( R \to \infty \))

Now there are no poles inside the contour, so the residue theorem gives that the integral vanishes. Thus

\[
G_{(v, t)}^{R} = \frac{1}{Z} \sum_{n,m} \left| \langle m | c^\dagger | n \rangle \right|^2 \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \cdot \frac{1}{2 \pi} \cdot (-2 \pi i) e^{i(E_n - E_m) t} \Theta(t)
\]

\[
= -i \Theta(t) \frac{1}{Z} \sum_{n,m} \left( e^{-\beta E_n} + e^{-\beta E_m} \right) \cdot e^{i(E_n - E_m) t} \left| \langle m | c^\dagger | n \rangle \right|^2
\]

\[\text{QED}\]
4. Fermi liquids.

(a) The spectral function of a Fermi liquid was given in the notes [Eq. (60)]:

\[
A_g(k\sigma, \omega) = Z_k \cdot \frac{1}{\pi} \left( \frac{1/2\tau_k}{(\omega - \xi_k)^2 + (1/2\tau_k)^2} \right) + A_{\text{incoherent}}(k\sigma, \omega). \tag{1}
\]

Here we have introduced the subscript \(g\) to stand for “general”, to indicate that this is the most general form of the spectral function for a Fermi liquid. On the other hand, the Fermi liquid spectral function given in the problem text is

\[
A_s(k\sigma, \omega) = Z \cdot \delta(\omega - \xi_k) + (1 - Z) \frac{\theta(W - |\omega|)}{2W}, \tag{2}
\]

where the introduced subscript \(s\) stands for “simplified”, as this expression is the spectral function associated with a simplified model of a Fermi liquid. We have the following correspondences between factors in the two expressions:

- \(Z_k \Rightarrow Z\). Thus in the simplified model, the \(k\)-dependence of \(Z_k\) is neglected.
- \(\frac{1}{\pi} \left( \frac{1/2\tau_k}{(\omega - \xi_k)^2 + (1/2\tau_k)^2} \right) \Rightarrow \delta(\omega - \xi_k)\). Thus in the simplified model, the finite lifetime \(\tau_k\) of the quasi-particles has been neglected, i.e. the quasi-particles have been taken to have an infinite lifetime. Also the renormalization (due to interactions) of the quasi-particle dispersion from \(\xi_k\) to some other function \(\xi_k^*\) has been neglected.
- \(A_{\text{incoherent}}(k\sigma, \omega) \Rightarrow (1 - Z) \frac{\theta(W - |\omega|)}{2W}\). Thus in the simplified model a simple function that is step-wise constant has been used to represent the “incoherent” background.

(b) \[
\int_{-\infty}^{\infty} d\omega \ A_s(k\sigma, \omega) = Z \int_{-\infty}^{\infty} d\omega \ \delta(\omega - \xi_k) + \frac{1 - Z}{2W} \int_{-\infty}^{\infty} d\omega \ \theta(W - |\omega|) = 1. \tag{3}
\]

(c) From Eq. (62) in the notes we have that the momentum distribution function of the Fermi liquid described by the spectral function \(A_s\) is given by

\[
\langle c_{k\sigma}^\dagger c_{k\sigma} \rangle = \int_{-\infty}^{\infty} d\omega \ A_s(k\sigma, \omega) n_F(\omega). \tag{4}
\]

At zero temperature, \(n_F(\omega) = \theta(-\omega)\), giving

\[
\langle c_{k\sigma}^\dagger c_{k\sigma} \rangle = \int_{-\infty}^{\infty} d\omega \ A_s(k\sigma, \omega) \theta(-\omega). \tag{5}
\]

Inserting for \(A_s\) and doing the \(\omega\) integration gives

\[
\langle c_{k\sigma}^\dagger c_{k\sigma} \rangle = Z \theta(-\xi_k) + \frac{1}{2}(1 - Z). \tag{6}
\]

From this expression one sees that when \(|k|\) crosses the Fermi surface from above, i.e. when \(\xi_k\) changes sign from positive to negative, the momentum distribution function jumps by \(Z\). Note that in the non-interacting case, \(Z = 1\), this expression for the momentum distribution function reduces to the (zero-temperature) Fermi-Dirac distribution \(n_F(\xi_k)\), for which the jump is 1 at the Fermi surface.