1. Matsubara and retarded Green functions for noninteracting bosons.

(a) Show that for noninteracting bosons with Hamiltonian

\[ H_0 = \sum_\nu \xi_\nu c_\nu^\dagger c_\nu, \]  

the Matsubara single-particle Green function \( G^{(0)}(\nu, \tau) = -\langle T_\tau (c_\nu(\tau)c_\nu^\dagger(0)) \rangle \) is given by

\[ G^{(0)}(\nu, \tau) = -[\theta(\tau)(1 + n_B(\xi_\nu)) + \theta(-\tau)n_B(\xi_\nu)]e^{-\xi_\nu \tau}. \]  

[To prove this result, use that for noninteracting bosons, \( \langle c_\nu^\dagger c_\nu \rangle = n_B(\xi_\nu) \), where

\[ n_B(\xi_\nu) \equiv \frac{1}{e^{\beta \xi_\nu} - 1} \]  

is the Bose-Einstein distribution function.]

(b) Show that the Fourier transformed function is given by

\[ G^{(0)}(\nu, i\omega_n) = \frac{1}{i\omega_n - \xi_\nu}. \]  

Use this to find the retarded Green function for noninteracting bosons.

2. Impurity scattering: Impurity average and Feynman diagrams.

In this problem \( j_1, j_2 \) etc. are defined as in Sec. 4.4 of the lecture notes, i.e. \( j_1 \) refers to the site summation variable in the first factor \( \rho(k - k_1) \) in the product of \( \rho \) functions, \( j_2 \) refers to the site summation variable in the second factor \( \rho(k_1 - k_2) \), etc.

(a) Calculate the contribution to \( \rho(k - k_1)\rho(k_1 - k_2)\rho(k_2 - k'_3) \) from the case \( j_1 = j_3 \neq j_2 \). Write down the corresponding contribution to \( \tilde{G}^{(3)}(k, k') \) and draw the corresponding Feynman diagram.

(b) Calculate the contribution to \( \rho(k - k_1)\rho(k_1 - k_2)\rho(k_2 - k_3)\rho(k_3 - k'_4) \) from the case \( j_1 = j_3 \neq j_2 = j_4 \). Write down the corresponding contribution to \( \tilde{G}^{(4)}(k, k') \) and draw the corresponding Feynman diagram.
3. Impurity scattering: Feynman diagrams at order $n = 4$.

(a) Draw all the Feynman diagrams that appear at order $n = 4$ in the perturbation expansion for the impurity-averaged single-particle imaginary-time (Matsubara) Green function. Group the diagrams according to the number of impurity crosses they contain.

(b) Identify (and draw) the new self-energy diagrams that appear at this order. Give the mathematical expression for each of them.