1. Dirac equation in two spatial dimensions with an external magnetic field.

In this problem we consider the Dirac equation in two spatial dimensions (described by cartesian coordinates $x$ and $y$). We set $c = 1$ and $\hbar = 1$ for simplicity.

(a) Show that a valid representation (to be used in the following) for the $\alpha$ and $\beta$ matrices is

$$\beta = \sigma_3, \quad \alpha_1 = \sigma_1, \quad \alpha_2 = \sigma_2.$$  \hspace{1cm} (1)

(b) An external magnetic field of magnitude $B$ is applied in the $z$ direction (i.e. perpendicular to the plane of motion): $\mathbf{B} = B\hat{z}$. In terms of electromagnetic potentials this can be represented as $\phi = 0, \mathbf{A} = (0, Bx, 0)$. Show that the Dirac equation for a particle of charge $q$ in the presence of the magnetic field can be rewritten as

$$(i\sigma_3 \partial_t - \sigma_2 \partial_x + \sigma_1 \partial_y - iqBx\sigma_1 - m)\Psi = 0.$$  \hspace{1cm} (2)

(c) Explain why it is natural to use the ansatz (here $p_y$ is a number)

$$\Psi(x, y, t) = e^{ip_y y - iEt} \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right).$$  \hspace{1cm} (3)

(d) Use the ansatz to derive an equation of the form $Q \left( \begin{array}{c} f(x) \\ g(x) \end{array} \right) = 0$, where $Q$ is a $2 \times 2$ matrix. Simplify expressions by introducing the operators

$$\xi_\pm = -i\partial_x \mp i(p_y - qBx).$$  \hspace{1cm} (4)

(e) Eliminate the function $g(x)$ from the problem to obtain an equation for $f(x)$ alone.

(f) Solve this equation, thus finding the solutions for both $f(x)$ and $E$ (hint: harmonic oscillator). Finally, find $g(x)$.

2. Green functions for a Heisenberg ferromagnet.

In this problem we consider a Heisenberg ferromagnet. We write the Hamiltonian as

$$H = \frac{1}{2} \sum_{i,j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j = \frac{1}{2} \sum_{i,j} J_{ij} \left[ \frac{1}{2}(S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z \right]$$  \hspace{1cm} (5)

with $J_{ij} \leq 0$. We also assume that $J_{ij}$ is only a function of $\mathbf{r}_i - \mathbf{r}_j$, so that $H$ is translationally invariant. Let us define the retarded Green function

$$R_{ij}(t) \equiv -i\theta(t) \langle [S_i^-(t), S_j^+(0)] \rangle$$  \hspace{1cm} (6)
where the time evolution of the operators is defined as $A(t) \equiv e^{iHt}A(0)e^{-iHt}$.

(a) Show that $R_{ij}(t)$ satisfies the equation of motion

$$i\frac{\partial R_{ij}(t)}{\partial t} = -2\delta_{ij}\delta(t)\langle S_i^z \rangle - i\theta(t)\langle [S_i^-(t),H],S_j^+(0)]\rangle.$$  \hfill (7)

(b) Show that

$$[S_i^-(t),H] = -\sum_{j'} J_{ij'}(S_{j'}^-S_i^z(t) - S_j^z(t)S_{j'}^-)$$ \hfill (8)

(c) Upon inserting (8) into (7), the rhs can be seen to involve expressions of the type $-i\theta(t)\langle [S_i^-(t),S_j^+(0)]\rangle$ which are retarded Green functions of higher order than $R_{ij}(t)$. To make progress, we will approximate these higher-order Green functions in terms of the $R$-functions. This will be done by replacing the operator $S_j^z$ by its expectation value $\langle S_j^z \rangle$ (here we omitted the site index in the expectation value, since this becomes site-independent due to the translational invariance of the Hamiltonian). Show that this “mean-field” approximation gives

$$i\frac{\partial R_{ij}(t)}{\partial t} = -\langle S_i^z \rangle \left[ 2\delta_{ij}\delta(t) + \sum_{j'} J_{ij'}(R_{j'j}(t) - R_{ij}(t)) \right].$$ \hfill (9)

(d) Next, we specialize to zero temperature, for which the ferromagnetic ground state has $\langle S_i^z \rangle = S$, where $S$ is the total quantum number of the spins. Introduce $R(k,t)$ as the space Fourier transform of $R_{ij}(t)$ and show that it satisfies

$$i\frac{\partial R(k,t)}{\partial t} = -2S\delta(t) - \Omega_k R(k,t)$$ \hfill (10)

where we defined

$$\Omega_k = S(J_k - J_0)$$ \hfill (11)

where $J_k$ is the Fourier transform of $J_{ij}$.

(e) Introduce $R(k,\omega)$ as the time Fourier transform of $R(k,t)$ and show that it is given by

$$R(k,\omega) = \frac{-2S}{\omega + \Omega_k + i\eta}$$ \hfill (12)

where $\eta = 0^+$ (introduced to ensure that $R(k,t) \propto \theta(t)$).

(f) Specialize now to the case of nearest-neighbour interactions on a hypercubic lattice, i.e. $J_{ij} = -|J|$ for all nearest neighbours and 0 otherwise. Show that $\Omega_k = 2|J|S\sum_\delta (1 - \cos k \cdot \delta)$ which is nothing but the spin-wave dispersion $\omega_k$ that we found earlier in the course when studying this problem using spin-wave theory.