Stability of Gaussian Process Learning Based Output Feedback Model Predictive Control

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Nowadays, model predictive control (MPC) has established itself as a state-of-the-art control approach in many different areas such as, for instance, the process industry, robotics, or building climate control. It is well known that a reasonable accurate prediction model is critical to achieve good performance and safe operation. However, for many applications no or only limited model knowledge is available. In this case learned models that are based on input-output data are a valuable alternative.

In this work we deal with nonlinear discrete time systems represented by a nonlinear autoregressive model with exogenous input (NARX) of the form

$$y_{k+1} = f(x_k, u_k, d_k) + \epsilon \tag{1a}$$

$$\text{s.t. } u_k \in \mathcal{U} \ , \tag{1b}$$

where x_k is the NARX state vector $x_k = [y_k, \ldots, y_{k-m_\mathrm{y}}, u_{k-1}, \ldots, u_{k-m_\mathrm{u}}]$ that consists of the current and past outputs y and inputs u. The output is corrupted by white Gaussian noise ϵ. The system is furthermore subject to hard input constraints $\mathcal{U}$ and soft output constraints $\mathcal{Y}$.

The control objective is to steer the system from an initial equilibrium to a target reference equilibrium, while satisfying the input and output constraints. To this end we employ model predictive control and a Gaussian process [1, 2] to represent the NARX model, which is then purely based on measured input-output data. Note that a GP prediction model is relatively easy and always possible to construct, as long as informative input-output data is available. We use input data $w = (x_k, u_k)$ and output observations y_{k+1} to train the GP model. The resulting posterior mean function $m^+(w)$ is then used as the next predicted output y_{k+1}. The resulting NARX prediction (2) is then used in the optimal control problem. Furthermore $\bar{f}(x, u) \triangleq f(x, u, 0)$ is the nominal model of (1a).

We establish stability by designing a suitable terminal cost function $V_\mathrm{f}(\cdot)$ and terminal control law $\kappa_\mathrm{f}(\cdot)$. We consider a formulation that does not require a terminal region by means of weighting V_f with a factor $\lambda \geq 1$, as proposed in [3]. We first establish stability for the nominal case, i.e., when the true system and the prediction/nominal model are exactly the same. Based on the nominal stability we show that the real process controlled by the proposed predictive controller is input-to-state (ISS) stable w.r.t. the one step estimation error (between the true and the nominal model) if this error is bounded and if the true model function and the nominal model, using the GP, are

$$
\begin{aligned}
x_{k+1} = \bar{F}(x_k, u_k) &= [\bar{f}(x_k, u_k), y_k, \ldots, y_{k+1-m_{\mathrm{y}}}, u_k, \ldots, u_{k+1-m_{\mathrm{u}}}] \\
&= [m^+(w), y_k, \ldots, y_{k+1-m_{\mathrm{y}}}, u_k, \ldots, u_{k+1-m_{\mathrm{u}}}]
\end{aligned} \tag{2}
$$

uniformly continuous.

Uniform continuity of the nominal model can be ensured if the posterior mean function of the GP is uniformly continuous. This can be achieved if the prior mean is uniform continuous and if continuously differentiable kernels (e.g. the squared exponential covariance function, the Matérn class covariance function with appropriate hyperparameters, or the rational quadratic covariance function) are employed. In that case the process is mean square differentiable [4, 1], i.e., the posterior mean function is differentiable and therefore also uniformly continuous.

Since the closed-loop system is ISS, there exists a stability margin in the form of an upper bound on the control error $\mathit{\Omega}(\mu)$ as a function of the norm of the one step estimation error μ. Furthermore, if the probability that the uncertainty is bounded by μ is ρ, then the system will converge to $\mathit{\Omega}(\mu)$ with that same probability.

The proposed combination of a GP-NARX prediction model and the output feedback MPC is illustrated using a continuous stirred-tank reactor example [5]. We compare the outcome with an output feedback MPC scheme that uses the true system dynamics for the prediction.

References

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