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Uncertainty quantification for fractional order PI control system: Polynomial chaos approach

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Abstract: Stability and performance of a system can be inferred from the evolution of statistical characteristic (i.e. mean, variance…) of system states. The polynomial chaos of Wiener provides a computationally effective framework for uncertainty quantification of stochastic dynamics in terms of statistical characteristic. In this work, polynomial chaos is used for uncertainty quantification of fractional order PI control system under the uncertainties both in parameters and additive stochastic disturbance.

Keywords: Polynomial Chaos, PI $^{\lambda}$, fractional calculus, fractional order system, stochastic process.

1. INTRODUCTION

In recent years, increasing attention has been given to fractional calculus as a powerful tool for more precise modelling of real world phenomena (Chen, 2006; Weron & Kotulski, 1996; Baleanu, Guvenc, & Machado 2010). So far, fractional order control systems have been studied mostly with deterministic setting. In reality, stochastic uncertainty may arise in systems when the physics governing the system is known and the system parameters are either unknown precisely or expected to vary in the operational lifetime. Such uncertainty also occurs when system models are built from experimental data using system identification techniques, where a system plant is represented by its transfer function with unknown parameters. As a result, the values of the parameters in the transfer function have a range of uncertainty. In order to include this uncertainty in the mathematical model, probabilistic methods have been developed.

Traditional probabilistic approaches to uncertainty quantification include the Monte Carlo (MC) method (Dunn & Shultis, 2011; Pupkov & Egupov, 2003) and its variants—for example, Latin Hypercube Sampling (McKay, Conover, & Beckman, 1979), which generate ensembles of random realizations for the prescribed random inputs and use repetitive deterministic solvers for each realization. Although such methods are straightforward to apply, their convergence rates can be relatively slow. For example, the mean value of MC typically converges as $1/\sqrt{K}$, where K is the number of realization. The need for a large number of samples for accurate results can cause an excessive computational burden.

The recently developed stochastic generalized polynomial chaos (gPC) methods (Xiu, 2010; Xiu & Karnidakis 2003; Pupkov, Egupov, Makarenkov, & Trofimov, 2003) offer faster convergence for problems with relatively large random uncertainties. With the gPC approach, stochastic solutions are approximated as series of orthogonal polynomial of the random uncertainties, and the statistical characteristic of solution can be obtained from gPC coefficients expansion.

This work extended the gPC framework to a fractional order model with PI $^{\lambda}$ controller to show how the method can be utilized to predict the stochastic performance of a fractional order controller with stochastic uncertainties.

The paper is organized as follows. Section 2 introduces fractional linear single input single output system. Section 3 briefly presents the gPC method for predicting the dynamic performance in terms of the mean and variance of output. Section 4 considers numerical examples for validating the method. Section 5 concludes the result.

2. FRACTIONAL ORDER DYNAMICS

Fractional calculus is the generalization of integration and differentiation to a fundamental operator of non-integer order (Monje , Chen, Vinagre, Xue, & Feliu, 2010):

$$
{}_0D_t^{\alpha} = \begin{cases} d^{\alpha}/dt^{\alpha} & \alpha > 0 \\ 1 & \alpha = 0 \\ \int_0^t (d\tau)^{-\alpha} & \alpha < 0 \end{cases} \tag{1}
$$

where $\alpha \in R$ is the order of the operator.
Among the several formulas of the generalized derivative, one of most commonly used is the Riemann-Liouville definition:

$$ {}_0D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} (\frac{d}{dt})^m \int_0^t \frac{f(\tau)}{(t-\tau)^{1-(m-\alpha)}} d\tau \quad (2) $$

The Rieman-Liouville fractional integral of a function, $f(t)$, is defined by:

$$ (I_0^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \quad (3) $$

The Laplace transform for fractional order under zeros initial condition is defined as:

$$ L\{D^{\alpha}{}_t f(t)\} = s^{\alpha} F(s) \quad (4) $$

Thus, a fractional order single input single output (SISO) system can be described by the fractional order differential equation:

$$ \begin{aligned} & a_1 D_0^{\alpha_1} y(t) + \ldots + a_l D_0^{\alpha_l} y(t) + y(t) \\ & = b_1 D_0^{\beta_1} u(t) + \ldots + b_m D_0^{\beta_m} u(t) + b_{m+1} u(t) \end{aligned} \quad (5) $$

or by the transfer function:

$$ G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + \ldots + b_0 s^{\beta_0}}{s^{\alpha_l} + \ldots + a_0 s^{\alpha_0}} \quad (6) $$

where α_i and β_j are arbitrary real positive numbers, *u(t)* and *y(t)* are the system's input and output, respectively.

3. STATISTICAL ANALYSIS OF CONTROL SYSTEM WITH GENERALIZED POLYNOMIAL CHAOS

3.1 Orthogonal polynomial with given distributions

Let $\rho(\xi)$ be the probability density function (pdf) of a scalar random variable ξ, which has finite moments of order up to. $2m$, $m \in \mathbb{N}$. Let P denotes the space of real polynomial; $\mathrm{P}_m \subset \mathrm{P}$ denotes the space of polynomial with degree up to *m*. The inner product of two polynomials p and q relative to measure $d\lambda = \rho(\xi)d\xi$ is defined by:

$$ (p,q)_{d\lambda} = \int_{\Gamma} p(\xi)q(\xi)\rho(\xi)d\xi \quad (7) $$

where Γ is the support domain of random variable.
Orthonormal polynomials relative to the probability measure $d\lambda$ are given by three term recurrence (Gautschi, 2003):

$$ \begin{aligned} & \sqrt{\beta_{k+1}}\phi_{k+1}(\xi) = (\xi - \alpha_k)\phi_k(\xi) - \sqrt{\beta_k}\phi_{k-1}(\xi), \; k = 0,1,\ldots, \\ & \phi_{-1}(\xi) = 0 \;, \phi_0(\xi) = 1/\sqrt{\beta_0} \end{aligned} \quad (8) $$

where the recurrence coefficients are given by:

$$ \begin{aligned} & \alpha_k(d\lambda) = \frac{(\xi\phi_k, \phi_k)_{d\lambda}}{(\phi_k, \phi_k)_{d\lambda}}, \\ & \beta_k(d\lambda) = \frac{(\xi\phi_k, \phi_k)_{d\lambda}}{(\phi_k, \phi_k)_{d\lambda}}, \qquad \beta_0 = \int_{\Gamma} d\lambda(\xi) \quad k = 0,1,\ldots, \end{aligned} \quad (9) $$

The index range is infinite $k \le \infty$ or finite $\mathrm{k} \le \mathrm{m}$, depending on whether the inner product is positive definite on P or P_m. Hence, the first *m* recursion coefficient pairs α_k and β_k are uniquely determined by the first *2m* moments of ξ. Classical methods such as a Chebyshev method, which are based on moments, could be used for obtaining the recurrence coefficients. Unfortunately, obtaining these coefficients from moments in general becomes severely ill conditioned and thus is not useful even for well-behaved measures for which there are no classical orthogonal polynomials. As an alternative, approximate methods based on discretization of an arbitrary measure can be employed to obtain the recursion coefficients for orthogonal polynomial set. The basic idea behind the discretization method is to approximate the inner product in (7) by a discrete measure:

$$ \begin{aligned} & (p,q)_{d\lambda} = \int_{\Gamma} p(\xi)q(\xi)\rho(\xi)d\xi = (p,q)_{d\lambda_S} \\ & = \sum_{\nu=1}^{S} w_{\nu} p(\xi^{(\nu)}) q(\xi^{(\nu)}) \end{aligned} \quad (10) $$

In the discretization method, the number of discrete S point measure increases until the discrete version of recurrence coefficients with desired accuracy is achieved. Gaustchi (2003) also showed the performance of the discretization method can be improved by utilizing the multi-component discretization. Note that this discretization procedure is for obtaining the polynomial set, which has nothing related to the discretized fractional order system.

3.2 Statistical analysis of fractional order system

Assume that a control system is governed by (5) with zeros initial conditions and subject to independent stochastic uncertainties. Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)$ denotes a random vector of mutually independent random components of system with pdfs of $\rho_i(\xi_i) : \Gamma_i \to \mathbb{R}^+$.Thus, the joint probability density of the random vector $\boldsymbol{\xi}$ is $\boldsymbol{\rho} = \prod_{i=1}^{N} \rho_i$, and the support of $\boldsymbol{\xi}$ is $\boldsymbol{\Gamma} \equiv \prod_{i=1}^{N} \Gamma_i \in \mathbb{R}^N$. The set of one dimensional orthonormal polynomials $\{\phi_i(\xi_i)_m^{d_i}\}$ can be defined in the dimensional space Γ_i , with respect to the weight $\rho_i(\xi)$ as in section 3.1. Based on the one-dimensional set of polynomials, an *N*-variate orthonormal set can be constructed with *P* total degrees in the space $\boldsymbol{\Gamma}$ by using the tensor product of the one-dimensional polynomials. Note that the basis polynomial of the *N*-variate also satisfies orthonormal condition:

$$ \int_{\Gamma} \Phi_m(\boldsymbol{\xi})\Phi_n(\boldsymbol{\xi})\boldsymbol{\rho}(\boldsymbol{\xi})d\boldsymbol{\xi} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \quad (11) $$

Considering a response function of output $f(y(t,\boldsymbol{\xi}))$, with statistics (e.g., mean, variance) of interest; the *N*-variate P^{th}

order approximation of this response function can be constructed as:

$$f_N^{\,P}(y(t,\xi)) = \sum_{i=1}^{M} \hat{f}_m(t)\Phi_m(\xi)$$
$$M+1 = \binom{N+P}{N} = \frac{(N+P)!}{N!P!} \qquad (12)$$

where P is the order of polynomial chaos, and $\hat{f}_m$ the coefficient of the gPC expansion that satisfies (13) as:

$$\hat{f}_m = \mathbf{E}[\Phi_m f(y)] = \int_\Gamma f(y)\Phi_m(\xi)\rho(\xi)d\xi \qquad (13)$$

where E[] denotes the expectation operator.

Since the function $f(y(t,\xi))$ is not known explicitly in general, the coefficients of gPC expansion in (13) have to be computed numerically as described below:

For each random parameter ξ_i, construct a uni-variate orthonormal polynomial using the algorithm in section 3.1 with respect to its pdf. Based on the orthogonal polynomial, construct the one dimensional quadrature $\left\{\xi_i^{(m)}, w_i^{(m)}\right\}_{m=1}^{q_i}$ so that one-dimensional integration can be accurately approximated by:

$$\int_{\Gamma_i} f(y(t,\xi_i))d\xi_i = \sum_{m=1}^{q_i} f(y(\xi_i^{(m)}))w_i^{(m)} \qquad (14)$$

Different type of one dimensional quadrature sets with respect to arbitrary set of orthogonal functions can be obtained using the algorithm and software suite OPQ from Gautschi (2003).

Construct a multi-dimensional quadrature set by tensorizing the one-dimensional quadrature set. Hence, the integration (13) is approximated by:

$$\hat{f}_m = \mathbf{E}[\Phi_m f(y)] = \int_\Gamma f(y)\Phi_m(\xi)\rho(\xi)d\xi$$
$$= \sum_{j_1=1}^{q_1}\cdots\sum_{j_N=1}^{q_N} g_m(y(\xi_1^{(j_1)},\ldots,\xi_n^{(j_n)}))(w_1^{(j_1)}\ldots w_n^{(j_n)}) \qquad (15)$$

where

$$g_m(y(\xi_1^{(j_1)},\ldots,\xi_n^{(j_n)}))$$
$$= f(y(\xi_1^{(j_1)},\ldots,\xi_n^{(j_n)}))\Phi_m(\xi_1^{(j_1)},\ldots,\xi_n^{(j_n)}) \qquad (16)$$

After obtaining the coefficients of gPC expansions, a post-processing procedure is carried out to obtain the mean and variance.

The mean of response function is estimated as:

$$M[f(t,y(\xi))] = \int_\Gamma f(t,\xi)\boldsymbol{\rho}(\xi)d\xi = \int_\Gamma \Phi_1(\xi)\sum_{i=1}^{M}\hat{f}_m(t)\Phi_m(\xi)d\xi$$
$$= \hat{f}_1(t) \qquad (17)$$

Note that the mean is the first coefficient of gPC expansion.

Next, the variance is obtained as:

$$\mathrm{var}[f(t,y(\xi))]$$
$$= \int_\Gamma \Big(\sum_{i=1}^{M}\hat{f}_m(t)\Phi_m(\xi) - \hat{f}_1\Big)\Big(\sum_{i=1}^{M}\hat{f}_m(t)\Phi_m(\xi) - \hat{f}_1\Big)\boldsymbol{\rho}(\boldsymbol{\xi})\mathbf{d}\boldsymbol{\xi} \qquad (18)$$
$$= \sum_{i=2}^{M}\hat{f}_i(t)^2$$

Thus, the variance is the sum of square of all coefficients in gPC expansion except the first. Note that the polynomial set starts with $\Phi_1(\xi) = 1$ and the probability density function is equivalent to the weighting function. These properties are utilized in deriving (17) and (18). If the response function $f(y(t,\xi)) = y(t,\xi)$, the mean and variance of system output are given by truncated series (17) and (18).

4. EXAMPLES

4.1 Example 1

Consider a closed-loop configuration in Fig.1. The gPC method is used to quantify uncertainty in the closed-loop configuration in Fig.1 for a fractional order system (Magin, 2006; Gyochuk & Hanggi 2004) with a PI$^{\lambda}$ controller taken from Luo, Chen, Wang, & Pi (2010):

$$P(s) = \frac{1}{0.4s^{0.5}+1}; C(s) = 0.1817(1+194.4/s^{1.216}) \qquad (19)$$

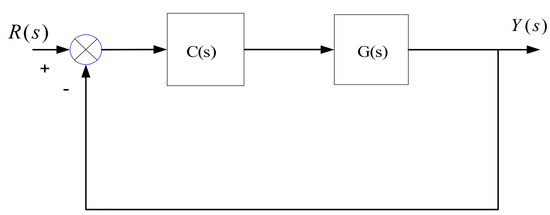

Fig. 1. A closed-loop control system.

The input *R(t)* is a band-limited Gaussian white noise process with zero mean and covariance function of:

$$\kappa_{RR} = 0.25\,\mathrm{sinc}\left(\frac{t_1 - t_2}{2\pi}\right) \qquad (20)$$

where the sinc function is defined by:

$$\mathrm{sinc}(x) = \begin{cases} \sin(\pi x)/(\pi x) \text{ elsewhere} \\ 1 \text{ for } x = 0 \end{cases} \tag{21}$$

Utilizing the non-canonical decomposition method (Chernhecski, 1969), the above band-limited white noise with bandwidth 0.5 *rad/s* is parameterized as

$$\begin{aligned} &R(t) = 0.5(\sin(\xi_2 t) + \xi_1 \cos(\xi_2 t)) \\ &E[\xi_1] = 0; E[\xi_1^{\,2}] = 1 \\ &\text{pdf of } \xi_2\text{: } \rho_{\xi_2}(\xi_2) = 1,\ \xi_2 \in [-0.5, 0.5] \end{aligned} \tag{22}$$

Since the variance is a measure of variability of random process, a large variance implies large deviations from nominal response. Hence, in this work the variance is used for characterizing the performance of system under the stochastic input. The variance estimated by the gPC is shown in Fig. 2. For validation, the estimated variance by the MC method is also shown in Fig. 2. The gPC method solves (6) with the nodes taken from the cubature and obtains statistical characteristics using the weight from the cubature set, while the MC solves it using random numbers generated from the given distributions and estimates statistical characteristics as a weighted sum, with equally weights, 1/M. Hence, in the gPC method the number of cubature node plays the same role as the number of samples M in the MC method. The computational time and simulation parameters for both methods are given in Table 1. The mean output is zero since the mean input is zero. From Fig .2 and Table 1, it is apparent that the computational effectiveness of the gPC is far superior to the MC.

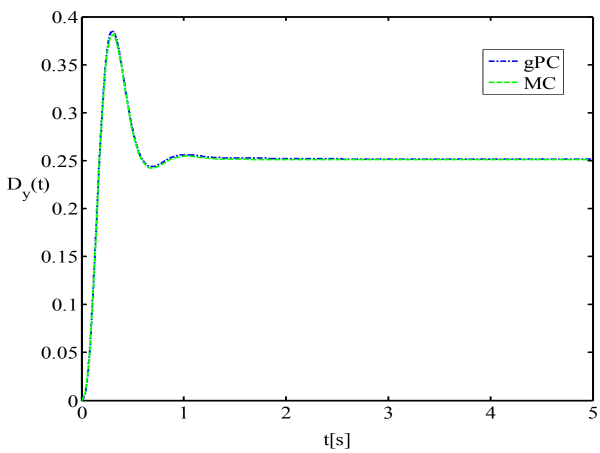

Fig. 2. Variances of system output.

Next, the gPC is used to analyze the effect of the bandwidth of limited bandwidth white noise on the variance of system output. Fig. 3 presents the variance of the above system output under excitation of band-limited white noises:

$$\kappa_{RR}(\tau) = 0.25\mathrm{sinc}(\frac{B}{\pi}\tau), B = 0.5, 1, 2, 4 \tag{23}$$

with parameterization

$$\begin{aligned} &R(t) = 0.5(\sin(\xi_2 t) + \xi_1 \cos(\xi_2 t)) \\ &E[\xi_1] = 0; E[\xi_1^{\,2}] = 1 \\ &\text{pdf of } \xi_2\text{: } \rho_{\xi_2}(\xi_2) = 1/2B,\ \xi_2 \in [-B, B] \end{aligned} \tag{24}$$

The fractional order was integrated using the code fodesol (Monje et.al, 2010). From the figure, it can be seen that as the bandwidth increases, the steady state variance increases while the maximum variance at transient response decreases.

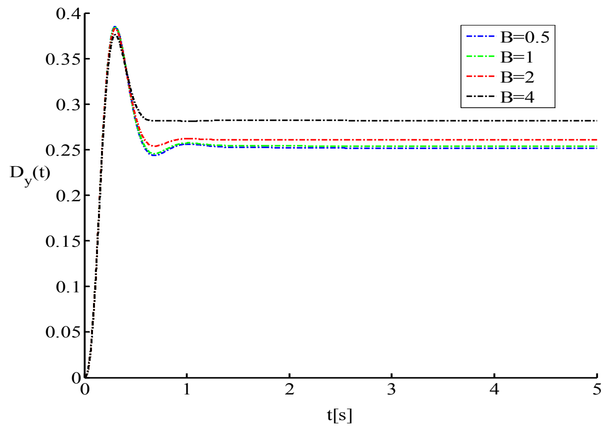

Fig. 3. Variances of system output with different bandwidth of the stochastic inputs.

Table 1. Simulation parameters and computational time for obtaining statistical characteristics by the MC and the gPC.

	Simulation parameters		Computational time (s)	
Example	MC (Samples)	gPC(cubature nodes)	MC	gPC
1	6400	36	2823.5 2	2.14
2	7225	225	2915.3	8.9
3	900	36	549.3	1.9

4.2 Example 2

The gPC is applied to quantify the uncertainties for the same system above where the input is zero mean and covariance functions:

$$\kappa_{RR} = 0.25\,\mathrm{e}^{(-2|t_1-t_2|)} \tag{25}$$

The above first order Markov noise can be parameterized by non-canonical decomposition as:

$$R(t) = 0.5(\sin(\xi_2 t) + \xi_1 \cos(\xi_2 t))$$
$$E[\xi_1] = 0; E[\xi_1^2] = 1 \tag{26}$$
$$\text{pdf of } \xi_2\text{: } \rho_{\xi_2}(\xi_2) = \frac{2}{\pi(4+\xi_2^2)},\ \xi_2 \in [-\infty, \infty]$$

The variance of system output by the gPC and the MC are presented in Fig.4. Since this signal can be interpreted as steady state output of a filter $F(s) = \frac{1}{s+2}$ under excitation of ideal white noise with covariance $\delta(t_1 - t_2)$, the exact steady state output can be easily calculated by frequency method as $\|FCG/(1+CG)\|_2^2$, where $\|\ \|_2$ denote the H_2 norm of the system. The H_2 norm for a fractional order system was calculated using numerical integration approach (Monje et.al, 2010). It can be seen that the variance estimated by the gPC method almost coincides with the exact value at steady state while the MC method not. This is because the MC method cannot give accurate results when the number of samples is not enough. The computational times for obtaining the system variances are given in Table 1. Since no orthogonal sets can be constructed with respect to the weight function for ξ_2, the Legendre polynomial has to be first used for constructing the Legendre quadrature set for uniform random component on $\varsigma_2 \in [0,1]$. Then, the iso-probabilistic transform (Lemaitre & Knio, 2010) is used to transform a quadrature set with respect to uniform distribution to a quadrature set for ξ_2.

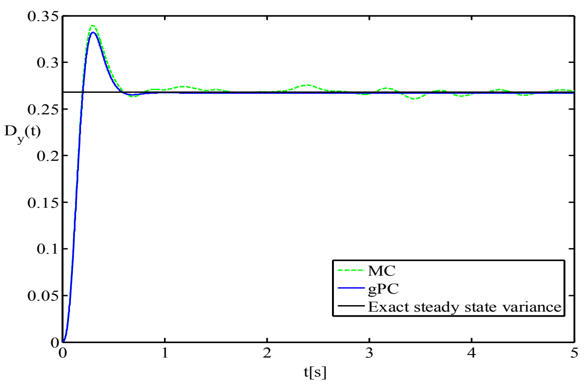

Fig. 4. Variance of output

Next, the system is driven by different random inputs with zero means and covariance functions:

$$\kappa_{RR} = 0.25\,\mathrm{e}^{(-\alpha|t_1-t_2|)}, \alpha = 1,2,3,4 \tag{26}$$

with parameterization:

$$R(t) = 0.5(\sin(\xi_2 t) + \xi_1 \cos(\xi_2 t))$$
$$E[\xi_1] = 0; E[\xi_1^2] = 1 \quad . \tag{27}$$
$$\text{pdf of } \xi_2\text{: } \rho_{\xi_2}(\xi_2) = \frac{\alpha}{\pi(\alpha^2+\xi_2^2)},\ \xi_2 \in [-\infty, \infty]$$

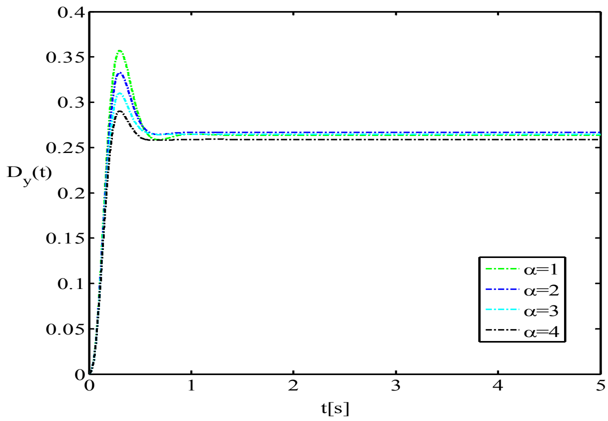

Fig. 5. Variances of system output under excitation of Markov process with different correlation length

Fig.5 demonstrates how the reciprocal correlation length α of the Markov process affects the variance of system output.

4.3 Example 3

In this example, the gPC methodology is applied for uncertainty quantifications of fractional order system with the same fractional PI$^{\lambda}$ controller as in the first two examples:

$$G(s) = \frac{k_1}{k_2 s^{0.5} + 1} \tag{28}$$

where k_1 and k_2 are random variables with triangular distributions Tr(0.5,1,1.5). The triangular distributions Tr(a,c,b) is defined by a cumulative distribution function (Forbes, Evans, Hating , & Peacock , 2011) :

$$Tr(a,c,b) = \begin{cases} \dfrac{(x-a)^2}{(b-a)(c-a)} & \text{if } a \le x \le c \\ 1 - \dfrac{(b-x)^2}{(b-a)(b-c)} & \text{if } c \le x \le b \end{cases} \tag{29}$$

The output means and variances for this example under unit step input are shown in Fig. 6. Computational time for this example is also given in Table 1. In contrast to the stochastic input case, the variance of the system output goes to zero as time increases because all output trajectories go to the same setpoint value at steady state.

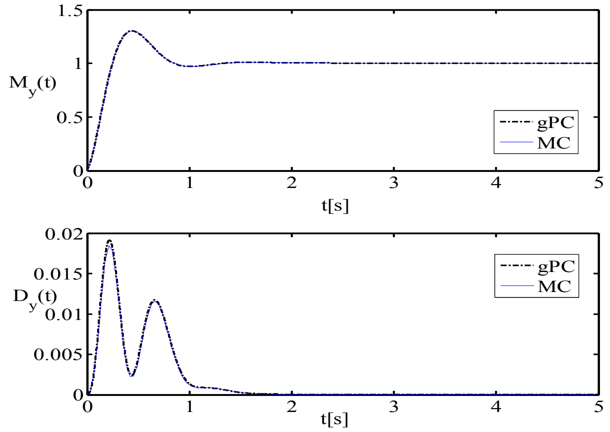

Fig. 6. Means and variances of system output under the stochastic parameter perturbations.

5. CONCLUSIONS

Variance is one of the mostly used indices for evaluating stochastic performance of system under stochastic perturbations. In this work, the stochastic performance of fractional order system measured by output variance was studied. The considered examples demonstrated the proposed method's accuracy and computational efficiency of the gPC method over the existing MC. The gPC method can give variance in both transient and steady state of system instead of steady state only as in the frequency method. From the advantages of the gPC method, it is suggested for performance analysis of stochastic fractional order system. The gPC was then utilized for analysing the stochastic performance of fractional order systems under different types of stochastic perturbation. Although the method was considered for a simple fractional order system with a PI^{λ} controller, the method can be easily extended to more complex type controllers and systems.

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REFERENCES

Baleanu, D., Guvenc Z.B., & Machado, T.J.A (eds.). (2010). *New trend in Nanotechnology and fractional calculus applications*. Springer, Dordrecht.

Chen, W. (2010). A speculative study of 2/3-order fractional Laplacian modelling of turbulence; some though and conjectures. *Chaos*, 16, 02316.

Chernhecki, V.I. (1969). *Analysis accuracy of nonlinear control systems*. Masintroenhie.

Dunn, L.W., & Shultis, J.K. (2011). *Exploring Monte Carlo Methods*. Elsevier.

Forbes, C., Evans, M., Hasting, N., & Peacock, B.(2011) *Statistical Distributions*. John Wiley & Son, Hoboken, New Jersey.

Gaustchi, W. (2003). *Orthogonal Polynomials: Computation and approximations*. Oxford University Press.

Gochyuk, I., & Hanggi, P. (2004). Fractional diffusion modelling of ion channel gatin. *Phys. Rev. E*, 70, 051915.

Luo, Y., Chen, Y.Q., Wang, C.Y., & Pi Y.G. (2010). Tuning fractional order proportional integral controllers for fractional order systems. *J. Process Control*, 20, 823-831.

Maitre, L.O.P, Knio, O.M. (2010).*Spectral method for uncertainties quantifications with application to computational fluid dynamics*.Springer.

Margin, L.(2006). *Fractional Calculus in Bioengineering*. Begell House Publisher Inc.

McKay, M., Conover, W., & Beckman, R. (1979). A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. *Technometrics*, 21, 239-245.

Monje, A.C., Chen, Y., Vingare, M.B., Xue, D., & Feliu,V. (2010). *Fractional-order systems and controls*: *Fundamental and applications*. Springer, London.

Pupkov , K.A., & Egupov , N.D. (eds.). (2003). *Classical and modern theory of control systems v.2*. BMSTU, Moscow.

Pupkov , K.A., Egupov, N.D., Makarenkov, A.M., Trofimov, A.I. (2003). *Theory and Numerical Methods for Studying Stochastic Systems*, Fizmatlits, Moscow.

Weron, K., & Kotulski, M. (1996). On the cole-cole relaxation functions and related Mittag-Leffler distribution. *Physica A*, 232,180-188.

Xiu, D. (2011). *Numerical method for stochastic Computation: Spectral approach*. Princeton University Press.

Xiu, D., & Karnidakis, G.E. (2003). The Wiener -Askey polynomial chaos for stochastic differential equation. *SIAM J.Sci.Comput.*, 24(2), 619-644.