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Nonlinear fractional PI control of a class of fractional-order systems

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Abstract: This paper deals with the design of nonlinear PI control techniques for regulating a class of fractional-order dynamics governed by a commensurate-order model, possibly nonlinear, perturbed by an external disturbance. The suggested control algorithm is the combination between a fractional-order PI controller and a nonlinear robust version of it, namely a second-order sliding mode control algorithm called ”super-twisting” controller in the literature. A key feature of the approach is the use of ad-hoc sliding manifolds whose construction involves fractional order derivatives. A constructive Lyapunov based synthesis is illustrated, which leads to simple tuning rules for the controller parameters guaranteeing the asymptotic rejection of the external disturbance under appropriate smoothness restrictions. Computer simulations illustrate the effectiveness of the proposed technique.

Keywords: Fractional order systems. Fractional order controllers. Nonlinear PI control. Sliding mode control.

1. INTRODUCTION

Fractional–order systems (FOSs), i.e. dynamical systems described using fractional (or, more precisely, non-integer) order derivative and integral operators, are studied with growing interest in recent years. It has been pointed out that a large number of physical phenomena can be modeled effectively by means of fractional–order models (see Sabatier *et al.* (2007)).

The long-range temporal or spatial dependence phenomena inherent to the FOS present unique and intriguing peculiarities, not supported by their integer-order counterpart, which raise numerous challenges and opportunities related to the development of control and estimation methodologies involving fractional order dynamics (see Vinagre *et al.* (2002); Ladaci *et al.* (2006); Podlubny (1999a)).

The pioneering applications of fractions calculus in control theory date back to the sixties (see Manabe (1961)). In the nineties, Oustaloup and his group proposed a non-integer robust control strategy named CRONE (*Commande Robuste d'Ordre Non-Entier*) (see Oustaloup *et al.* (1996)). Another well-known fractional control algorithm is the fractional-order PID (FPID, or $PI^{\lambda}D^{\mu}$) controller introduced by Podlubny (see Podlubny (1999a,b)). Recently, fractional calculus is penetrating other nonlinear control paradigms as well such as the model-reference adaptive control (see Vinagre *et al.* (2002); Ladaci *et al.* (2006)).

It is the task of this paper to study the properties of a control scheme for FOSs that combines together a fractional PI controller and a nonlinear version of it. The latter is a Sliding Mode Control (SMC) algorithm called ”super-twisting” controller in the literature (see Levant (1993)).

Although fractional calculus has been previously combined with the sliding mode control methodology in the controller design for integer-order systems (see Efe *et al.* (2008); Calderno *et al.* (2006)), SMC techniques have been applied to fractional-order systems only recently, (see Si-Ammour *et al.* (2009); Efe (2009)). In Efe (2009), nonlinear single-input fractional-order dynamics expressed in a form that can be considered as a fractional-order version of the chain-of-integrators “Brunowsky” normal form were studied, which will be the class of reference in this work, too. Noticeably, sliding manifolds containing fractional-order derivatives were used in Si-Ammour *et al.* (2009) in combination with conventional relay control techniques. The same type of sliding manifolds has been later used, in combination with second-order sliding mode control methodologies, to address control, observation and fault detection tasks for certain classes of uncertain linear FOS (see Pisano *et al.* (2010, 2011)).

In this paper we consider a class of nonlinear FOSs expressed in the previously mentioned chain-of-integrators

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form and we investigate the stability properties of a control scheme combining the fractional PI controller and a fractional nonlinear PI algorithm. Unlike the standard *stand-alone* PI controller, which can asymptotically reject constant disturbances only, this combined scheme proves to be capable of asymptotically rejecting a nonvanishing disturbance of arbitrary shape and fulfilling the unique constraint that its time derivative as uniformly bounded by an a-priori known constant starting from some finite time instant on. Convergence to zero of the system variables, and asymptotic rejection of a class of matched disturbances, will be demonstrated by means of a Lyapunov approach.

The paper is structured as follows: in Section II the main definitions and properties of fractional order derivatives and integrals are recalled, with emphasis on their composition which plays an important role in the present developments. Section III states the control problem under investigation and presents the main result, namely the Lyapunov based stability analysis of the combined linear/nonlinear PI controller in question. Section IV presents some computer simulations, and the final Section V gives some concluding remarks and perspectives for next related research activities.

2. FRACTIONAL OPERATORS AND THEIR PROPERTIES

Definition 1. (Left) **Riemann-Louville fractional integral** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \qquad (1)$$

where $\Gamma(\cdot)$ denotes the Euler gamma function (see Kilbas *et al.* (2006)).

For integer values of α, relation (1) reduces to the well-known Cauchy repeated integration formula (see Podlubny (1999a)). It can also be shown that when α approaches zero the fractional integral (1) reduces to the identity operator (see (Saichev *et al.* (1996))). In the current paper, the fractional integral of order zero is taken to represent the identity operator by definition, i.e.

$$I^0 f(t) = f(t). \qquad (2)$$

Definition 2. (Left) **Riemann-Liouville fractional derivative** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the n^{th} derivative of the left Riemann-Liouville fractional integral of order $n-\alpha$, where n is the smallest integer greater than, or equal to, α

$$^{RL}D^{\alpha} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha} f(t). \qquad (3)$$

Definition 3. (Left) **Caputo fractional derivative** of order $\alpha > 0$ of a given signal $f(t)$ at time instant $t \geq 0$ is defined as the left Riemann-Liouville fractional integral of order $n-\alpha$ of the n^{th} derivative of f, where n is the smallest integer greater than, or equal to, α

$$^{C}D^{\alpha} f(t) = I^{n-\alpha} \left(\frac{d}{dt}\right)^n f(t). \qquad (4)$$

For $\alpha \in (0,1)$ the Riemann-Liouville and Caputo derivatives are related by the next equation, that will be used in the sequel

$$^{RL}D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(0)}{t^{\alpha}} + {}^{C}D^{\alpha} f(t) \qquad (5)$$

Since the fractional integral of order zero is the identity operator, in accordance with (2), it is obvious that both definitions of fractional derivative reduce to the classical derivative of order n when $\alpha = n$. Particularly, when the differentiation order is zero, both definitions of fractional derivatives reduce to the identity operator. The next useful properties of the fractional integral and differential operators will be used in the sequel. The proofs can be found in a number of well-known textbooks (see e.g. (Kilbas *et al.* (2006)) and (Podlubny (1999a))).

Lemma 1. The left Riemann-Liouville fractional integral satisfies the **semigroup property**. Let $\alpha > 0$ and $\beta > 0$, then

$$I^{\alpha}\left\{I^{\beta} f(t)\right\} = I^{\beta}\left\{I^{\alpha} f(t)\right\} = I^{\alpha+\beta} f(t) \qquad (6)$$

Lemma 2. The left Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$ is the left inverse of the Left Riemann-Liouville fractional integral of the same order,

$$^{RL}D^{\alpha}\left\{I^{\alpha} f(t)\right\} = f(t), \qquad (7)$$

for almost all $t \geq 0$. The opposite is, however, not true, since

$$I^{\alpha}\left\{^{RL}D^{\alpha} f(t)\right\} = f(t) - \frac{f_{1-\alpha}(0)}{\Gamma(\alpha)} t^{\alpha-1}, \qquad (8)$$

where $f_{1-\alpha}(0) = \lim_{t\to 0} I^{1-\alpha} f(t)$.

Lemma 3. The following is true when applying left Riemann-Liouville fractional integral operator to the left Caputo fractional derivative of the same order of a signal $f(t)$

$$I^{\alpha}\left\{^{C}D^{\alpha} f(t)\right\} = f(t) - f(0). \qquad (9)$$

It is important to notice that, unlike the classical derivative, the fractional derivative operators **do not commute**. In general, in fact, one has that

$$^{RL}D^{\alpha}\left\{^{RL}D^{\beta} f(t)\right\} \neq {}^{RL}D^{\beta}\left\{^{RL}D^{\alpha} f(t)\right\} \neq {}^{RL}D^{\alpha+\beta} f(t),$$
$$^{C}D^{\alpha}\left\{^{C}D^{\beta} f(t)\right\} \neq {}^{C}D^{\beta}\left\{^{C}D^{\alpha} f(t)\right\} \neq {}^{C}D^{\alpha+\beta} f(t)$$

However, the following equalities hold true for all $\alpha > 0$ and $n \in \mathbb{N}$

$$\frac{d^n}{dt^n}\left\{^{RL}D^{\alpha} f(t)\right\} = {}^{RL}D^{n+\alpha} f(t), \qquad (10)$$

$$^{C}D^{\alpha}\left\{\frac{d^n}{dt^n} f(t)\right\} = {}^{C}D^{n+\alpha} f(t). \qquad (11)$$

The next Lemma, that will be instrumental in the present treatment, was proven in (Pisano *et al.* (2010)).

Lemma 4. Consider an arbitrary signal $z(t) \in \mathbb{R}$. Let $\beta \in (0,1)$. If there exists $T < \infty$ such that

$$I^{\beta} z(t) = 0 \qquad \forall t \geq T \qquad (12)$$

then

$$\lim_{t\to\infty} z(t) = 0. \qquad (13)$$

3. NONLINEAR FRACTIONAL PI CONTROL FOR SISO FOS

We consider nonlinear uncertain commensurate-order fractional systems governed by the "chain of (fractional) integrators" dynamic model

$$\begin{aligned} {}^C D^\alpha x_1 &= x_2 \\ {}^C D^\alpha x_2 &= x_3 \\ &\vdots \\ {}^C D^\alpha x_{n-1} &= x_n \\ {}^C D^\alpha x_n &= f(\mathbf{x}, t) + u(t) + \psi(t). \end{aligned} \tag{14}$$

where $\alpha \in (0,1)$ is the commensurate order of differentiation of (14), vector $\mathbf{x}(t) = [x_1(t),\ x_2(t), ..., x_n(t)] \in \mathbb{R}^n$ collects the process internal variables (the notion of state variables is inappropriate and generally not used in the context of FOS), $u(t) \in \mathbb{R}$ is the control input, $\psi(t) \in \mathbb{R}$ is an exogenous disturbance, and $f(x,t) : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ is a nonlinear functions referred to as the "drift term". We refer to the Caputo definition of fractional derivatives as it allows to take into account a finite and physically reasonable initial condition $\mathbf{x}(0)$ for the process internal variables.

The external disturbance $\psi(t)$ is supposed to fulfill the next restriction

Assumption 1. The exists an a-priori known constant M such that

$$\left| \frac{d}{dt}\psi(t) \right| \leq M, \qquad t \geq 0. \tag{15}$$

The aim is that of finding a control law capable of steering the variables of the closed loop process to the origin regardless of the presence of the unknown disturbance term $\psi(t)$, satisfying the Assumption 1.

Consider the fractional order sliding variable

$$\sigma(t) = I^{(1-\alpha)} \left[x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t) \right], \tag{16}$$

where the constants $c_1, c_2, ..., c_{n-1}$ are selected in such a way that all the roots p_i of the polynomial

$$P(s) = s^{(n-1)} + \sum_{i=0}^{n-2} c_{i+1} s^i = \Pi_{i=1}^{n-1}(s - p_i) \tag{17}$$

satisfy the next relation

$$\alpha\frac{\pi}{2} < arg(p_i) \leq \pi. \tag{18}$$

The stability of system (14) once constrained to evolve along the sliding manifold $\sigma(t) = 0$ is analyzed in the next Lemma 5. A controller capable of steering the considered dynamics onto the sliding manifold in finite time will be illustrated later on.

Lemma 5. Consider system (14) and let the zeroing of the sliding variable (16) be fulfilled starting from the finite moment t_1, i.e. let

$$\sigma(t) = 0, \qquad t \geq t_1, \quad t_1 < \infty, \tag{19}$$

with the c_i parameters in (16) satisfying (17)-(18). Then, the next conditions hold

$$\lim_{t\to\infty} x_i(t) = 0, \qquad i = 1, 2, ..., n \tag{20}$$

Proof of Lemma 5 Define the quantity

$$\xi(t) = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t). \tag{21}$$

By taking into account Lemma 4 specialized with $\beta = 1-\alpha$ and $z(t) = \xi(t)$, it yields that the finite time zeroing of $\sigma(t)$ guarantees that signal $\xi(t)$ decays asymptotically to zero. We then simply derive from (21) that

$$x_n(t) = -\sum_{i=1}^{n-1} c_i x_i(t) + \xi(t) \tag{22}$$

where

$$\lim_{t\to\infty} \xi(t) = 0. \tag{23}$$

Now, in light of (22) we rewrite the first $n-1$ equations of (14) as

$$\begin{aligned} {}^C D^\alpha x_1 &= x_2 \\ {}^C D^\alpha x_2 &= x_3 \\ &\vdots \\ {}^C D^\alpha x_{n-1} &= -\sum_{i=1}^{n-1} c_i x_i(t) + \xi(t) \end{aligned} \tag{24}$$

and notice that (24) form a reduced-order (as compared to (14)) fractional order system with an asymptotically decaying input term $\xi(t)$. It readily follows from (17)-(18) that system (24) is Mittag-Leffler stable when $\xi(t) = 0$ (see Podlubny (1999a)), thereby the input decay property (23) implies the same for the process variables $x_i(t)$ with $i = 1, 2, ..., n-1$. We now conclude from (22) that $x_n(t)$ asymptotically decays, too. Lemma 5 is proved. □

It is worth to remark that the enforcement of conditions (21), (23) actually "cancels" the last equation of (14) by making the system to behave as the reduced order one (24). We shall treat the sliding variable $\sigma(t)$ in (16) as the error variable, and we seek for a control law expressed in the form

$$u(t) = u^p(t) + u^i(t) + u^{eq}(t) \tag{25}$$

where $u^p(t)$ and $u^i(t)$ are, respectively, nonlinear versions of the proportional and integral control actions taking the form

$$u^p(t) = -k_1\sigma - k_2|\sigma|^{1/2} sign\,(\sigma) \tag{26}$$

$$\dot{u}^i(t) = -k_3\sigma - k_4 sign(\sigma), \quad u^i(0) = 0 \tag{27}$$

and $u^{eq}(t)$ is a control component that will be specified later on. By setting constants k_2 and k_4 to zero then the sum of the control components (26)-(27) reduces to the standard PI controller. On the other hand, by setting k_1 and k_3 to zero one obtains the well-known "super-twisting" controller (see Levant (1993)), which belongs to the family of second-order sliding mode controllers. The similarity

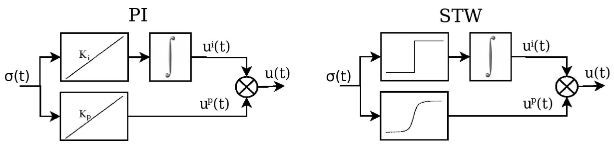

Fig. 1. Architecture Comparison between linear (left) and nonlinear PI.

between a classical PI controller and the super-twisting (STW) one are evident (see Figure 1) in that they both possess a static component (a pure gain, for the PI, and a nonlinear gain with infinite slope at 0 for the STW) and an integral component (a pure integration, for the PI, and the integration of the sign of the sliding variable, for the STW).

We are now in position to state the next main result.

Theorem 1. Consider system (14) along with the sliding variable (16)-(18), and let Assumption 1 be in force. Then, the control law (25)-(27), specified with

$$u^{eq}(t) = -f(\mathbf{x}, t) - \sum_{i=1}^{n-1} c_i x_{i+1}(t), \tag{28}$$

and with the tuning parameters chosen according to

$$k_1 > 0, \qquad k_2 > 2\sqrt{\rho}, \qquad k_4 > \rho, \tag{29}$$

$$k_3 > k_1^2 \frac{k_2^2 + \frac{5}{2}[(\frac{1}{4}k_2^2 - \rho) + k_2 k_4]}{(\frac{1}{4}k_2^2 - \rho) + k_2 k_4}. \tag{30}$$

where

$$\rho > M, \tag{31}$$

provides the asymptotic decay of the state $x(t)$.

Proof of Theorem 1 By virtue of Definition 2, specified with $n = 1$ and $f(t) = x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t)$, and exploiting as well the linearity of the fractional derivative operator, one can easily derive that

$$\begin{aligned} \frac{d}{dt}\sigma(t) &= {}^{RL}D^{\alpha}\left[x_n(t) + \sum_{i=1}^{n-1} c_i x_i(t)\right] \\ &= {}^{RL}D^{\alpha}x_n(t) + \sum_{i=1}^{n-1} c_i\, {}^{RL}D^{\alpha}x_i(t) \end{aligned} \tag{32}$$

In light of relation (5), eq. (32) can be rewritten in terms of Caputo derivatives as follows

$$\frac{d}{dt}\sigma(t) = {}^{C}D^{\alpha}x_n(t) + \sum_{i=1}^{n-1} c_i\, {}^{C}D^{\alpha}x_i(t) + \varphi(t) \tag{33}$$

where

$$\varphi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{x_n(0) + \sum_{i=1}^{n-1} c_i x_i(0)}{t^{\alpha}} = \frac{K_0}{t^{\alpha}} \tag{34}$$

with implicitly defined constant $K_0 = \frac{x_n(0) + \sum_{i=1}^{n-1} c_i x_i(0)}{\Gamma(1-\alpha)}$.

The system equations (14) can be now substituted into (33), yielding the simplified expression

$$\dot{\sigma}(t) = f(\mathbf{x}, t) + u(t) + \psi(t) + \sum_{i=1}^{n-1} c_i x_{i+1}(t) + \varphi(t) \tag{35}$$

Although the disturbance (34) and all its time derivatives are unbounded at $t = 0$, one has that the first-order time derivative

$$\frac{d}{dt}\varphi(t) = -\frac{\alpha K_0}{t^{\alpha+1}} \tag{36}$$

is bounded, in magnitude, along any time interval $t \in [t_1, \infty)$, $t_1 > 0$, according to

$$\left|\frac{d}{dt}\varphi(t)\right| \le \frac{\alpha K_1}{t_1^{\alpha+1}} \equiv \Psi_1, \quad K_1 = \left|x_n(0) + \sum_{i=1}^{n-1} c_i x_i(0)\right|. \tag{37}$$

We now substitute the control (25)-(28) into (35), yielding

$$\frac{d}{dt}\sigma = -k_1\sigma - k_2|\sigma|^{1/2} sign\,(\sigma) + u^i(t) + \psi(t) + \alpha(t) \tag{38}$$

$$\frac{d}{dt}u^i = -k_3\sigma - k_4 sign(\sigma) \tag{39}$$

Define

$$z(t) = u^i(t) + \psi(t) + \alpha(t) \tag{40}$$

and rewrite (38)-(39) as

$$\frac{d}{dt}\sigma = -k_1\sigma - k_2|\sigma|^{1/2} sign\,(\sigma) + z(t) \tag{41}$$

$$\frac{d}{dt}z = -k_3\sigma - k_4 sign(\sigma) + \frac{d}{dt}\psi(t) + \frac{d}{dt}\alpha(t) \tag{42}$$

Notice that, by Assumption 1 and by relation (37), the perturbation terms in (39) fulfill the next estimation

$$\left|\frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t)\right| \le M + \Psi_1, \quad t \ge t_1 > 0 \tag{43}$$

with arbitrary $t_1 > 0$. Thus, by setting ρ as in (31) it readily follows that there exist a finite moment $t_2 > 0$ such that $|\frac{d}{dt}\psi(t) + \frac{d}{dt}\varphi(t)| \le \rho$ at every $t \ge t_2$.

Stability of a class of systems including the dynamics (41)-(43) above was already investigated in the literature (cfr. Moreno *et al.* (2008), Th. 5), where, particularly, the global finite time stability of the uncertain system trajectories was demonstrated by means of a positive definite and radially-unbounded non-smooth Lyapunov function which specifies as follows in the present context

$$V = \xi^T \Pi \xi, \quad \xi = \begin{bmatrix} |\sigma|^{1/2}\mathrm{sign}(\sigma) \\ \sigma \\ z \end{bmatrix}, \tag{44}$$

$$\Pi = \frac{1}{2}\begin{bmatrix} (4k_4 + k_2^2) & k_1 k_2 & -k_2 \\ k_1 k_2 & 2k_3 + k_1^2 & -k_1 \\ -k_2 & -k_1 & 2 \end{bmatrix}. \tag{45}$$

It turns out after the appropriate computations (cfr. Moreno *et al.* (2008), Proof of Th. 5) that the tuning conditions (29)-(31) imply the existence of a positive constant γ_1 such that

$$\frac{d}{dt}V \le -\gamma_1\sqrt{V}, \qquad t \ge t_1. \tag{46}$$

Inequality (46) guarantees the global **finite time** convergence of V to zero, and, hence, the same property for the $\sigma(t)$ and $z(t)$ variables. By (38), the finite time convergence to zero of $\frac{d}{dt}\sigma(t)$ can be easily concluded, too. The asymptotic decay of $\mathbf{x}(t)$, thus, readily follows from Lemma 4. Theorem 1 is proven. □.

3.1 Uncertain drift term

Now assume that the uncertain drift term $f(\mathbf{x},t)$ is imprecisely known by means of a certain estimate $\hat{f}(\mathbf{x},t)$. We shall devise a sufficient condition guaranteeing that the previously presented control, with the estimate $\hat{f}(\mathbf{x},t)$ used in (28) in place of the actual function $f(\mathbf{x},t)$, guarantees the same robust performance demonstrated in Theorem 1. Denote

$$\epsilon(\mathbf{x},t) = f(\mathbf{x},t) - \hat{f}(\mathbf{x},t) \tag{47}$$

and assume what follows

Assumption 2 There is an a-priori fixed constant W such that

$$\left| \frac{d}{dt}\epsilon(\mathbf{x},t) \right| \leq W \tag{48}$$

Under Assumption 2, it can be developed a synthesis procedure similar to that in Theorem 1, which leads to the same controller and tuning inequalities (29)-(30) for the gains (k_1,k_2,k_3,k_4) , with the new restriction $\rho > M + W$ to be imposed instead of (31). A rigorous treatment could be be straightforwardly derived from the previously given one, and it is omitted for brevity. It is, however, complicated to get an a-priori estimate of W as generally the time derivative of $\epsilon(\mathbf{x},t)$ may explicitly depend not only on the process variables vector $\mathbf{x}$ but on the control input u as well, which is why this modified control deserves more accurate investigations that will be made in next research.

4. SIMULATION RESULTS

Consider system

$$\begin{aligned} {}^{C}D^{0.5}x_1 &= x_2 \\ {}^{C}D^{0.5}x_2 &= x_3 \\ {}^{C}D^{0.5}x_3 &= f(\mathbf{x},t) + u(t) + sin(\pi t). \end{aligned} \tag{49}$$

with nonlinear drift term function $f(\mathbf{x},t) = -0.5x_1 - 0.5x_2^3 - 0.5x_3|x_3|$ and a sinusoidal time dependent matched uncertainty. This system was considered in the related publication (Efe (2009)). Constant M upperbounding the sinusoidal uncertainty time derivative according to Assumption 1 can be evaluated as $M = \pi$. The initial conditions are $x_1(0) = x_2(0) = x_3(0) = 2$. Let us bear in mind that the Caputo definition of the fractional derivative in (49) allows to take into account finite and physically meaningful initial conditions of the process variables, in opposition to what happens with the RL definition which brings infinite values for the initial conditions.

The fractional order sliding variable is defined according to (16) as

$$\sigma(t) = I^{0.5}\left[x_3(t) + c_2x_2(t) + c_1x_1(t)\right], \tag{50}$$

An effective choice for the c_1 and c_2 constants which guarantees conditions (17)-(18) is as follows

$$c_1 = \lambda^2, \quad c_2 = 2\lambda, \quad \lambda > 0 \tag{51}$$

and, particularly, is such that $p_1 = p_2 = -\lambda$.

The perfect knowledge of the drift term function $f(\mathbf{x},t)$ is assumed in the first TEST 1, and control (25)-(31), (50)-(51) has been implemented with the parameter values $\lambda = 10$, $k_1 = 10$, $k_2 = 10$, $k_3 = 15$, $k_4 = 15$.

Figure 2 shows the time evolutions of the sliding variable $\sigma(t)$ and of signal $\xi(t) = x_3(t) + 2\lambda x_2(t) + \lambda^2 x_1(t)$. Particularly, the left plots show the entire history of the signals while the right plots display a zoom of the corresponding steady-state behaviour. The upper plots of Figure 2 then confirm the finite-time convergence to zero of the chosen sliding variable, while the lower plots show that, according to Lemma 4, signal $\xi(t)$ goes to zero once the system motion is constrained along the sliding manifold $\sigma(t) = 0$ featuring the slow "creeping" behaviour exhibited by fractional order dynamics (see Podlubny (1999a)).

Figure 3 shows the time evolutions of the process variables, and figure 4 displays the control variable which, as expected, is a continuous, although non smooth function of time.

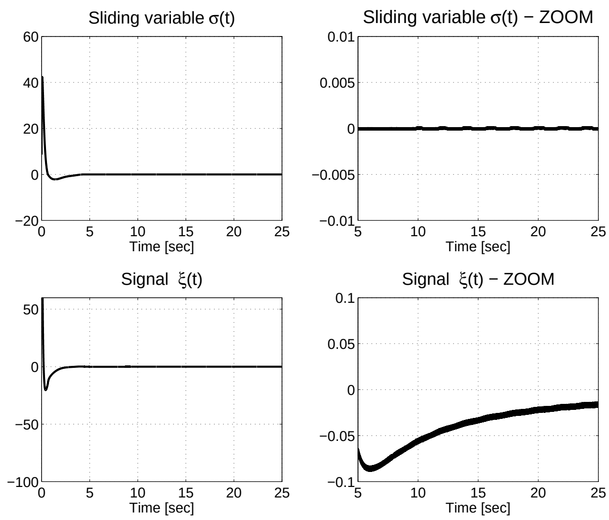

Fig. 2. TEST 1. Fractional sliding variable $\sigma(t)$ and signal $\xi(t)$.

In TEST 2, the control strategy has been implemented by assuming an uncertain drift term, according to the treatment of Subsection 3.1, and by letting its estimate be $\hat{f}(\mathbf{x},t) = 0$. Thus, in TEST 2 the control strategy $u(t) = u^p(t) + u^i(t) - c_1x_2(t) - c_2x_3(t)$ is applied, which is a mixture of fractional PI control (the first two components) and conventional proportional feedback (the last two elements). Controller parameters are increased, as compared to the setting used in TEST 1, to cope with the higher uncertainty and the values $k_1 = 20, k_2 = 20, k_3 = 30, k_4 = 40$ are used. Figure 5 shows the convergence to zero of the process variables in the TEST 2. It is concluded from the presented analysis that the proposed control scheme possesses strong robustness properties and provides good

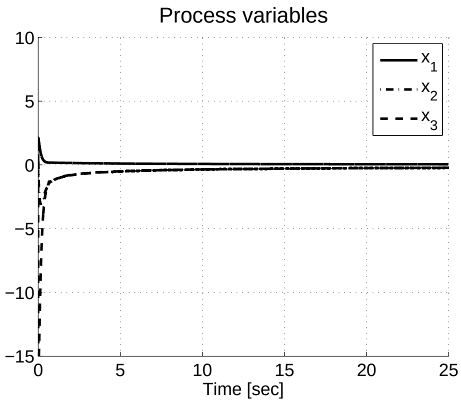

Fig. 3. TEST 1. Process variables $x_i(t)$, $i = 1, 2, 3$.

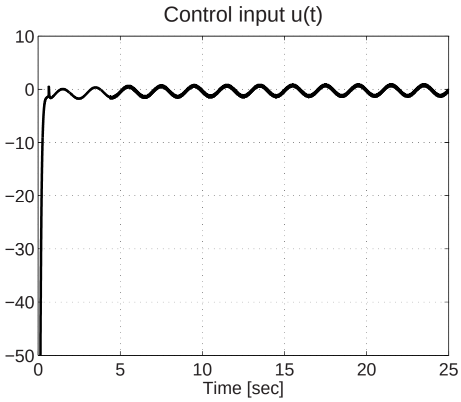

Fig. 4. TEST 1. Control input $u(t)$.

performance in presence of significant model uncertainties.

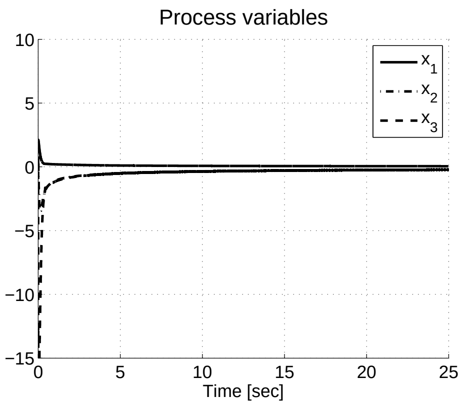

Fig. 5. TEST 2. Process variables $x_i(t)$, $i = 1, 2, 3$.

5. CONCLUSIONS

A nonlinear PI control technique has been suggested and analyzed in the framework of the regulation problem for a class of nonlinear fractional-order processes. The proposed methodology is capable of asymptotically rejecting a class of arbitrarily shaped external disturbances with uniformly bounded time derivative. Distinctive issue of the approach is a fractional order siding surface tailored to the considered class of systems. More complex tracking control problems and more general classes of plants will be investigated in the future. Additionally, new types of sliding surfaces will be sought to speed up the convergence of the process variables, which actually suffer of the "creeping" effect, namely an extremely slow convergence to the desired operating point.

REFERENCES

Calderon, A.J., Vinagre, B.M., Felix, V. (2006). On Fractional Sliding Mode Control. 7th Portuguese Conf. on Aut. Contr. (*CONTROLO 2006*), Lisbon, Portugal.

Efe, M.O., Kasnakoğlu, C. (2008). A Fractional Adaptation Law for Sliding Mode Control. *Int. J. Adapt. Contr. Sign. Proc.* 22, 968–986.

Efe, M.O. (2009). Fractional Order Sliding Mode Controller Design for Fractional Order Dynamic Systems. in Z.B. Guvenc, D. Baleanu, J.A. Tenreiro Machado, eds., *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer Verlag, Berlin.

Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations.* Elsevier, The Netherlands.

Ladaci, S. and Charef, A. (2006). On fractional adaptive control. *Nonl. Dyn.*, 43(4), 365–378.

Levant, A. (1993). Sliding order and sliding accuracy in sliding mode control", *Int. J. Contr.*, 58, 1247-1263.

Manabe, S. (1961). The non-integer integral and its application to control systems. *Electrotechnical J. of Japan* 6 (3-4), 83-87.

Moreno, J.A. and Osorio, M. (2008). A Lyapunov approach to second-order sliding mode controllers and observers. Proc. 47th IEEE Conference on Decision and Control (*CDC 2008*), Cancun (MX), 2856 - 2861.

Oustaloup, A., Moreau, X., Nouillant, M. (1996). The CRONE Suspension. *Contr. Eng. Pract.* 4(8), 1101-1108.

Pisano, A., Rapaić, M.R., Jeličić, Z.D., Usai, E. (2010). Sliding mode control approaches to the robust regulation of linear multivariable fractional-order dynamics. *Int. J. Rob. Nonlin. Contr.* 20(18), 2045–2056.

Pisano, A., Rapaić, M.R., Jeličić, Z.D., Usai, E. (2011). Second-order sliding mode approaches to disturbance estimation and fault detection in fractional-order systems. Proc. 18th IFAC World Congress, 2436–2441.

Podlubny, I (1999). Fractional Differential Equations. Academic Press

Podlubny, I. (1999) Fractional Order Systems and $PI^{\lambda}D^{\mu}$-Controllers. *IEEE Trans. Aut. Contr.* 44 (1), 208-214.

Sabatier, J., Agrawal, O.P. and Tenreiro Machado, J.A. (2007). *Advances in Fractional Calculus – Theoretical Developments and Applications* Physics and Engineering Series. Springer, Berlin.

Saichev, A., Woyczynski, W.A. (1996). Distributions in the Physical and Engineering Sciences. Vol I: Distributional and Fractal Calculus, Integral Transforms and Wavelets. Birkhauser.

Si-Ammour, A., Djennoune, S., Bettayeb, M. (2009). A sliding mode control for linear fractional systems with input and state delays. *Communicat. Nonlin. Sc. Num. Simulat.*, 14(5), 2310-2318.

Vinagre, B. M., Petras, I., Podlubny, I. and Chen, Y. Q. (2002) Using fractional order adjustment rules and fractional order reference models in model reference adaptive control. *Nonlin. Dyn.*, 29, 269–279.