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On the Stability of an Event-based PI Controller for FOPDT processes

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Abstract: This paper deals with the stability of an event-based proportional-integral controller. In particular, necessary and sufficient conditions on the controller parameters for the existence of equilibrium points without limit cycles are given for a first-order-plus-dead-time process. Practical issues related to the controller implementation are also addressed. The presented conditions enable a simpler tuning of the controller. Simulation and experimental results are provided as illustrative examples.

1. INTRODUCTION

It is well known that in some processes a small steady-state control error of the process output around the set-point does not constitute a hard design constraint but, however, the reduction of the information exchanged between the agents that take part in the control loop (sensors, controllers, actuators) is one of the tightest requirements. Indeed, the reduction of the information flow is a relevant issue especially in the presence of wireless sensors, as there are constraints on the communication rate [Otanez et al., 2002, Miskowicz, 2006]. In these situations, cutting down the traffic load is a key point because the more traffic, the higher possibility of lost data and stochastic time delays. This prevents the occurrence of large latencies and delay jitters.

In this context, one of the most convenient strategies is the use of event-based sampling and control approaches. Indeed, during last years event-based sampling and control techniques have been addressed by a large number of researchers (see, for example, [Åström, 2008, Heemels et al., 2008]) also in the context of Proportional-Integral-Derivative (PID) controllers [Årzèn, 1999, Rabi and Johansson, 2008, Sánchez et al., 2011], as these kind of controllers are the most employed controllers in industry owing to their advantageous cost\benefit ratio. Among the different event-based sampling strategies, one of the most common is the so called send-on-delta (SOD) sampling (also known as deadband sampling [Vasyutynskyy and Kabitzsh, 2007] or level crossing sampling [Kofman and Braslavsky, 2006]) where the measured value of the process variable is sent to the controller when the process variable (or some function of it) crosses predefined quantization levels [Sánchez et al., 2009].

However, it has to be recognized that the great success of (time-based) PID controllers is motivated by the fact that they are capable to provide a satisfactory performance for many processes with a relatively easy design, also because of the large number of tuning rules that are available [O'Dwyer, 2006]. Conversely, in event-based control the events occur asynchronously and therefore the tuning of the PID controller parameters is in general more challenging, as the timing of the events influences the system performance and limit cycles may arise [Vasyutynskyy, 2008] (note that the presence of limit cycles is a typical problem in general event-based control systems [Cervin and Åström, 2007]). Further, in addition to the PID gains, there are other parameters (threshold values) employed in the control algorithm that have to be tuned, thus making the overall control design more complex. Indeed, the tuning of a PID controller with deadband sampling has not been explicitly addressed in the literature until now, at least to the authors' knowledge.

In this paper we consider an event-based PI (EB-PI) control scheme in which a particular SOD sampling is applied to the control. This approach is particularly effective when it is necessary to reduce the number of transmissions from the controller to the actuator.

In this context, we find necessary conditions on system instability and necessary and sufficient conditions on the controller parameters for the existence of equilibrium points without limit cycles when the process can be modelled as a first-order-plus-dead-time (FOPDT) transfer function. Note that, although this can be considered as a special case, it is significant from a practical point of view as many industrial processes can be modelled effectively in this way. The cases of event-based P, I, and PI controllers are considered. It is believed that these conditions can be usefully employed for the tuning of the overall controller.

It is worth stressing that, with respect to the results presented in [Beschi et al., 2011], here we consider a different EB-PI control strategy and we are interested in providing conditions for the absence of limit cycles and exploring a new type of architecture for EB-PI controllers with the aim of reducing the number of parameters.

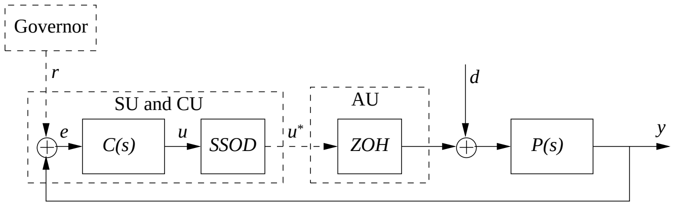

Fig. 1. Control architecture. SU: Sensor Unit, CU: Control Unit, AU: Actuator Unit.

2. CONTROL ARCHITECTURE

The architecture of controller can be divided into three main parts: the sensor unit, which samples the process variable, calculates the error and conveys its value to the control unit, which elaborates the new value of the control action. Finally, the new value of the control action is conveyed from the control unit to the actuator unit, which changes appropriately the actuator value.

In an event-based strategy these three tasks can run separately, and in some cases the three parts can be implemented by hardware located in different places of the plant. In order to reduce the communications between the different units, typically the signal values are not sent at constant intervals (notice that this operation requires time synchronization of the hardware) but only when the signal changes significantly with respect to the last sent value and without the necessity of synchronization.

It is important to notice that the different units can also be implemented in the same hardware, and clearly in this case the communications between these parts can be done at each interval. In this work, we consider the control architecture where the information between the sensor unit and the actuator unit can be exchanged at constant intervals and the new value of the control action is sent only when an event occurs . This architecture is shown in Figure 1. As said before, the communication between the control unit and the actuator unit is done only if there is an event. The event generator rules used in this work is a special case of the send-on-delta sampling method (see [Miskowicz, 2006]) which can be considered as a generalization of a relay with hysteresis where there are an infinite number of thresholds [Kofman and Braslavsky, 2006]. We call this technique symmetric send-on-delta (SSOD) sampling

Denote as $u(t)$ the control action before the SSOD sampling block and as $u^*(t)$ the sampled control action, which can assume only values multiple of a predefined threshold Δ, namely $u^*(t) = j\Delta$ with $j \in \mathbb{Z}$. The sampled signal changes its value to the upper quantization level when the control action $u(t)$ increases more than Δ, or to the lower quantization level when $u(t)$ decreases more than Δ.

The mechanism can also be analyzed by means of a state-machine representation, where j is the state number, $u(t) \geq (j+1)\Delta$ is the condition to jump to the upper state $j+1$ and $u(t) \leq (j-1)\Delta$ is the condition to jump to the lower state $j-1$. Also in the standard send-on-delta (SOD) technique, it is possible to define a state machine representation, but in this case the value assumed by u^* in the state j is equal to $j\Delta + q$ (see [Beschi et al., 2011]), where $q \in [0, \Delta]$ is an unquantizable quantity which depends on the initial value. The control action is calculated with the standard PI controller, namely:

$$C(s) = K_p + \frac{K_i}{s} \tag{1}$$

where $K_p \geq 0$ is the proportional gain and $K_i \geq 0$ is the integral gain.

It is worth stressing that the main differences between the presented technique (for short PI-SSOD, to indicate that the SSOD sampling is done on the control action) and the SOD-PI controller (see [Beschi et al., 2011]) are:

1) In the SOD-PI controller, the sensor unit has to calculate the integral error each time, and sends, with SOD sampling rules, the error and integral error signals to the control and actuator unit, which elaborates the new control action. In this technique, the control action is quantized in the SOD-PI because both e^* and IE^* are quantized. This makes impossible to compensate exactly a generic constant load disturbance of amplitude d, and if the controlled process has a pole at the origin then there are surely limit cycles (see [Beschi et al., 2011]). Moreover, there are three parameters that describe the SOD sampling in a SOD-PI controller. In addition, Δ_i has not a clear physical meaning, therefore it is complicated to tune it. Finally, another important disadvantage is that in this technique it is very complicated to implement the anti-windup methods used in time-based PI control theory, because they require transmissions between the two units.

2) In the PI-SSOD controller, the sensor and control unit has to execute standard time-driven controller, the SSOD algorithm and to send the SSOD sampling control action to the actuator unit, which acts as a zero-order-hold block. Also with this technique, the control action is quantized, but and therefore it is not possible to prevent the occurrence of limit cycles when the controlled process has a pole at the origin. To eliminate this disadvantage. a load disturbance estimator, described in Section 3, is implemented. Further, in this control strategy, only a parameter is necessary to describe the communication between the two parts, namely Δ, and this parameter has a clear physical meaning and it is easy to tune. Finally, the standard anti-windup methods can be easily applied because the computation of the integral error and the control action is done in the same hardware.

3. STABILITY AND LIMIT CYCLES ANALYSIS FOR A FOPDT PROCESS

This section addresses the stability properties of the controlled system, where the considered process is a FOPDT system, described by the following transfer function:

$$P(s) = \frac{K}{\tau s+1} e^{-Ls} \tag{2}$$

where K is the process gain (which is assumed to be positive without loss of generality), $\tau > 0$ is the time constant and $L \geq 0$ is the apparent dead time. Then, we can write:

$$Y(s) = \frac{K}{\tau s+1} e^{-Ls} \left(U(s) + \frac{d}{s} \right) \tag{3}$$

where $Y(s)$ is the Laplace transform of the process output $y(t)$, $U(s)$ is the Laplace transform of the control action $u(t)$ and d is the amplitude of a constant load disturbance. It is important to notice that, because the PI-SSOD technique presents a nonlinearity, the system can reach an equilibrium point, or can present a limit cycle around an equilibrium point or can be unstable. In general, the behavior of each equilibrium point can be different. In this section, we demonstrate that it is possible to find a region of the parameter space $K_p - K_i$ where the system is certainly marginally or asymptotically stable for all

the equilibrium points. This region can then be divided into two other regions: one where the system can present certainly a limit cycle, (and therefore the trajectory can tend to it for certain values of reference signal r, load disturbance d and the system initial conditions) and one where the system surely does not present a limit cycle.
We prove that if the PI-SSOD control system is unstable then also the continuous time-driven PI-controlled system with the same proportional and integral gains is unstable, therefore the instability region of PI-SSOD controller is included in time-based PI controller one.
To prove this necessary condition, we have first to demonstrate that if the control action diverges, then also the control action derivative diverges for all the positive proportional and integral gains (which are the only ones with a physical meaning).

Proposition 1. For all the positive proportional and integral gains, if the absolute value of the control action tends to infinity then also the absolute value of control action derivative tends to infinity.

Proof. To proof the proposition, we have to demonstrate the following implication:

$$\lim_{t\to\infty} |u(t)| = \infty \Longrightarrow \lim_{t\to\infty} |\dot{u}(t)| = \infty$$

By contradiction, we suppose that:

$$\lim_{t\to\infty} |u(t)| = \infty \ \text{ and } \ \dot{u}(t) \in [a,b], \ \text{ with } a<b, \ a,b\in\mathbb{R},$$

and therefore:

$$\lim_{t\to\infty} \left|\frac{\dot{u}(t)}{u(t)}\right| = 0, \ \text{almost everywhere.} \tag{4}$$

Consider first the proportional controller case (namely $K_i = 0$ and $K_p > 0$), we can write the following equations:

$$u(t) = K_p e(t) = K_p\left(r(t) - y(t)\right), \tag{5}$$

$$u^*(t) = SSOD(u(t),\Delta) = u(t) + l_1(t), \tag{6}$$

$$\tau\dot{y}(t) = -y(t) + Ku^*(t-L) = -y(t) + Ku^*(t) + Kl_2(t), \tag{7}$$

where: $e(t) = r(t) - y(t)$ is the error signal, the reference signal $r(t)$ is a bounded signal, $l_1(t)$ is a signal bounded in $[-\Delta,\Delta]$ and $l_2(t)$ is a bounded signal because the control action derivative is bounded. Using (4) and (5) it is trivial to obtain:

$$\lim_{t\to\infty} \left|\frac{\tau\dot{y}(t)}{y(t)}\right| = 0.$$

Using (5)-(7) it is possible to obtain:

$$\dot{y}(t) = -(1+KK_p)y(t) + KK_p r(t) + K(l_1(t) + l_2(t))$$

Finally, using the last two equations, we can write:

$$\lim_{t\to\infty} \left| -(1+KK_p) + \frac{KK_p r(t) + Kl_1(t) + Kl_2(t)}{y(t)} \right| = 0$$

and

$$\lim_{t\to\infty} \left| -(1+KK_p) \right| = 0$$

which implies $K_p = -\frac{1}{K} < 0$ which is absurd.
Consider now the case of PI or I controller (namely, $K_i > 0$ and $K_p \geq 0$). It is possible to rewrite the series of the process and the sensor and control unit in a state space form as:

$$\begin{cases} \dot{x}_1(t) = (\frac{K_p}{\tau} - K_i)x_2(t) - \frac{KK_p}{\tau}u^*(t-L) \\ \dot{x}_2(t) = -\frac{1}{\tau}x_2(t) + \frac{K}{\tau}u^*(t-L) \end{cases}$$
$$y(t) = x_2(t),$$
$$u^*(t) = SSOD(u(t),\Delta). \tag{8}$$

It is important to notice that $x_1(t)$ is the control action before the delay and, as the first case, $u^*(t-L) = x_1(t) + l(t)$, where $l(t) = l_1(t) + l_2(t)$. Another important consideration is that the limit (4) becomes:

$$\lim_{t\to\infty} \left|\frac{\dot{x}_1(t)}{x_1(t)}\right| = 0$$

Thus, we can describe the control system as:

$$\begin{cases} \dot{x}_1(t) = \left(\frac{K_p}{\tau} - K_i\right)x_2(t) - \frac{KK_p}{\tau}x_1(t) - \frac{KK_p}{\tau}l(t) \\ \dot{x}_2(t) = -\frac{1}{\tau}x_2(t) + \frac{K}{\tau}x_1(t) + \frac{K}{\tau}l(t) \end{cases} \tag{9}$$

We can *then* divide the first equation by $x_1(t)$, excluding the time instant when $x_1(t) = 0$, and determine the limit for $x_1(t) \to \infty$. By taking into account that $\dot{x}_1(t)$ is bounded, we obtain:

$$\lim_{|x_1|\to\infty} \left|\frac{\dot{x}_1(t)}{x_1(t)}\right| = \left| -\frac{KK_p}{\tau} + \left(\frac{K_p}{\tau} - K_i\right)\frac{x_2(t)}{x_1(t)} \right| = 0, \tag{10}$$

where the parts which tend to zero are neglected. If K_2 is equal to $K_1\tau$, we obtain $K_1 = K_2 = 0$, but this is a trivial case. Conversely, the limit is true only when the ratio between $x_1(t)$ and $x_2(t)$ is a bounded quantity. Thus, we can state that $x_1(t)$ has an infinity order smaller or equal than $x_2(t)$, and therefore:

$$\lim_{t\to\infty} \left|\frac{\dot{x}_2(t)}{x_1(t)}\right| = \lim_{t\to\infty} \left| -\frac{1}{\tau}\frac{x_2(t)}{x_1(t)} + \frac{K}{\tau} \right| = 0$$

where the parts which tend to zero are neglected. Thus $\frac{x_2(t)}{x_1(t)} \to K$ and (10) becomes:

$$\lim_{|x_1|\to\infty} \left|\frac{\dot{x}_1(t)}{x_1(t)}\right| = |-KK_i| = 0,$$

which is an absurd because $K_i > 0$. □

Using Proposition 1 it is possible to state the following necessary condition.

Proposition 2. If the closed-loop system of Figure 1 is unstable then the same system controlled by a time-driven PI controller with the same value of K_p and K_i is also unstable.

Proof. The time interval Δt between two events can be calculated as the ratio between Δ and the mean value of the control action derivative, called $\bar{\dot{u}}$, in the interval, in fact:

$$\Delta = \left| \int_{\tau}^{\tau+\Delta t} \dot{u}(t)dt \right| = |\bar{\dot{u}}|\Delta t.$$

From Proposition 1 we know that if the absolute value of the output tends to infinity also the absolute value of its derivative diverges to infinity. Thus, Δt decreases to zero and the PI-SSOD algorithm becomes a continuous-time controller, which proves the proposition. □

The parameters region for which there are not surely limit cycles and the system converges to the equilibrium state can be found with the following propositions, where we prove that if the system is in the state $j = j_{eq} \pm n$, where j_{eq} is the equilibrium state and $n > 0$, then it cannot evolve to the state $j = j_{eq} \mp n$ and therefore it has to tend to the equilibrium state. It is possible to derive a detailed analysis of the equilibrium points of a P-SSOD controlled system from [Beschi et al., 2011], while it is clear that for I-SSOD and PI-SSOD controlled system the only possible steady state is $j = 0$.

Proposition 3. In a P-SSOD controlled system, if $KK_p < 1$ then limit cycles cannot occur.

Proof. Denoting as j_{eq} the equilibrium state, we have to prove that if the system is in an initial state $j = j_{eq} \pm n$, with $n > 0$, the state machine cannot evolve to the state $j = j_{eq} \mp (n+1)$, and at least one of the states $j = j_{eq} + n$ and $j = j_{eq} - n$ cannot evolve to the state $j = j_{eq} - n$ and $j = j_{eq} + n$, respectively. In this way the limit cycle is surely avoided.

For the proof it is important to notice that if j_{eq} is the equilibrium state then the steady-state control action is $u_{ss,j_{eq}} = j_{eq}\Delta + \alpha\Delta$, with $\alpha \in [-1, 1]$ and that the steady-state control action for a generic state $j = j_{eq} + m$, with $m \in \mathbb{Z}$, is $u_{ss,j_{eq}+m} = j_{eq}\Delta + \alpha\Delta - mKK_p\Delta$. Another important property of the FOPDT system is that if the steady-state control actions of each state, which are involved in the limit cycle, are bounded in a interval $[a, b]$, with $a, b \in \mathbb{R}$, then for all time instants the control action before the SSOD sampling is bounded in the same interval. Now we demonstrate that if the initial state is $j = j_{eq} + n$, with $n > 0$, and $KK_p < 1$ then the system cannot reach the state $j = j_{eq} - n - 1$, and if $\alpha \geq 0$ then the system cannot reach the state $j = j_{eq} - n$. In other words, we have to prove that:

$$u_{ss,j_{eq+n}} = j_{eq}\Delta + \alpha\Delta - nKK_p\Delta > (j_{eq} - n - 1)\Delta, \quad \forall n > 0, \forall\alpha \in [-1, 1]$$

or equivalently:

$$n(1 - KK_p) > -(1 + \alpha), \ \forall n > 0, \forall\alpha \in [-1, 1]$$

which is always true.

Moreover if $\alpha \geq 0$, we can state that:

$$u_{ss,j_{eq+n}} = j_{eq}\Delta + \alpha\Delta - nKK_p\Delta > (j_{eq} - n)\Delta, \quad \forall n > 0, \forall\alpha \in [0, 1]$$

or equivalently:

$$n(1 - KK_p) > -\alpha, \ \forall n > 0, \forall\alpha \in [0, 1]$$

which is always true.

In the same way we can prove that if the initial state is $j = j_{eq} - n$, with $n > 0$, and $KK_p < 1$ then the system cannot reach the state $j = j_{eq} + n + 1$, and if $\alpha \leq 0$ then the system cannot reach the state $j = j_{eq} + n$. □

It is worth noting that the PI-SSOD control action is quantized, therefore it is not possible to guarantee a null steady-steady error for all the possible value of the load disturbance. Thus therefore PI-SSOD strategy (because of the presence of the integral part) is not enough to avoid the presence of limit cycles and it is necessary to add additional features to the controller. In this work we propose a solution where an algorithm estimates the "unquantized" part of the load disturbance and compensates it, allowing the controlled system to have a null steady-state error.

For this reason, we present necessary and sufficient conditions on the presence of limit cycles when the load disturbance is zero, finding the region of PI parameters where there are not limit cycles. Afterwards, we present the load disturbance compensation algorithm. The next proposition addresses the cases of I and PI controllers.

Proposition 4. In a PI-SSOD controlled system, with $K_p \geq 0$, $K_i > 0$ and $r = d = 0$ limit cycles cannot occur if the following conditions are satisfied: if $KK_p \geq KK_i\tau$, the parameters K_1 and K_2 are inside the portion of the first quadrant delimited by the following curve:

$$\begin{aligned} K_1(\tilde{t}_1) &= \frac{\tilde{t}_1 - 2\,l + 2\,e^l - e^{l+\tilde{t}_1}\,\left(2\,l - \tilde{t}_1 + 2\right) + 4\,e^l\,\sinh(l)}{2\,l\,e^l - 2\,l\,e^{l+\tilde{t}_1} + 2\,\tilde{t}_1\,e^l\,\sinh(l)} \\ K_2(\tilde{t}_1) &= -\frac{2\,\left(e^{2\,l} - e^{l+\tilde{t}_1} + e^l - 1\right)}{\tilde{t}_1 - \tilde{t}_1\,e^{2\,l} + 2\,l\,e^{l+\tilde{t}_1} - 2\,l\,e^l} \end{aligned} \tag{11}$$

where $K_1 = KK_p$, $K_2 = KK_i\tau$, $a = K_1 - K_2$, $l = \frac{L}{\tau}$ and $\tilde{t}_1 \in [l, \infty[$;

if $KK_p < KK_i\tau$ the parameters K_1 and a are inside the portion of the first quadrant delimited by the curve defined by the follow equation:

$$K_1 = K_2 - \frac{K_2 l - 1}{1 - e^{l - 2K_2^{-1}}}.$$

Proof. To prove the proposition, we find the locus of the parameters $K_p - K_i$ which allow the smaller limit cycle to arise. This locus delimits a part of the first quadrant where the PI parameters guarantee the absence of limit cycles. The system is described using equations (8), which can be normalized in:

$$\begin{cases} \dot{\tilde{x}}_1(t) = a\tilde{x}_2(t) - K_1\tilde{u}^*(t - l) \\ \dot{\tilde{x}}_2(t) = -\tilde{x}_2(t) + \tilde{u}^*(t - L) \end{cases} \tag{12}$$
$$\tilde{u}^*(t) = SSOD(\tilde{x}_1(t), 1).$$

by choosing $\tilde{t} = \frac{t}{\tau}$, $l = \frac{L}{\tau}$, $\tilde{x}_2 = \frac{x_2}{\Delta}$, $\tilde{x}_1 = \frac{Kx_1}{\Delta}$, $\tilde{y} = \frac{y}{\Delta}$, $\tilde{u} = \frac{Ku}{\Delta} = j$, $K_1 = KK_p$, $K_2 = KK_i\tau$ and $a = K_1 - K_2$.

The control variable $\tilde{u}^*$ assumes the values:

1) 1 for a time interval equal to $\tilde{t}_1$; at the beginning of this interval the state of the system assume the value $\tilde{x}^0$ and at the end the state is equal to $\tilde{x}^1$; l instants before the beginning, the state assumes the value $\tilde{x}^d$ with $\tilde{x}^d_1 = 1$;

2) 0 for a time interval $\tilde{t}_2$; at the end of this interval the state is equal to $\tilde{x}^2$; l instants before the beginning the state assumes the value $\tilde{x}^a$ with $\tilde{x}^a_1 = 0$;

3) -1 for a time interval $\tilde{t}_3$; at the end of this interval the state is equal to $\tilde{x}^3$; l instants before the beginning the state assumes the value $\tilde{x}^b$ with $\tilde{x}^b_1 = -1$;

4) 0 for a time interval $\tilde{t}_4$; at the end of this interval the state is equal to $\tilde{x}^0$; l instants before the beginning the state assumes the value $\tilde{x}^c$ with $\tilde{x}^c_1 = 0$.

Because of the symmetric nature of the nonlinearity, we study symmetric limit cycles (see [Åström, 1995]), namely $\tilde{t}_1 = \tilde{t}_3$, $\tilde{t}_2 = \tilde{t}_4$, $x^2 = -x^0$ and $x^3 = -x^1$. Experimental and simulation tests confirm this hypothesis.

During an interval, the system (12) evolves as a linear system with constant input and the evolution of the state can be written as the sum of the free movement and the forced movement which is the unitary step response multiplied by the amplitude of the input. Thus, we can write:

$$\tilde{x}(\tilde{t} - \tilde{t}_0) = R(\tilde{t} - \tilde{t}_0; a, K_1)\tilde{x}(t_0) + T(\tilde{t} - \tilde{t}_0; a, K_1)\tilde{u}^*(\tilde{t}_0)$$

where $\tilde{t}_0$ is the initial time,

$$R(\tilde{t}; a, K_1) = \begin{bmatrix} 1 & -ae^{-\tilde{t}} \\ 0 & e^{-\tilde{t}} \end{bmatrix} \text{ and}$$

$$T(\tilde{t}; a, K_1) = \left[K_1\tilde{t} + a\left(\tilde{t} + e^{-\tilde{t}} - 1\right), 1 - e^{-\tilde{t}}\right]^T.$$

Because the trajectory is periodic we can write the following equations:

$$\begin{aligned} \tilde{x}^1 &= R(\tilde{t}_1; a, K_1)\tilde{x}^0 + T(\tilde{t}_1; a, K_1) \\ \tilde{x}^2 &= -x^0 = R(\tilde{t}_2; a, K_1)\tilde{x}^1 \end{aligned} \tag{13}$$

and obtain:

$$\tilde{x}^0 = -(I + R_2R_1)^{-1}R_2T_1, \ \tilde{x}^1 = (I + R_2R_1)^{-1}T_1 \tag{14}$$

where: $R_1 = R(\tilde{t}_1; a, K_1)$, $R_2 = R(\tilde{t}_2; a, K_1)$ and $T_1 = T(\tilde{t}_1; a, K_1)$.

It is interesting to note that, when the input is zero (namely, in the time intervals $\tilde{t}_2$ and $\tilde{t}_4$.) the trajectory

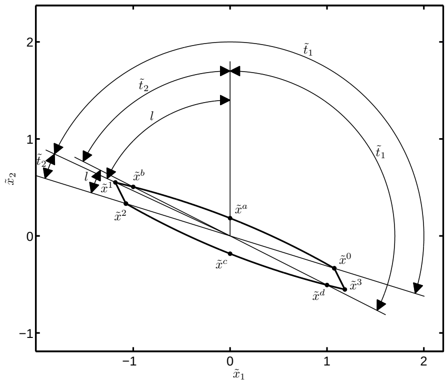

Fig. 2. Example of limit cycle trajectory with $a \geq 0$ where the arc arrows represent the time intervals.

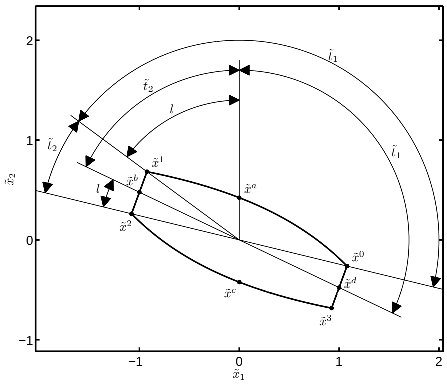

Fig. 3. Example of limit cycle trajectory with $a < 0$ where the arc arrows represent the time intervals.

is a straight line with an angular coefficient equal to $-a$ (obtained by the ratio $\frac{\dot{\tilde{x}}_1}{\tilde{x}_2}$). Thus, the maximum value assumed by $\tilde{x}_1$ is $\tilde{x}^3$ when $a \geq 0$ (see Figure 2), and $\tilde{x}^0$ when $a < 0$ (see Figure 3). It is important to note that in the limit cycle shown in Figure 2, for small amplitudes, we have $\tilde{t}_2 \leq l \leq \tilde{t}_1$. Conversely, in the case of Figure 3, for small amplitudes, we have $l \leq \tilde{t}_2 \leq \tilde{t}_1$. Consider now the case $a \geq 0$ and apply the condition of the smallest limit cycle (namely $\max \tilde{x}_1(t) = 1$, therefore $\tilde{x}_1^3 = 1$). In this way we obtain that $\tilde{x}^b = \tilde{x}^1$, therefore $\tilde{t}_2 = l$. We can find $\tilde{x}^a$ from the equation:

$$\tilde{x}^1 = R(l; a, K_1)\tilde{x}^a + T(l; a, K_1).$$

By the imposition of $\tilde{x}_1^1 = -1$ and $\tilde{x}_1^a = 0$ we can find the parametrization (11) of the locus $K_1 - K_2$ with respect to t_1 presented in the proposition.

Consider now the case $a < 0$ and apply the condition of $\max \tilde{x}_1(t) = 1$, therefore $\tilde{x}_1^0 = 1$. In this way we obtain that $\tilde{x}^b = -\tilde{x}^0$. We can find $\tilde{x}^a$ from the equation:

$$\tilde{x}^1 = R(l; a, K_1)\tilde{x}^a + T(l; a, K_1).$$

By the imposition of $\tilde{x}_1^0 = -1$ and $\tilde{x}_2^a = 0$ we can find that $\tilde{t}_2 \to \infty$, $\tilde{t}_1 = \frac{2}{K_1 - a}$ and the representation (4) of the locus $K_1 - K_2$. □

Note also that Propositions 3-4 are necessary and sufficient conditions to avoid the presence of limit cycles. In other words, in the region of the plane $K_p - K_i$ delimited by these conditions there are surely no limit cycles, whereas out of this region there is surely at least a possible limit cycle. Actually, for some values of initial conditions, reference signal and load disturbance it is possible that the system state does not reach the limit cycle trajectory but tends to the equilibrium point.

Another important consideration is that the parameter Δ does not have influence on the stability analysis and therefore it can be chosen using only considerations about the desired precision. It is worth to remember that the number of events increases with the decreasing of Δ.

As already mentioned, because of its quantized nature, the presented control strategy cannot compensate the unquantized part of the load disturbance, therefore a bimodal limit cycle surely arises. This limit cycle with period T involves two consecutive states $j_l + 1$ and j_l, where t^* is the interval time where $j = j_l + 1$. Thus, the control action assumes only the values $j_l\Delta$ and $(j_l + 1)\Delta$. By applying the definition of limit cycle, the trends of the process and the controller quantities during a cycle are periodic. Considering now the integrated error $Ie(t)$ and its Laplace transform $IE(s)$, which can be calculated as

$$IE(s) = \frac{R(s)}{s} - \frac{K}{s(\tau s+1)}(U(s) + d)$$

which corresponds to the following differential equation:

$$\tau \ddot{Ie}(t) + \dot{Ie}(t) = r(t) - K(u(t) + d + \tau \dot{r}(t)).$$

By integrating over a period and remembering that, for hypothesis, r and d are constant and $Ie(t)$ is periodic, we obtain:

$$rT - KTd - K\int_0^T u(t)dt = 0$$

and finally:

$$\bar{u} = j_l\Delta + \Delta\frac{t^*}{T} = \frac{r}{K} - d$$

where $\bar{u}$ is the mean value of $u(t)$ during a period and $\hat{d} := \Delta\frac{t^*}{T}$ is the "unquantized" part of the control action necessary to compensate the load disturbance.

In the real plants, the trajectory tends to the limit cycles but does not reach it. Hence, the hypothesis of periodicity is only an approximation, therefore it is impossible an exact compensation of the disturbances. For this reason, a small deadband is introduced, the width of this band is upper-limited by Δ and it can be set by taking into account the numerical errors in the calculus of t^* and T and the presence of noise.

4. PRACTICAL ISSUES

The control technique can be implemented easily in very simple industrial controllers, because it does not require a great computational effort or complex routines. The algorithm for the sensor and the control unit with the load disturbance compensator, can be outlined using the following pseudocode (for sake of brevity the initialization is not shown), where h is the control unit sampling period. Note that the actuator unit task has to only keep the last received value until the next event.

Sensor and Control Unit Task

1) calculate e; 2) calculate $u_k = u_{k-1} + K_p e + K_i eh$;
3) set $u_{k-1} = u_k$; 4) if $|u - u^*| \leq \Delta$ then go to 4;
5) set $u^{*,2} = u^{*,1}$, $t^{*,2} = t^{*,1}$, $u^{*,1} = u^*$ and $t^{*,1} = t^*$
6) save in t^* the time interval between the last and the current event.
7) if $u > u^*$ then $u^* = \Delta\lfloor\frac{u}{\Delta}\rfloor$ else $u^* = \Delta\lceil\frac{u}{\Delta}\rceil$;
8) if $u^{*,2} = u^*$ then go to 4 else go to 4;
9) if $u^* > u^{*,1}$ then $\hat{D} = \Delta\frac{t^*}{t^{*,1}+t^*}$ else $\hat{D} = \Delta\frac{t^{*,1}}{t^{*,1}+t^*}$;
10) send $u^* + \hat{D}$ to the actuator unit; 11) end.

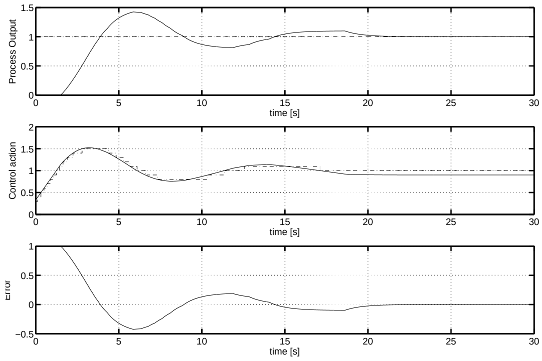

Fig. 4. Simulation results for PI-SSOD controller with the first set of parameters. First plot (from the top): process variable for the PI-SSOD controller (solid line). Second plot: PI-SSOD control variable before the SSOD sampling (solid line) and after the SSOD sampling (dashed line). Third plot: control error for the PI-SSOD controller (solid line).

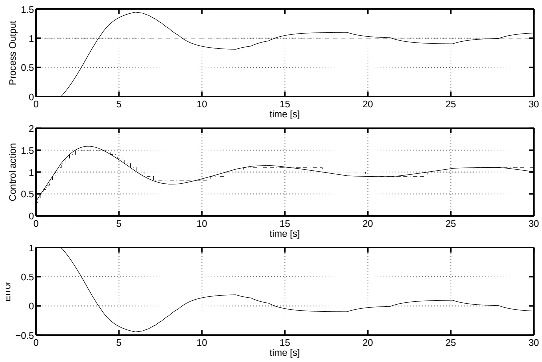

Fig. 5. Simulation results for PI-SSOD controller with the second set of parameters. First plot (from the top): process variable for the PI-SSOD controller (solid line). Second plot: PI-SSOD control variable before the SSOD sampling (solid line) and after the SSOD sampling (dashed line). Third plot: control error for the PI-SSOD controller (solid line).

5. SIMULATION RESULTS

In this section simulation results are presented with the aim to illustrate the validity of the propositions stated in Section 3. The considered FOPDT process is:

$$P(s) = \frac{1}{s+1}e^{-1.5s}.$$

Two sets of PI-SSOD parameters have been chosen: the first set ($K_p = 0.333$ and $K_i = 0.537$), which respects the conditions stated in Proposition 4; the second set ($K_p = 0.333$ and $K_i = 0.569$) which does not respect the conditions. The threshold parameter Δ is chosen as $\Delta = 0.1$. A step set-point signal of amplitude $r = 1$ is applied at time $t = 0$.

Figure 4 shows the simulation results when the first set is selected. The number of events is 24. Conversely, by selecting the second set, even if the response is faster, a limit cycle occurs, as shown in Figure 5. In this case the number of events is 27.

6. CONCLUSIONS

In this paper, we have addressed the stability issue and the presence of limit cycles for an EB-PI control system. The main advantages of this technique with respect to other formulations of EB-PI control algorithm have been outlined. Then, we presented necessary conditions on the system instability and necessary and sufficient conditions to avoid the limit cycles. These conditions are an important aid to tune the controller. Practical issues on the implementation of the algorithm have been also discussed. Simulation and experimental results have confirmed the significance of the work which can represent a valuable step in the development of easy-to-use tuning rules for this kind of controllers.

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