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Robust Gain-Scheduled Smith PID Controllers for Second Order LPV Systems with Time Varying Delay

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Abstract: In this paper, a new approach to design a robust gain scheduled linear parameter varying (LPV) PID controller with pole placement constraints (through LMI regions) is proposed for LPV systems with second order structure and time-varying delay. The controller structure includes a Smith predictor, real time estimated parameters that schedule the controller and Smith predictor (using the known part of the delay) and unstructured dynamic uncertainty which covers the delay uncertainty. Finally, the proposed control technique is validated in a real case study based on a piece of a sewer system.

Keywords: Gain-scheduling, LPV control, PID, Smith Predictor, Sewer network

1. INTRODUCTION

Since the 60's, the empirical (or classical) gain-scheduling (GS) control has been used for controlling non-linear and time-varying systems. But, this control methodology achieves closed-loop stability, without guarantees, for slowly varying parameters. In order to overcome this deficiency, linear parameter-varying gain-scheduling (LPV GS) controllers are introduced to allow arbitrarily smooth or discontinuous variations of plant dynamics. The LPV GS method guarantees closed-loop stability based on the concept of quadratic stability (QS) (Becker & Packard,1994) for all real parameter trajectories inside a given region. This methodology allows multi-objective criteria (H_∞, H_2, pole-placement) as well (Apkarian *et al.*, 1995a,b). Under additional hypotheses, LPV control problems can be transformed to a convex optimisation problem involving linear matrix inequalities (LMI's). This results in a well behaved and computationally tractable problem. For analysis, when the LMI conditions depend on the system parameter vector in a multi-affine way, it suffices to verify these conditions only at the vertices of the parameter polytope. Usually for time invariant systems with long time delays, a Smith predictor is an effective method to control the process, because the time delay is fixed. Nevertheless, the Smith predictor has the inherent drawback that its performance is sensitive to the process model uncertainty, especially to the time delay. If a process model deviates from the process dynamics the system performance deteriorates. Applications of the Smith predictor are, therefore, often limited in industrial processes. To alleviate this limitation, it is necessary to find a mechanism to compensate or take into account model errors.

The main contribution of this paper is to design a Linear Parameter Varying PID Smith Predictor controller (LPV PID+SP) for second order plus time delay linear parameter varying systems, taking into account robust stability, performance, closed-loop pole constraints and essentially the time varying nature of the plant. The varying parameters are measured (estimated) in real time and used to schedule PID parameters. A "*delay scheduling*" Smith predictor scheme is used to compensate most of the estimated delay. However, there is still a remaining delay due to the inaccuracy in its estimation that it will be represented as unstructured dynamic uncertainty in a robust control framework. For a general LPV system case, the design of a LPV PID controller should be formulated as an output feedback control that usually derives in solving a non-convex optimization problem based on BMI's (Mattei, 2003). But, because of the special structure of the plant model considered (second order plus delay), the basic idea in our approach is to tune the PID controller reformulating it as a convex state-feedback problem. Finally, the proposed control technique is validated in a real case study based on a piece of a sewer system.

The structure of the paper is the following: In *Section 2*, the statement of the gain scheduled Smith PID controller for second order plus time delay is presented. The formulation, synthesis and implementation of this controller in the framework of the LPV control theory recalled in *Appendix* is introduced in *Section 3*. To validate the proposed methodology, it is applied to a case study based on the control of a sewer system in *Section 4*. In *Section 5*, final conclusions are drawn.

2. PROBLEM STATEMENT

2.1 Problem statement

Let us consider the following second order LPV system plust time delay

$$G(s,\theta) = \frac{b_0(\theta)}{s^2 + a_1(\theta)s + a_0(\theta)} e^{-s\tau(\theta)} \tag{1}$$

whose parameters $a_0(\theta)$, $a_1(\theta)$ and $b_0(\theta)$ are varying-parameters functions of some scheduling variable $\theta(t)$ that can be measured on-line. The parameter range Θ is a box defined by $[b_{0\min} \;\; b_{0\max}]$ for the gain $b(\theta)$, and $[a_{0\min} \;\; a_{0\max}]$ $[a_{1\min} \;\; a_{1\max}]$, for denominator coefficients $a_0(\theta)$, $a_1(\theta)$ and $[\tau_{min} \;\; \tau_{max}]$ for the time delay $\tau(\theta)$.

The objective is to design a gain-scheduling PID controller with the following structure

$$G_{PID}(s,\theta) = k_p(\theta) + \frac{k_i(\theta)}{s} + k_d(\theta)s \qquad (2)$$

using LPV theory for the plant model described by (1). Model (1) is an usual representation of many industrial and environmental processes. By including the parameter measurements/estimations, this controller adjusts to the variations in the plant dynamics in order to maintain stability and high performance along all trajectories $\theta(t)$. In other words, the controller is 'self-scheduled', that is automatically gain-scheduled with respect to $\theta(t)$.

The variable delay in (1) can be handled in two different ways: (a) As an LTI dynamic uncertainty covered conveniently by a weight W_Δ as in (Skogestad *et al.*, 1997) (b) As a time-varying parameter which updates a Smith Predictor. The first approach could be conservative, and unnecessarily decrease the overall performance. On the other hand, the second approach could provide a far better performance, but it does not take into account the measurement error of the time-varying delay $\tau(\theta)$. In this paper, it is proposed to combine both approaches by assuming that a real time estimation $\hat{\tau}(\theta)$ of the delay is available, which will be used to update a Smith Predictor (Fig. 1). The difference between the actual and the estimated delay is considered as global dynamic uncertainty as in (Skogestad *et al.*, 1997) (Sánchez-Peña *et al.*, 1998) (Morari & Zafiriou, 1989) and is used in the design and robustness conditions. Therefore, we assume that the time delay dynamics has a time varying nature although its estimation error dynamics is time invariant, with a constant bound. The latter can be explained as follows: sensors are usually modeled as time invariant systems, with a bounded error provided by the manufacturer, as we have assumed here. The dependence of the delay with the operating point can be determined by physical modeling or identification and is measured (estimated) in real time. Proceeding in such a way, most of the delay is compensated and the remaining portion, denoted as

$$\Delta\tau(\theta) = \tau(\theta) - \hat{\tau}(\theta) \qquad (3)$$

can be covered by LTI unstructured uncertainty. This measurement error is always smaller that the actual delay, therefore the uncertainty is less conservative, which in turn has a lower impact on performance. This uncertainty is handled here as multiplicative output uncertainty and the following weight "covers" the delay measurement error frequency response as tightly as possible (see Chapter 11, (Sánchez-Peña *et al.*, 1998).):

$$W_\Delta(s,\Delta\tau) = \frac{2.05\Delta\tau_{\max} s}{\Delta\tau_{\max} s + 1} \quad with \;\; \Delta\tau(\theta) \le \Delta\tau_{max}. \qquad (4)$$

Although the delay is time varying, by assuming that the delay measurement error is time invariant, the same robust stability analysis of the Smith predictor can be performed, following the approach proposed in (Sánchez-Peña *et al.*, 1998) (Morari & Zafiriou, 1989) for the LTI case. This is due to the fact that the remaining system, after the cancellation of delay with the use of its estimation, can be considered as finite dimensional LTI, according to this assumption. Therefore, the delay scheduled Smith Predictor eliminates the infinite dimensional as well as the time varying nature of the delay, reducing it to a LTI dynamic uncertainty. This is one of the main contributions of this work as compared to previous approaches (Ge *et al.*, 2002).

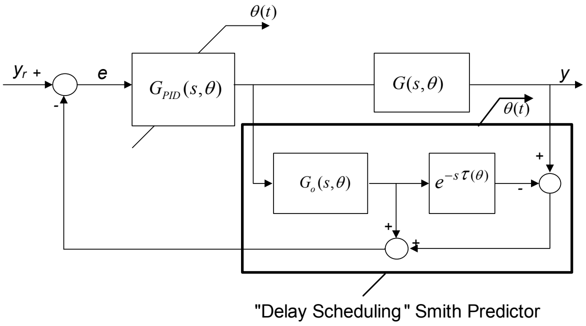

Fig.1. "Delay scheduling" Smith Predictor scheme.

2.2 Control specifications

The control design specifications that will be considered are a mixture of performance and robustness objectives arranged as a mixed sensitivity problem (MSP) (Skogestad *et al.*, 1997) (see Fig.2), as follows:

$$\left\| \begin{bmatrix} W_e S & W_u KS & W_\Delta T \end{bmatrix} \right\|_\infty^T < \gamma \le 1 \qquad (5)$$

Here S is the sensitivity and T is the complementary sensitivity functions. These transfer functions represent weighted tracking error (or disturbance rejection), weighted control action and robust stability, respectively. In order to limit the control energy and bandwidth of the controller, a weight W_u is included in the design. Such weight is a transfer function with a crossover frequency approximately equal to that of the desired closed-loop bandwidth. The weight for the complementary sensitivity, W_Δ, captures the uncertainty of the plant model (in this case coming from the delay measurement error) and also limits the closed-loop bandwidth. Typically, a disturbance in the system output is a low frequency signal, and therefore it will be successfully rejected if the minimum value of S is achieved over the same frequency band. This is performed by selecting a weight W_e, with a bandwidth equal to that of the disturbance in the controller design specifications. Robustness is presented as an H_∞ bound and is related with the dynamic uncertainty coming from the real time delay estimation error. Performance is a combination of weighted error and control action minimization measured in terms of the energy integrals of the input and output signals involved. A PID controller is a good approximation of a robust high order controller at low frequencies, especially because of the inclusion of the integral action. Then, the resulting PID controller is expected to preserve the disturbance rejection performance of a high-order controller. Furthermore, the time response is tuned via a selected closed-loop pole placement LMI region (Chilali *et al.*, 1999). This control design problem will be solved using the notion of QS and closed-loop pole placement applied to an MSP, considering the delay measurement error as multiplicative dynamic uncertainty (see *Section 2.1*). An MSP can always be formulated as a Linear Fractional Transformation (LFT), and solved recasting two previous theoretical results (see *Appendix*): 1) Quadratic H_∞ performance. 2) Robust and Quadratic D–Stability.

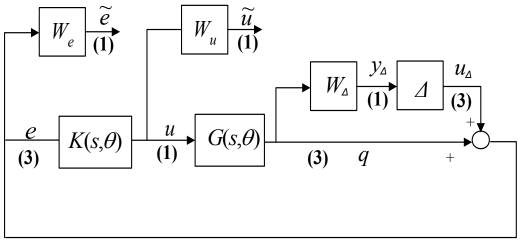

Fig.2. Proposed LPV feedback system scheme (MSP scheme).

The problem statement is as follows:

Problem 1. *Given the LPV system in Eq. (1), find a gain-scheduling PID controller combined with the delay scheduled Smith Predictor presented in Fig. 1, that guarantees QS and an H_∞ norm bound less than a positive number γ on the w-z input-output channel $\forall\theta\in\Theta$, and pole placement requirements applied to the MSP in Eq. (5).*

3. PROBLEM SOLUTION

3.1 Preliminary considerations

The LPV PID controller design of the system described in Problem 1 will now be formulated and embedded in a self-scheduled LPV control framework developed by (Apkarian, 1995a) (Becker&Packard, 1994) briefly summarised in *Appendix*. For a general LPV system case, the design of a LPV PID controller in (2), should be formulated as an output feedback control that usually derives in solving a non-convex optimisation problem based on BMI's. However if the system to be controlled has the second order structure in (1), a convex state feedback problem can be formulated (see (Ge *et al.*,2002) (Zheng *et al.*, 2002) for details), leading to the following state space description:

$$\begin{aligned} \dot{x} &= A(\theta)x + B(\theta)u + B_r r, \\ u &= -K(\theta)x + k_p(\theta)r + k_d(\theta)\dot{r}, \\ y &= C\,x, \end{aligned} \tag{6}$$

where y is the system output, $x = [x_1 \quad x_2 \quad x_3]^T$ the state with variables defined by $x_1 = y, x_2 = \dot{x}_1$, $x_3 = -\int e dt, e = r - y$, r the reference input, and

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 \\ -a_0(\theta) & -a_1(\theta) & 0 \\ 1 & 0 & 0 \end{bmatrix}, B(\theta) = \begin{bmatrix} 0 \\ b_0(\theta) \\ 0 \end{bmatrix}, B_r(\theta) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$C = [1 \;\; 0 \;\; 0], \; K(\theta) = \begin{bmatrix} k_p(\theta) & k_d(\theta) & k_i(\theta) \end{bmatrix}. \tag{7}$$

In this state-space model, the PID controller design becomes a static state feedback controller, and the static feedback gain $K(\theta)$ simply contains all the PID controller parameters. Note also that there are three varying parameters in (1) (Ge *et al.*, 2002). The control design scheme proposed for *Problem 1*, which combines measured (estimated) LPV parameters and unstructured output uncertainties is presented in Fig.2, and it is represented as an LFT. In such an LFT representation, the second order LPV model (1) is formulated as in (7) to achieve a PID controller as a state feedback. Moreover, it is necessary to consider the following issues:

1. The performance and control effort weight functions need to be constants ($W_e = D_e$, $W_u = D_u$), so that the order of the augmented model does not increase the one of model (1).
2. Additionally, in order to not increase the augmented model's order, the uncertainty weight in (4) is modified as follows:

$$\tilde{W}_\Delta(s, \Delta\tau) = 2.05\Delta\tau_{max}s, \tag{8}$$

so that $W_\Delta = \tilde{W}_\Delta G_f$.

Then, using a Smith Predictor scheme (Fig. 1) and the uncertainty weight introduced in (8) bounding the delay measurement error in (3), the following LFT LPV system representation is obtained:

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + B_{u_\Delta}(\theta)u_\Delta(t) \\ z(t) &= C_z(\theta)x(t) + D_{zu}(\theta)u(t) + D_{zu_\Delta}(\theta)u_\Delta(t) \\ q(t) &= C_q(\theta)x(t) + D_{qu}(\theta)u(t) + D_{qu_\Delta}(\theta)u_\Delta(t) \end{aligned} \tag{9}$$

with $x = [x_1 \quad x_2 \quad x_3]^T = [y \quad x_\Delta \quad x_I]^T = [y \quad \dot{x} \quad x_I]^T$, $u = [u_\Delta \quad u]^T$ $z = [y_\Delta \quad \tilde{u} \quad \tilde{e}]^T$, and

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 \\ -a_0(\theta) & -a_1(\theta) & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad B(\theta) = \begin{bmatrix} 0 & b_0(\theta) & 0 \end{bmatrix}^T,$$

$$B_{u_\Delta}(\theta) = [0 \quad 0 \quad 0]^T, \; C_z(\theta) = \begin{bmatrix} D_\Delta & C_\Delta & 0 \\ 0 & 0 & 0 \\ -D_e & 0 & 0 \end{bmatrix},$$

$$C_q(\theta) = [-1 \quad 0 \quad 0], D_{zu}(\theta) = [0 \quad D_u \quad 0]^T,$$

$$D_{zu_\Delta} = [0 \quad 0 \quad -D_e]^T, \; D_{qu_\Delta} = -1, \; D_{qu} = 0. \tag{10}$$

3.2 Problem solution

Since the gain, $b_0(\theta)$ of the LPV system (1) varies with parameter θ, to fulfill hypothesis (ii) associated to *Remark 1* in *Appendix*, the time varying gain of the system can be compensated in the following way. First, the LPV gain-scheduling PID controller $G_{PID}(s,\theta)$ is designed taking into account only the variation of the parameters $a_0(\theta)$ and $a_1(\theta)$, and assuming that the parameter $b_0(\theta)$ has a nominal value b_{0nom}. Finally, keeping the same inner loop through equation

$$\tilde{G}_{PID}(s,\theta) = G_{PID}(s,\theta)\frac{b_{0nom}}{b_0(\theta)} \tag{11}$$

the variation of parameter $b_0(\theta)$ is considered in the design of the controller. Due to the fact that the time varying parameters enter affinely in the augmented model equations (see (9) and (10)), the parameter region is polytopic and since condition (ii) is fulfilled through the transformation introduced by (11), the model of the LPV system can be represented by:

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \sum_{i=1}^{r} \lambda_i \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}.$$

The delay $\tau(\theta)$ has already been considered as a scheduled (time varying) parameter in the Smith Predictor implementation, and the delay estimation error bounded by a multiplicative uncertainty in the design process, as explained in *Section 3.1*. Next, a static time varying state feedback controller is computed, which satisfies QS and the quadratic H_∞ performance specifications. Such a controller can be transformed by the equivalence introduced in (6), in a PID controller as in (2). This controller schedules the parameters $a_0(\theta)$, $a_1(\theta)$ and by means of the transformation in (11), the scheduling of parameter $b_0(\theta)$ is added. This controller guarantees QS and Quadratic H_∞ Performance, as well as ("frozen") closed-loop pole location inside the desired LMI region. Since the plant is polytopic, the controller $K(s,\theta) = K(\theta)$ is designed as a polytopic model and implemented according to:

$$K(\theta) \in Co\{K(v_1), K(v_2), \ldots, K(v_r)\} := \{\sum_{i=1}^{r} \lambda_i K_i \,;\, K_i = K(v_i)\}$$

where: $\sum_{i=1}^{r} \lambda_i(\theta) = 1$, $\lambda_i(\theta)>0$. (12)

This technique is known as a convex decomposition technique, and Co is the function that generates the convex hull of the polytope vertices. The polytopic coordinates are calculated by fast algorithms in such a way that each vertex v_i, i=1,...,r has coordinates:

$$\lambda_i = \prod_{j=1}^{l} \tilde{\vartheta}_j^i \quad \text{with}$$

$$\tilde{\vartheta}_j^i = \begin{cases} \vartheta_j^i & \text{if } \underline{\vartheta}_j^i \text{ is a coordinate of } v_i \\ 1-\vartheta_j^i & \text{if } \overline{\vartheta}_j^i \text{ is a coordinate of } v_i \end{cases} \quad \text{where}$$

$$\tilde{\vartheta}_j^i = \frac{(\overline{\theta}_j^i - \theta_j^i)}{(\overline{\theta}_j^i - \underline{\theta}_j^i)},\ j=1,\ldots,l, \qquad (13)$$

and $(\underline{\theta}_j^i, \overline{\theta}_j^i)$ represent the upper and lower bounds of θ_j^i, and l fulfils that $r=2^l$. Finally, the closed-loop system is $\dot{x}_{cl} = A_{cl}(\theta)x_{cl} + B_w w$, with matrices $A_{cl}(\theta)$ and $C_{cl}(\theta)$ that depend on the parameter vector θ described as follows:

$$A_{cl} = \left\{\sum_{i=1}^{r} \lambda_i(\theta)(A_i + B_w K_i)\right\},\ B_{cl} = \left\{\sum_{i=1}^{r} \lambda_i(\theta)(C + D_{zw} K_i)\right\}.$$

4. APPLICATION TO A SEWER SYSTEM

4.1 Description and LPV Model

The case study to illustrate the proposed LPV PID design method is a piece of a sewer system of Barcelona city (see Fig. 3). It is composed of a real detention tank (Fig. 4) and a single pool sewer equipped with an upstream sluice gate and a downstream spillway. An electromotor is driving gate position and two sensors located upstream and downstream of the gate are measuring the flows. The total length of the sewer is L=2km, a gate discharge coefficient C_{dg}= 0.6, a Manning roughness coefficient n=0.014, gate width and sewer width B=2.5m, a downstream spillway of height y_s = 0.7m, a spillway coefficient C_{ds}= 2.66, and a bottom slope I_0= 5.10^{-4}.

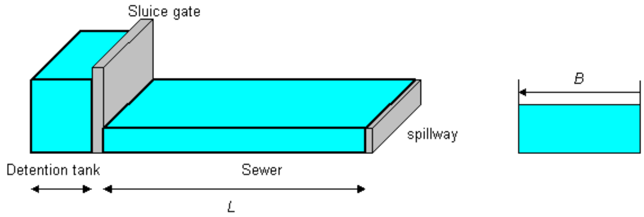

Fig.3. Sewer system scheme. (Left) Longitudinal and (right) cross section.

The dynamics of a single sewer is classically modelled with the Saint-Venant equations since it behaves as a single reach open-flow canal. However such set of partial differential equations are too complex for control design. According to (Litrico et al., 1999), the following transfer function, known as Hayami model, linking the upstream flow Q_{ups} and downstream flow Q_{dns} for a single reach canal of length X can be derived

$$G(s) = \frac{Q_{dns}(s)}{Q_{ups}(s)} = \frac{e^{-s\tau(\theta)}}{1 + k_1(\theta)s + k_2(\theta)s^2} \qquad (14)$$

where the scheduling parameter $\theta = Q_{dns}$. In the considered case study, the time varying parameters of the model (14) can be bounded taking into account that operating range of the scheduling variable is $\theta \in [0, 5]$: $k_1(\theta) \in [495.82, 875.90]$, $k_2(\theta) \in [61459.86, 191799.32]$ and the time delay $\tau(\theta) \in [341.94, 604.06]$ (s).

Fig.4. Detention tank construction (left). Detention tank inside (right)

4.3 Controller Design and Results[1]

Comparing (14) with (1), the following relations follow: $b_0(\theta) = \frac{1}{k_2(\theta)}$, $a_0(\theta) = \frac{1}{k_2(\theta)}$ and $a_1(\theta) = \frac{k_1(\theta)}{k_2(\theta)}$. Then, the control methodology presented in *Section 3* can be applied. Additionally, the delay estimation error is considered to be

[1] The real sewer behavior is accurately reproduced by Saint-Venant's equations using a simulator developed by the group of "Modeling and Control of Hydraulic Systems" at the UPC.

bounded $\Delta\hat{\tau}(\theta)\in[0,5]$ and comes from the experimental identification of model (14). The error in the time delay is taken into account in the control design as an LTI unstructured multiplicative uncertainty $W_{\Delta}(s,\Delta\hat{\tau})=10.25s$. Once the main time varying delay has been compensated by the Smith Predictor (Fig. 1) and the remaining delay error considered as the weight $W_{\Delta}(s,\Delta\hat{\tau})$ of a multiplicative dynamic uncertainty (see *Section 3.1*), a PID controller is designed as a state feedback. This controller should guarantee closed-loop stability and the following (step response) performance specifications: 1) tracking error of 0.1, 2) control signal within [0, 5] and 3) closed-loop damping of $\xi\geq 0.5$ and settling time in $t_{ss}\approx 230$ s, for any arbitrarily fast parameter variation. The tracking error and the bounded control signal are represented by performance weights $W_e = 1$ and W_u = 0.4, respectively. Furthermore, to achieve this desired transient behaviour and prevent controller fast dynamics, a pole clustering constraint is added. To this end, a LMI region $S(h_1,h_2,\alpha)$ is defined as a combination of three subregions: 1) a conic sector with apex at x = 0 and angle α = $3\pi/4$, which captures the closed-loop damping constraint $\xi\geq 0.5$; 2) a left half plane that guarantees the maximum settling time (h_1 = -0.016). 3) Left half plane that guarantees the minimum settling time (h_2 = -0.0018). Once formulated the MSP with pole placement constraint problem (see Problem 1), the gains of LPV PID controller are obtained by solving LMI's coming from *Theorem* 1 and 2 recalled in *Appendix* at each vertex of the interval box of $b_0(\theta)$, $a_1(\theta)$ and $a_0(\theta)$ as indicated in *Theorem* 3 (also in the *Appendix*). Fig. 4 presents graphically the gains of the LPV PID controller obtained at the different vertices and their interpolation using (13).

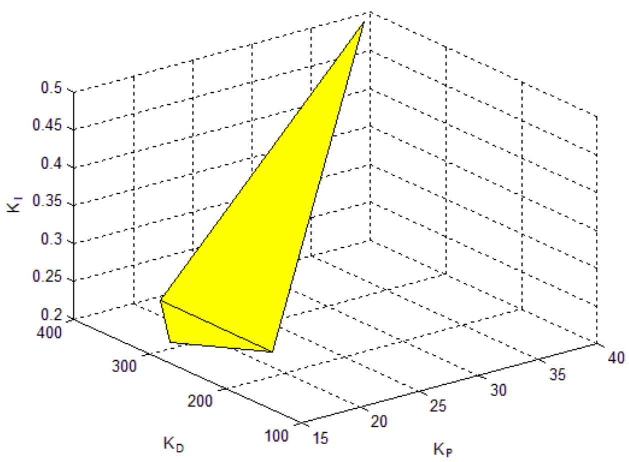

Fig. 4 Controller gains of the LPV PID controller.

The obtained closed-loop responses in simulation using the LPV PID+SP design are shown in Fig. 5. Fig. 6 presents the evolution of the model parameters. It can be observed that the performance specifications are achieved for the whole admissible operating range. On the other hand, if a robust LTI H_∞ PID controller with a standard LTI Smith Predictor designed for the worst case set of parameters is used, the time response is slower than the one obtained by its LPV counterpart and do not satisfy the performance specifications for the whole admissible operating range.

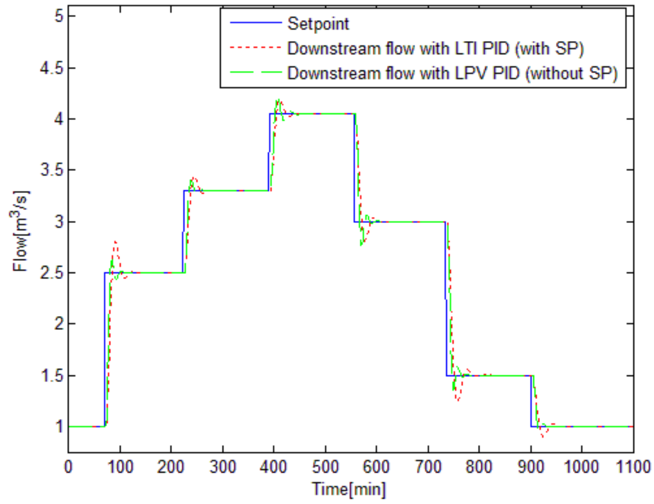

Fig. 5 Closed-loop response for different operating points.

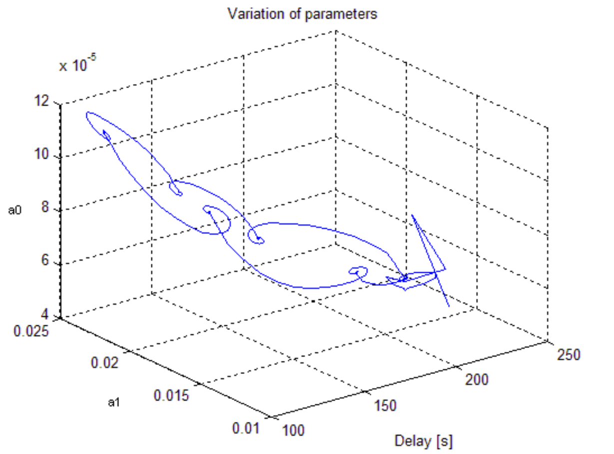

Fig.6 Evolution of the model parameters corresponding to the operating point changes presented in Fig. 5.

5. CONCLUSIONS

The main contribution of this paper is the development of a new approach to design a gain-scheduled Smith PID controller for LPV second order systems plus delay solving a MSP problem with closed-loop pole placement constraints. The time varying delay is handled by a "delay-scheduling" Smith predictor and the estimated delay error is treated as an unstructured dynamic uncertainty. Thanks to the second order system structure, the PID controller design can be viewed as an state-feedback controller whose design can be transformed to a convex optimisation problem involving LMI's. This approach has successfully been applied to a real case study based on the control of a piece of a sewer system with satisfactory results.

APPENDIX: BACKGROUND ON LPV CONTROL THEORY

Given an LPV system described by state-space equations of the form

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + B(\theta)u(t) + B_w(\theta)w(t) \\ z(t) &= C_z(\theta)x(t) + D_{zu}(\theta)u(t) + D_{zw}(\theta)w(t) \\ q(t) &= C_q(\theta)x(t) + D_{qu}(\theta)u(t) + D_{qw}(\theta)w(t) \end{aligned} \qquad (15)$$

where $x \in \Re^n$ is the state vector, $u \in \Re^{m_1}$ and $w \in \Re^{m_2}$ are the control and disturbance input vectors, respectively, $z \in \Re^{p1}$ and $q \in \Re^{p_2}$ are the measured and controlled output vectors, respectively. $A(\cdot)$, $B(\cdot)$, $B_w(\cdot)$, $C_z(\cdot)$, $C_q(\cdot)$, $D_{qu}(\cdot)$, $D_{zu}(\cdot)$, $D_{zw}(\cdot)$, $D_{qw}(\cdot)$ are continuous matrix valued functions of the time varying parameter vector $\theta(t) \in \Theta \subset \Re^l$, Θ being a polytope with r vertices. We assume the time varying parameters $\theta(t)$ can be measured (or estimated in the case of quasi-LPV models) in real time as in (Apkarian *et al.*, 1995b) (Becker and Packard, 1994). Performance is defined as requiring a bounded output $q(t)$ for any bounded external signal $w(t)$, both measured by their energy integral. The synthesis technique for LPV systems is based on the following results:

Theorem 1. *(Quadratic H_∞ Performance) (see Apkarian et al., 1995b). The LPV system given by Eq.(1) is QS and has quadratic H_∞ performance if there exists a positive definite matrix X>0 such that*

$$B^0_{[A(\theta),B(\theta),C(\theta),D(\theta)]}(X,\gamma) := \begin{bmatrix} A^T(\theta)X + XA(\theta) & XB(\theta) & C^T(\theta) \\ B^T(\theta)X & -\gamma I & D^T(\theta) \\ C(\theta) & D(\theta) & -\gamma I \end{bmatrix} < 0$$

for all admissible values of the parameter θ.

Remark 1. According to the self-scheduled H_∞ control synthesis problem for LPV systems developed by (Apkarian et al., 1995a), a control design which guarantees the Quadratic H_∞ performance for the closed-loop system, should fulfill the following necessary and sufficient conditions:

(i) $D_{qu}(\theta) = 0$ or equivalently $D_{qu_i} = 0$ for i=1,2,..,r.

(ii) $B(\theta), C_q(\theta), D_{zu}(\theta), D_{qw}(\theta)$ are parameter independent or equivalently $B_i = B, C_{q_i} = C, D_{zu_i} = D_{zu}, D_{qw_i} = D_{qw}$ for i=1,2,..,r.

(iii) The pairs $(A(\theta), B)$ and $(A(\theta), C_q)$ are quadratically stabilizable and detectable over Θ, respectively.

Theorem 2 (*Quadratic $\mathcal{D}$ stability*) (see *Chilali et al.*, 1999). *Consider the LPV system $\dot{x} = A(\theta)x$ with parameter θ, when θ is a fixed value ("frozen" time). Its pole location in the LMI-Region*[2] *$\mathcal{D}$ at each time t ("frozen" time) can be described by: $M_{\mathcal{D}}$= $\left[\ \alpha_{kl}X + \beta_{kl}A(\theta)X + \beta_{lk}XA(\theta)^T\ \right]_{1\le k,l\le m}$, where X is a positive definite matrix, and $M_{\mathcal{D}}[A(\theta),X]$ and $f_{\mathcal{D}}(z)$ can be related by the following substitution, $\left[X, A(\theta)X, XA(\theta)^T \leftrightarrow (1, z, \bar{z})\right]$. Then, the matrix $A(\theta)$ is quadratic $\mathcal{D}$ stable if and only if there exists a symmetric positive definite matrix X such that $M_{\mathcal{D}}[A(\theta),X]<0$ for all admissible values of the parameter θ.*

[2] A subset $\mathcal{D}$ of the complex plane is called an LMI-Region if there exists a symmetric matrix $\alpha = [\alpha_{kl}] \in \Re^{m\times m}$ and a matrix $\beta = [\beta_{kl}] \in \Re^{m\times m}$ such that: $\mathcal{D} = \{z \in C : f_D(z) < 0\}$, $f_D(z) := \alpha + z\beta + \bar{z}\beta^T = [\alpha_{kl} + \beta_{kl}z + \beta_{kl}\bar{z}]_{1\le k,l\le m}$

Based on the fact that a finite set of LMI can be solved in the multi-affine case when the parameters vary in a polytope, a computationally feasible solution to the problem exists, first formulated in (Becker and Packard, 1994), as follows.

Theorem 3. (Vertex Property) (see Apkarian *et al.*, 1995b). Consider a polytopic linear parameter-varying plant as in Eq. (1), where

$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \in \Theta := Co\left\{ \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}, \quad i = 1,\dots,r \right\}$$

and assume A,B,C,D are affine functions of θ, then the following items are equivalent:

i. *The system is quadratic $\mathcal{D}$–stable with Quadratic H_∞ performance γ.*

ii. *There exists a positive definite matrix X>0, which satisfies the following LMI's:*

$$M_D(A_i, X) < 0$$
$$B^0_{[A_i,B_i,C_i,D_i]}(X,\gamma) < 0, \quad i = 1,2,\dots,r$$

If *Theorem 3* is fulfilled, *Theorem 1* and *2* only should be verified on the vertices of the parameter polytope Θ. This implies that the number of inequalities needed to test the analysis conditions of these theorems can be reduced to a finite one, which makes such an approach appealing.

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