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Closed Form Expressions of Linear Discrete Time System Responses with an Application to PID controllers

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Abstract: General closed–form expressions of linear discrete–time system responses of arbitrary order are presented without proof. The system poles can be real and/or complex, and may be repeated. While these expressions are readily computed from the system in standard forms, they are based on a backward difference formulation, shown to offer some important simplifications and a closer analogy with the continuous case. Expressions are also derived for Lyapunov (Sylvester) equations, whose solution is the corresponding (cross) Grammian matrix, thus allowing evaluation of it without direct reference to the poles of the system. Finally, an example is presented of how these expressions may be utilized to obtain an expression for zero-optimized open-loop PID coefficients.

1. INTRODUCTION

Closed form transfer functions for discrete time systems are of considerable interest in the area of control systems and filter design, see e.g., Gajić [2003], Brogan [1991], Oppenheim et al. [1997], Feuer et al. [1996] and Goodwin et al. [2001]. In many cases, it is beneficial to carry out the entire procedure from system identification through controller design and implementation in discrete time, despite the fact that the actual process to be controlled may by nature be continuous.

In this paper we present without proof due to space limitations general closed–form expressions of linear discrete–time system responses of arbitrary order. While these expressions are readily computed from the system in standard forms, they are based on a backward difference formulation, shown to offer some important simplifications. Our approach follows closely an approach for obtaining closed–form expressions for linear continuous–time responses presented, e.g., in Herjólfsson et al. [2006] and Hauksdóttir et al. [2007]. By working with backward differences we obtain a closer analogy than in some earlier work on discrete time systems. See, e.g., Herjólfsson et al. [2004] for some earlier work on PID controllers in discrete–time.

The discrete–time responses are presented in Section 2. The calculation of Grammian matrices is discussed in Section 3. One application is the computation of PID controllers by minimizing impulse, step, ramp, etc., open-loop response deviations from a reference response, effectively presenting design requirements. This is dealt with in Section 4, including examples.

2. DISCRETE–TIME SYSTEM RESPONSES

Consider the n-th order discrete-time difference equation

$$\sum_{i=0}^{n} a_{-i}y[k-i] = \sum_{i=0}^{m} b_{-i}u[k-i], \quad a_0 = 1, \quad k \geq 0, \tag{2.1}$$

with the initial conditions $y[k] = 0, \quad k = -1, -2, \ldots, -n$, corresponding to the transfer function

$$\frac{Y(z)}{U(z)} = \frac{b_0 + b_{-1}z^{-1} + \cdots + b_{-m}z^{-m}}{1 + a_{-1}z^{-1} + \cdots + a_{-n}z^{-n}} = z^{n-m}\frac{b_0 z^m + b_{-1}z^{m-1} + \cdots + b_{-m}}{z^n + a_{-1}z^{n-1} + \cdots + a_{-n}}. \tag{2.2}$$

Expressing this equation in backward difference form we have

$$\sum_{i=0}^{n} \alpha_{-i}\nabla^i y[k] = \sum_{i=0}^{m} \beta_{-i}\nabla^i y[k], \tag{2.3}$$

where $\nabla y[k] = y[k] - y[k-1]$,

$$\mathcal{A}_n^T = [\, \alpha_0 \;\, \alpha_{-1} \;\, \cdots \;\, \alpha_{-n} \,] = [\, a_0 \;\, a_{-1} \;\, \cdots \;\, a_{-n} \,]\, P_{n+1}, \tag{2.4}$$

$$\mathcal{B}_m^T = [\, \beta_0 \;\, \beta_{-1} \;\, \cdots \;\, \beta_{-m} \,] = [\, b_0 \;\, b_{-1} \;\, \cdots \;\, b_{-m} \,]\, P_{m+1}, \tag{2.5}$$

and P_k denotes a $k \times k$ Pascal matrix, whose $(p+1, q+1)$-th element is $(-1)^q \binom{p}{q}$ with $\binom{p}{q} = 0$ if $q > p$.

Equivalently we are replacing the transfer function (2.2) with

$$\frac{\beta_0 + \beta_{-1}\hat{z} + \cdots + \beta_{-m}\hat{z}^m}{1 + \alpha_{-1}\hat{z} + \cdots + \alpha_{-n}\hat{z}^n}, \tag{2.6}$$

where

$$\hat{z} = \frac{z-1}{z}. \tag{2.7}$$

We are interested in having a closed formula for the solution to such an equation when $u[k]$ is a forcing function of a given order, γ, denoted by $I_\gamma[k]$. We choose to define

$$I_0[k] = \delta[k] \text{ and}$$
$$I_\gamma[k] = \binom{k+\gamma-1}{\gamma-1} H[k], \quad \gamma \geq 1, \tag{2.8}$$

rather than defining it as

$$\frac{k^{\gamma-1}}{(\gamma-1)!} H[k], \quad \gamma \geq 1, \tag{2.9}$$

as is more common. Here $\delta[k]$ denotes the Dirac delta function, and $H[k]$ denotes the Heaviside unit step function. The underlying reason for adopting the choice (2.8), rather than (2.9), is the result of the following lemma.

Lemma 1.

$$\nabla I_{\gamma+1}[k] = I_\gamma[k], \tag{2.10}$$
$$I_{\gamma+1}[k] = \sum_{i=0}^{k} I_\gamma[i], \quad \gamma \geq 0, k \geq 0. \tag{2.11}$$

Proof: We have from (2.8) and Pascal's identity that when $\gamma \geq 1,\ k \geq 0$

$$\begin{aligned} \nabla I_{\gamma+1}[k] &= \binom{k+\gamma}{\gamma} H[k] - \binom{k+\gamma-1}{\gamma} H[k-1] \\ &= \binom{k+\gamma-1}{\gamma-1} H[k] = I_\gamma[k]. \end{aligned} \tag{2.12}$$

It follows from this result that

$$\begin{aligned} I_{\gamma+1}[k] &= I_{\gamma+1}(0) + \sum_{i=1}^{k} \nabla I_{\gamma+1}[i] \\ &= 1 + \sum_{i=1}^{k} I_\gamma[i] = \sum_{i=0}^{k} I_\gamma[i]. \end{aligned} \tag{2.13}$$

□

Furthermore we have the following result:

Theorem 1: Let $y[k]$ denote the solution to (2.1) when $u[k] = p_m[k]H[k]$ and $p_m[k]$ is a general polynomial of degree m in k.

Let $y_\gamma[k]$ denote the solution to (2.1) when $u[k] = I_\gamma[k]$, $\gamma \geq 1$. Then

$$y[k] = \sum_{\gamma=0}^{m} (\nabla^\gamma p_m[-1])\, y_{\gamma+1}[k]. \tag{2.14}$$

Proof: By Newton's backward difference interpolation formula, $p_m[k]$ is exactly expressed by

$$p_m[k] = \sum_{\gamma=0}^{m} (\nabla^\gamma p_m[-1]) \binom{k+\gamma}{\gamma}, \tag{2.15}$$

choosing the interpolation points as $k = -1, -2, \ldots, -m-1$. This follows from the fact that there is a unique polynomial of degree m that satisfies such $m+1$ conditions. The result follows directly from (2.8) and (2.15). □

We can, in particular, make use of (2.14), if we wish to obtain a solution when the forcing function, $u[k]$, is of form (2.9). Denoting the solution when $u[k] = \frac{k^{\gamma-1}}{(\gamma-1)!} H[k]$ by $\hat{y}_\gamma[k]$ we thus get

$$\begin{aligned} \hat{y}_2[k] &= -y_1[k] + y_2[k], \\ \hat{y}_3[k] &= \frac{1}{2} y_1[k] - \frac{3}{2} y_2[k] + y_3[k], \\ \hat{y}_4[k] &= -\frac{1}{6} y_1[k] + \frac{7}{6} y_2[k] - 2y_3[k] + y_4[k], \text{ etc.} \end{aligned} \tag{2.16}$$

Before stating the main result of this section, we introduce some notation. Let $\lambda_1, \lambda_2, \ldots, \lambda_\nu$ denote the poles of the transfer function

$$\frac{1}{a(z)} = \frac{1}{z^n + a_{-1} z^{n-1} + \cdots + a_{-n}}, \tag{2.17}$$

repeated $d_1, d_2, \ldots, d_\nu$ times, respectively and κ_{ij} denote the basic partial fraction coefficients given by the standard formula

$$\kappa_{ij} = \frac{1}{(d_i - j)!} \frac{d^{(d_i-j)}}{dz^{(d_i-j)}} \left[\frac{(z-\lambda_i)^{d_i}}{a(z)} \right] \Bigg|_{z=\lambda_i}, \begin{matrix} j = 1, 2, \ldots, d_i \\ i = 1, 2, \ldots, \nu. \end{matrix} \tag{2.18}$$

Thus,

$$\begin{aligned} \frac{1}{a(z)} &= \frac{1}{(z-\lambda_1)^{d_1} (z-\lambda_2)^{d_2} \cdots (z-\lambda_\nu)^{d_\nu}} \\ &= \sum_{i=1}^{\nu} \sum_{j=1}^{d_i} \frac{\kappa_{ij}}{(z-\lambda_i)^j}. \end{aligned} \tag{2.19}$$

Next we introduce the $n \times n$ Jordan matrix

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & J_\nu \end{bmatrix} \tag{2.20}$$

with the diagonal blocks

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix} \tag{2.21}$$

each a $d_i \times d_i$ matrix, as well as the matrix

$$\tilde{J} = I - J^{-1} = J^{-1}(J - I), \tag{2.22}$$

where I denotes the $n \times n$ unit matrix and assuming, for the time being, that $\lambda_i \neq 0, i = 1, 2, \ldots, \nu$. Then introduce the n-vector

$$\kappa = [\, \kappa_{11} \cdots \kappa_{1d_1} \cdots \kappa_{\nu 1} \ldots \kappa_{\nu d_\nu} \,]^T \tag{2.23}$$

and the $n \times (m+1)$ matrices

$$\mathcal{K}_{\gamma,m} = [\, \tilde{J}^{-\gamma}\kappa \;\; \tilde{J}^{1-\gamma}\kappa \;\cdots\; \tilde{J}^{m-\gamma}\kappa \,] \tag{2.24}$$

and

$$\mathcal{K}_{\gamma,m,n} = J^{n-1} \mathcal{K}_{\gamma,m}, \tag{2.25}$$

where we note that the latter matrix is well defined for $n + \gamma \geq m + 1$, even if $\lambda_i = 0$ for some $i = 1, 2, \ldots, \nu$.

For a given vector $\mathbf{c}_\gamma = [c_\gamma, c_{\gamma-1}, \ldots, c_1]^T$, denote by $D_{\mathbf{c}_\gamma}$ the following $\gamma \times \gamma$ upper triangular Hankel matrix

$$D_{\mathbf{c}_\gamma} = \begin{bmatrix} c_\gamma & c_{\gamma-1} & \cdots & \cdots & c_1 \\ c_{\gamma-1} & c_{\gamma-2} & \cdots & c_1 & 0 \\ \vdots & \vdots & \cdots & \cdots & \ldots \\ c_1 & 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{2.26}$$

Finally, introduce the n-vector function

$$\mathcal{E}[k] = \left[\mathcal{E}_1[k]^T, \mathcal{E}_2[k]^T, \ldots, \mathcal{E}_\nu[k]^T \right]^T, \tag{2.27}$$

where

$$\mathcal{E}_i[k] = \begin{bmatrix} \lambda_i^k \\ \frac{d}{d\lambda}\lambda^k\Big|_{\lambda=\lambda_i} \\ \vdots \\ \frac{1}{(d_i-1)!}\frac{d^{(d_i-1)}}{d\lambda^{d_i-1}}\lambda^k\Big|_{\lambda=\lambda_i} \end{bmatrix} = \begin{bmatrix} \lambda_i^k \\ \binom{k}{1}\lambda_i^{k-1} \\ \vdots \\ \binom{k}{d_i-1}\lambda_i^{k-d_i+1} \end{bmatrix} \tag{2.28}$$

and the γ-vector function

$$\rho_\gamma[k] = \left[1 \ \ (k+1) \ \ \binom{k+2}{2} \ \ \cdots \ \ \binom{k+\gamma-1}{\gamma-1}\right]^T. \tag{2.29}$$

We can now state without proof:

Theorem 2. Denote by $y_\gamma[k]$ the solution to the difference equation (2.1) with $u[k] = I_\gamma[k]$, $\gamma = 0, 1, \ldots$. Then

$$y_\gamma[k] = (\mathcal{K}_{\gamma,m,n}\mathcal{B}_m)^T\,\mathcal{E}[k] - \left(D_{\mathbf{c}_\gamma}\mathcal{B}_{\gamma-1}\right)^T \rho_\gamma[k] \tag{2.30}$$

with $\mathcal{K}_{\gamma,m,n}$, $\mathcal{B}_m$ and $D_{\mathbf{c}_\gamma}$, defined as in (2.25), (2.5) and (2.26), respectively, setting $\beta_{-i} = 0$ if $i > m$, and where the coefficients c_j, $j = 1, 2, \ldots, \gamma$, in (2.26) are the solution of the following system

$$\begin{bmatrix} \alpha_0 & 0 & \cdots & \cdots & 0 \\ \alpha_{-1} & \alpha_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \alpha_{-\gamma+1} & \alpha_{-\gamma+2} & \cdots & \cdots & \alpha_0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\gamma \end{bmatrix} = - \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.31}$$

assuming that the system (2.1) has no poles equal to one.

Remark 1. Given the ν poles of (2.17), the n-vector $\mathcal{K}_{\gamma,m,n}\mathcal{B}_m$ in (2.30) can be calculated by $\mathcal{O}(n^2)$ operations. Firstly the partial fraction coefficients (2.18) can be calculated by $\mathcal{O}(n^2)$ operations. Secondly, the matrix $\mathcal{K}_{\gamma,m}$ in (2.24) can be calculated by $\mathcal{O}(mn)$ operations from the vector κ in (2.23) by recursively forming matrix-vector products of the form $u = \tilde{J}v$, by solving the system $Ju = (J - I)v$, and/or solving linear systems of the form $\tilde{J}u = v$, by solving the system $(J - I)u = Jv$, each requiring $\mathcal{O}(n)$ operations. Finally, the matrix-vector product $J^{n-1}(\mathcal{K}_{\gamma,m}\mathcal{B}_m)$ requires $\mathcal{O}(n^2)$ operations. The modifications required in the case of zero poles, do not increase this total complexity. The calculation of the γ-vector $D_{\mathbf{c}_\gamma}\mathcal{B}_{\gamma-1}$ in (2.30) requires further $\mathcal{O}(n\gamma)$ operations (if $\gamma \le n$). When the system is stable the calculations of α_i, $i = 0, 1, \ldots, \gamma-1$ from the a-coefficients by (2.4) requires $\mathcal{O}(n\gamma)$ operations, the solution of $\mathbf{c}_\gamma$ from (2.31) further $\mathcal{O}(\gamma^2)$ operations.

3. CALCULATION OF GRAMMIAN MATRICES

In this section we consider the calculations of Grammian matrices associated with stable equations of form

$$\sum_{i=0}^{n} a_{-i}y[k-i] = I_\gamma[k],\ k \ge 0,\ \gamma = 0, 1, 2, \ldots \tag{3.1}$$

with $y[k] = 0$, $k = -1, -2, \ldots, -n$ and $I_\gamma[k]$ being defined as in (2.8), referred to as the basic response of order γ. Let $y_b^{(-\gamma)}[k]$ denote the transient part of this basic response. Then, it follows from (2.30) that

$$y_b^{(-\gamma)}[k] = \left(J^{n+\gamma-1}(J-I)^{-\gamma}\kappa\right)^T \mathcal{E}[k],\ k \ge 0, \tag{3.2}$$

for $\gamma = 0, 1, 2, \ldots$. Now let $y_b^{(\gamma)}$ denote the backward difference of order γ of the basic response of order zero, $y_b[k]$, i.e.

$$y_b^{(\gamma)}[k] = \nabla^\gamma y_b[k] = \left(J^{n-\gamma-1}(J-I)^\gamma\kappa\right)^T \mathcal{E}[k],\ k \ge 0, \tag{3.3}$$

for $\gamma = 1, 2, \ldots$, noting that $\nabla\mathcal{E}[k] = J^{-1}(J-I)\mathcal{E}[k]$.

Remark 2: This symmetry between higher order differences and higher order transient basic responses, analogous to such a symmetry in the continuous case, with the backward differences being replaced by derivatives, is indeed the main motivation for working with backward differences.

Next let $\hat{Y}_{\gamma,m}$ denote the m-vector function, $m \ge 0$,

$$\hat{Y}_{\gamma,m}[k] = \left[y_b^{(-\gamma)}[k]\ y_b^{(-\gamma)}[k-1]\cdots y_b^{(-\gamma)}[k-m+1]\right], \tag{3.4}$$

and let $Y_{\gamma,m}$ denote the m-vector function

$$Y_{\gamma,m}[k] = \left[y_b^{(-\gamma)}[k]\ y_b^{(-\gamma+1)}[k]\cdots y_b^{(-\gamma+m-1)}[k]\right] \tag{3.5}$$

related to $\hat{Y}_{\gamma,m}$ by the Pascal matrix

$$Y_{\gamma,m}[k] = P_m\hat{Y}_{\gamma,m}[k]. \tag{3.6}$$

Then let $\hat{G}_{\gamma,m_1,m_2}$ denote the $m_1 \times m_2$ cross-Grammian matrix associated with the solution of two separate equations of form (3.1), identified by the subscripts 1 and 2, such that the (i,j)-th element of G_{γ,m_1,m_2} is given by

$$\sum_{k=0}^{\infty} y_{1,b}^{(-\gamma)}[k-i]y_{2,b}^{(-\gamma)}[k-j],\ i = 0, 1, 2, \ldots, m_1-1,\quad j = 0, 1, 2, \ldots, m_2-1. \tag{3.7}$$

Similarly, let G_{j,m_1,m_2} denote the corresponding cross-Grammian matrix based on backward differences whose (i,j)-th element is

$$\sum_{k=0}^{\infty} y_{1,b}^{(-\gamma+i)}[k]y_{2,b}^{(-\gamma+j)}[k],\ i = 0, 1, \ldots, m_1-1,\quad j = 0, 1, \ldots, m_2-1. \tag{3.8}$$

These Grammians depend on the coefficients $a_{1,0}, a_{1,1}, \ldots, a_{1,n_1-1}, a_{2,0}, a_{2,1}, \ldots, a_{2,n_2-1}$, although we do not denote that explicitly. Equivalently, we have

$$\hat{G}_{\gamma,m_1,m_2} = \sum_{k=0}^{\infty}\hat{Y}_{1,\gamma,m_1}[k]\hat{Y}_{2,\gamma,m_2}[k]^T \tag{3.9}$$

$$G_{\gamma,m_1,m_2} = \sum_{k=0}^{\infty}Y_{1,\gamma,m_1}[k]Y_{2,\gamma,m_2}[k]^T \tag{3.10}$$

and then from (3.6)

$$G_{\gamma,m_1,m_2} = P_{m_1}\hat{G}_{\gamma,m_1,m_2}P_{m_2}^T. \tag{3.11}$$

These cross-Grammians can be calculated as solutions to Sylvester-systems of size $n_1 \times n_2$, provided $m_1 \le n_1$ and $m_2 \le n_2$, where n_1 and n_2 denotes the order of (3.1) in each case. This is shown by the next theorem and corollary, but note that even if $m_1 < n_1$ and/or $m_2 < n_2$, we must solve a system of size $n_1 \times n_2$ and then obtain the desired Grammian matrix as the $m_1 \times m_2$ principal submatrix of the solution. Also note, that if $m_1 \ge n_1$ or $m_2 \ge n_2$, we have the option of increasing n_1 and/or n_2 by adding extra zero poles.

Theorem 3. The cross-Grammian matrix $\hat{G}_{\gamma,n_1,n_2}$ (3.10) is the solution of the following discrete Sylvester equation

$$C_1XC_2^T - X + P_{n_1}u_{1,\gamma}u_{2,\gamma}^TP_{n_2}^T = 0,\ \gamma = 0,1,2,\ldots, \quad (3.12)$$

where C denotes the $n \times n$ companion matrix

$$\begin{bmatrix} -a_{-1} & -a_{-2} & \cdots & -a_{-n+1} & -a_{-n} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad (3.13)$$

P_n denotes the Pascal matrix and u_γ is an n-vector whose i-th element is $1 + \sum_{j=1}^{\gamma+1-i} c_j$ where $c_i, i = 1,2,\ldots,\gamma$ are determined by (2.31).

Proof: $y_b^{(-\gamma)}[k]$, the transient part of the basic response of order γ, satisfies the homogeneous difference equation (3.1) setting $I_\gamma[k] \equiv 0$, and it can be deduced from (2.30), setting $m = 0$ and $b_0 = 1$, and (2.26), that it further satisfies the initial conditions

$$\nabla^i y[0] = \begin{cases} 1 + \sum_{j=1}^{\gamma-i} c_j, & i = 0,1,\ldots,\gamma-1 \\ 1, & i = \gamma,\gamma+1,\ldots,n-1. \end{cases} \quad (3.14)$$

This follows from the fact that $\nabla^i y_\gamma[0] = 1$, $i = 0,1,\ldots,n-1$ by the definition of the basic response of order γ, and

$$\nabla^i \rho_\gamma[k] = \begin{cases} \left[\, 0 \cdots 0 \ \rho_{\gamma-i}[k] \,\right]^T & \text{if } \gamma > i \\ \left[\, 0 \cdots 0 \,\right]^T & \text{if } \gamma \le i. \end{cases} \quad (3.15)$$

Equivalently $y^{(-\gamma)}[k]$ satisfies the same homogeneous difference equation and the initial conditions

$$\begin{bmatrix} y[0] \\ y[-1] \\ \vdots \\ y[-n+1] \end{bmatrix} = P_n \begin{bmatrix} y[0] \\ \nabla y[0] \\ \vdots \\ \nabla^{n-1}y[0] \end{bmatrix} = P_n u_\gamma.$$

Thus, yet another equivalent formulation is that $y^{(-\gamma)}[k]$ satisfies the first order system

$$Y[k] = CY[k-1] \quad Y[0] = P_n u_\gamma, \quad (3.16)$$

where $Y[k] = [y[k]\ y[k-1] \cdots y[k-n+1]]^T$ and C is the companion matrix (3.13). The results now follows from a straight forward well known argument. □

Corollary 1: The cross-Gramian matrix G_{γ,n_1,n_2} is the solution of the following Sylvester equation

$$\tilde{C}_1X\tilde{C}_2^T - X + u_{1,\gamma}u_{2,\gamma}^T = 0, \quad \gamma = 0,1,2,\ldots \quad (3.17)$$

where $\tilde{C}_1 = P_{n_1}C_1P_{n_1}$, $\tilde{C}_2 = P_{n_2}C_2P_{n_2}$.

Proof: The result follows from (3.11) and (3.12) and the fact that $P_n^{-1} = P_n$. □

Remark 3. The cross-Grammian matrix G_{γ,m_1,m_2} (3.9) is also given by the following expression

$$G_{\gamma,m_1,m_2} = \mathcal{K}_{1,\gamma,m_1-1,n_1-1}^T W_{n_1,n_2} \overline{\mathcal{K}_{2,\gamma,m_2-1,n_2-1}}, \quad (3.18)$$

where $\mathcal{K}_{\gamma,m,n}$ is the $n \times (m+1)$ matrix (2.25) and W_{n_1,n_2} is an $n_1 \times n_2$ matrix whose (i,j)-th element with

$$i = \sum_{k=1}^{t-1} d_{1,k} + r \text{ and } j = \sum_{k=1}^{u-1} d_{2,k} + s \quad (3.19)$$

is given by

$$\frac{\sum_{p=1}^{\min(r,s)} \binom{r-1}{p-1} \lambda_{1,t}^{r-p} \binom{s-1}{p-1} \bar{\lambda}_{2,u}^{s-p}}{(1-\lambda_{1,t}\bar{\lambda}_{2,u})^{r+s-1}}, \quad (3.20)$$

$r = 1,\ldots,d_{1,t}, t = 1,\ldots,\nu_1, s = 1,\ldots,d_{2,u}, u = 1,\ldots,\nu_2$, the poles and their multiplicities associated with the two equations being denoted as in (2.19).

We have directly from Theorem 3 that

$$W_{n_1,n_2} = \sum_{k=0}^{\infty} \mathcal{E}_1[k]\mathcal{E}_2[k]^H \quad (3.21)$$

and the result now follows from the fact that by (2.27) and (2.28), the (i,j)-th element of W_{n_1,n_1} with i and j given by (3.19) is

$$\sum_{k=0}^{\infty} \binom{k}{r-1} \lambda_{1,t}^{k-r+1} \binom{k}{s-1} \lambda_{2,s}^{k-s+1}. \quad (3.22)$$

4. AN APPLICATION TO PID ZEROS

As an example of how the expressions above may be utilized, we derive an expression for zero–optimized open–loop PID coefficients, with respect to a response of arbitrary order. Here we follow an approach, presented, e.g., in Herjólfsson et al. [2006] and Hauksdóttir et al. [2007] for continuous systems. While open loop zero-optimization may lead to instability in the closed loop, it remains a useful tool for providing initial values for the PID coefficients in a two stage optimization approach, cf. Herjólfsson et al. [2012].

When applying PID control in open loop we are replacing (2.3) with the equation

$$\sum_{i=0}^{n} \alpha_{-i}\nabla^i y[k] = \left(k_I + k_P\nabla + k_D\nabla^2\right) \sum_{i=0}^{m} \beta_{-i}\nabla^i u[k], \quad (4.1)$$

k_I, k_P and k_D being the discrete PID coefficients. Introducing the $(m+3) \times 3$ matrix

$$B = \begin{bmatrix} \beta_0 & 0 & 0 \\ \beta_{-1} & \beta_0 & 0 \\ \beta_{-2} & \beta_{-1} & \beta_0 \\ \vdots & \vdots & \vdots \\ \beta_{-m} & \beta_{-m+1} & \beta_{-m+2} \\ 0 & \beta_{-m} & \beta_{-m+1} \\ 0 & 0 & \beta_{-m} \end{bmatrix}, \quad (4.2)$$

and the $\gamma \times 3$ matrix B_γ corresponding to the first γ rows of B, B being padded by zero-rows if $\gamma > m+3$, and the 3-vector $p = [k_I\ k_P\ k_D]^T$, it follows directly from (2.30) that the solution to (4.1) can be expressed in terms of the PID-coefficients as follows

$$\begin{aligned} y_\gamma[k] &= (\mathcal{K}_{\gamma,m+2,n}Bp)^T\mathcal{E}[k] - (D_{\mathbf{c}_\gamma}B_\gamma p)^T\rho_\gamma[k] \\ &= (Bp)^T Y_{\gamma,m+3}[k] - (D_{\mathbf{c}_\gamma}B_\gamma p)^T\rho_\gamma[k], \end{aligned} \quad (4.3)$$

with $Y_{\gamma,m}[k]$ being defined by (3.5).

We wish to track a system with a given transfer function

$$\frac{\beta_{r,0} + \beta_{r,-1}\hat{z} + \cdots + \beta_{r,-m_r}\hat{z}^{m_r}}{\alpha_{r,0} + \alpha_{r,-1}\hat{z} + \cdots + \alpha_{r,-n_r}\hat{z}^{n_r}}, \quad (4.4)$$

to which we apply the same forcing function $I_\gamma[k]$. Denoting it by $y_{r,\gamma}[k]$, we now wish to choose the PID-coefficients in such a way that

$$\sum_{k=0}^{\infty}(y_\gamma[k] - y_{r,\gamma}[k])^2 \tag{4.5}$$

is minimized. Since the infinite sums of the non-transient parts of $y_\gamma[k]$ and $y_{r,\gamma}[k]$ are unbounded as $k \to \infty$, if $\gamma \geq 1$, whereas the infinite sums of the transient parts are bounded, assuming that both systems are stable, this implies that the PID-coefficients have to be chosen so that

$$\sum_{k=0}^{\infty}((Bp)^T Y_{\gamma,m+3}[k] - \mathcal{B}_r^T Y_{r,\gamma,0}[k])^2 \tag{4.6}$$

is minimized subject to the constraint that

$$(D_{\mathbf{c}_\gamma} B_\gamma p)^T \rho_\gamma[k] \equiv (D_{\mathbf{c}_{r,\gamma}} \mathcal{B}_{r,m_r,\gamma})^T \rho_{r,\gamma}[k], \tag{4.7}$$

if $\gamma \geq 1$. The additional suffix, r, denotes that the corresponding expression is for the reference system. The vector $\mathcal{B}_{r,m_r,\gamma}$ consists of the first γ elements of the vector $\mathcal{B}_{r,m_r}$. Equivalently the PID-coefficients can be chosen so that

$$p^T B^T G_\gamma Bp - 2p^T B^T H_\gamma \mathcal{B}_r \tag{4.8}$$

is minimized, subject to the same constraint, where G_γ is the $(m+3) \times (m+3)$ Grammian matrix

$$\sum_{k=0}^{\infty} Y_{\gamma,m+3}[k] Y_{\gamma,m+3}[k]^T, \tag{4.9}$$

which can be determined from Theorem 3 as the $(m+3) \times (m+3)$ principal minor of the $n \times n$ solution of the given Lyapunov equation where we have increased n, if necessary, such that $n \geq m+3$, by adding $m+3-n$ zero poles. H_γ is the $(m+3) \times (m_r+1)$ cross Grammian matrix

$$\sum_{k=0}^{\infty} Y_{\gamma,m+3}[k] Y_{r,\gamma,m_r+1}[k]^T, \tag{4.10}$$

which can again be determined from Theorem 3 as the $(m+3) \times (m_r+1)$ principal minor of the $n \times n_r$ solution of the given Sylvester equation, increasing n and n_r as before, if necessary. Thus the PID-coefficients will be determined directly by the following set of equation

$$\begin{bmatrix} A_\gamma & U_\gamma^T \\ U_\gamma & 0 \end{bmatrix} \begin{bmatrix} p \\ \frac{1}{2}\Lambda \end{bmatrix} = \begin{bmatrix} V_\gamma \\ W_\gamma \end{bmatrix}, \tag{4.11}$$

where Λ is a $\gamma \times 1$ vector of Lagrange multipliers,
$A_\gamma = B^T G_\gamma B$, a 3×3 matrix,
$U_\gamma = D_{\mathbf{c}_\gamma} B_\gamma$, a $\gamma \times 3$ matrix,
$V_\gamma = B^T H_\gamma \mathcal{B}_{r,m_r}$, a 3×1 vector and
$W_\gamma = D_{r,\mathbf{c}_{r,\gamma}} \mathcal{B}_{r,m_r,\gamma}$, a $\gamma \times 1$ vector.

Example 1: Impulse response minimization - $\gamma = 0$.

Consider the third order system having poles at 0.5, 0.6 and 0.8 and a unity gain given by

$$\frac{Y(z)}{U(z)} = z^2 \frac{0.04}{z^3 - 1.9z^2 + 1.18z - 0.24}, \tag{4.12}$$

where $n = 3$ and $m = 1$. We then have

$$\mathcal{A}_3 = \begin{bmatrix} \alpha_0 \\ \alpha_{-1} \\ \alpha_{-2} \\ \alpha_{-3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1.9 \\ 1.18 \\ -0.24 \end{bmatrix} = \begin{bmatrix} 0.04 \\ 0.26 \\ 0.46 \\ 0.24 \end{bmatrix} \tag{4.13}$$

and

$$\mathcal{B}_1 = \begin{bmatrix} \beta_0 \\ \beta_{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.04 \end{bmatrix} = \begin{bmatrix} 0.04 \\ -0.04 \end{bmatrix}. \tag{4.14}$$

The reference system is given by

$$\frac{Y_r(z)}{U(z)} = \frac{0.3}{z - 0.7}, \tag{4.15}$$

where $n_r = 1$ and $m_r = 1$. Then

$$\mathcal{A}_{r,1} = \begin{bmatrix} \alpha_{r,0} \\ \alpha_{r,-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.7 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} \tag{4.16}$$

and

$$\mathcal{B}_{r,1} = \begin{bmatrix} \beta_{r,0} \\ \beta_{r,-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.3 \\ -0.3 \end{bmatrix}. \tag{4.17}$$

We now compute the solution to the discrete Sylvester equation

$$\tilde{C} G_0 \tilde{C}^T - G_0 + u_0 u_0^T = 0, \tag{4.18}$$

where

$$\tilde{C} = P_4 \begin{bmatrix} 1.9 & -1.18 & 0.24 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} P_4 \tag{4.19}$$

where we have added a zero in the top row since $n < m+3$. We further have

$$u_0 = [\,1\ 1\ 1\ 1\,]^T \tag{4.20}$$

and using Matlab's `dlyap` to solve the discrete Sylvester equation we obtain

$$G_0 = \begin{bmatrix} 44.1553 & 1.3143 & -1.9502 & -0.8784 \\ 1.3143 & 2.6285 & 0.6783 & -0.2001 \\ -1.9502 & 0.6783 & 1.3567 & 1.1566 \\ -0.8784 & -0.2001 & 1.1566 & 2.3131 \end{bmatrix}. \tag{4.21}$$

Then

$$B = \begin{bmatrix} 0.04 & 0 & 0 \\ -0.04 & 0.04 & 0 \\ 0 & -0.04 & 0.04 \\ 0 & 0 & -0.04 \end{bmatrix} \tag{4.22}$$

and

$$A_0 = \begin{bmatrix} 0.0706 & 0.0021 & -0.0031 \\ 0.0021 & 0.0042 & 0.0011 \\ -0.0031 & 0.0011 & 0.0022 \end{bmatrix}. \tag{4.23}$$

Next, we compute the solution to the discrete Sylvester equation

$$\tilde{C} H_0 \tilde{C}_r - H_0 + u_0 u_{r,0}^T = 0, \tag{4.24}$$

which results in

$$H_0 = \begin{bmatrix} 3.8734 & 1.0000 \\ 1.4620 & 1.0000 \\ 0.7386 & 1.0000 \\ 0.5216 & 1.0000 \end{bmatrix}. \tag{4.25}$$

Then, we have

$$V_0 = B^T H_0 \mathcal{B}_{r,1} = [\,0.0289\ 0.0087\ 0.0026\,]^T \tag{4.26}$$

and

$$p = [\,k_I\ k_P\ k_D\,]^T = A_0 \backslash V_0 = [\,0.4048\ 1.6096\ 0.9769\,]^T. \tag{4.27}$$

The resulting step responses and pole (`x`) zero (`o`) diagram for the open loop is shown in Figure 1. The resulting step responses and root locus for the closed loop is shown in Figure 2. All closed loop poles are labeled by an $*$ on the corresponding root locus.

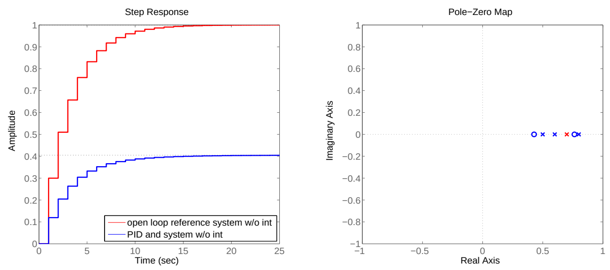

Fig. 1. Example 1. A third order system with impulse optimized PID zeros running in open loop.

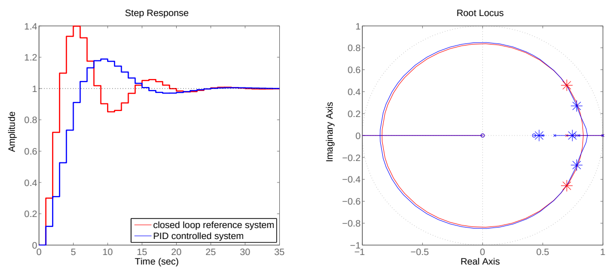

Fig. 2. Example 1. A third order system with impulse optimized PID zeros running in closed loop.

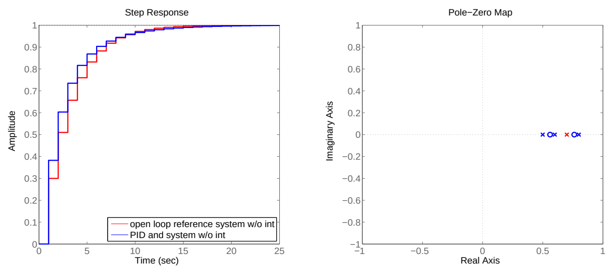

Fig. 3. Example 2. A third order system with step optimized PID zeros running in open loop.

Example 2: Step response minimization - $\gamma = 1$.

In the case of step response minimization, we have

$$u_1 = \left[\, 1 + c_1 \;\; 1 \;\; 1 \;\; 1 \,\right]^T \left[\, -24 \;\; 1 \;\; 1 \;\; 1 \,\right]^T \tag{4.28}$$

and

$$u_{r,1} = \left[\, 1 + c_{r,1} \;\; 1 \,\right]^T = \left[\, -2.3333 \;\; 1 \,\right]^T . \tag{4.29}$$

Then,

$$D_{\mathbf{c}_\gamma} = [c_1] = -25, \text{ where } \alpha_0 c_1 = -1 \tag{4.30}$$

and

$$D_{\mathbf{c}_{\gamma,\gamma_\mathbf{r}}} = [c_{r,1}] = -3.3333, \text{ where } \alpha_{r,0} c_{r,1} = -1. \tag{4.31}$$

We now have

$$U_1 = D_{\mathbf{c_1}} B_1 = \left[\, -1 \;\; 0 \;\; 0 \,\right] \tag{4.32}$$

and

$$W_1 = D_{r,\mathbf{c}_{r,\gamma_r}} \beta_{r,m_{r,1}} = c_{r,1}\beta_{r,0} = -1. \tag{4.33}$$

This results in

$$p = \left[\, k_I \;\; k_P \;\; k_D \,\right]^T = \left[\, 1.0000 \;\; 4.4761 \;\; 4.0838 \,\right]^T , \tag{4.34}$$

the open loop results are shown in Figure 3 and the corresponding closed loop results are shown in Figure 4.

Comparing the pole-zero maps in Figures 1 and 3, one of the PID zeros (labeled `o`) is located to the left of the

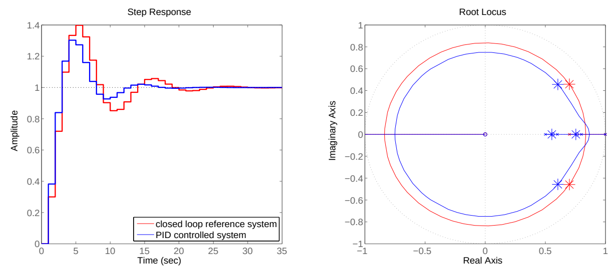

Fig. 4. Example 2. A third order system with step optimized PID zeros running in closed loop.

open loop poles (labeled `x`) in the impulse optimization, while in the step optimization the corresponding zero is located between the faster two system poles. While the interpretation of this is not obvious, it is clear that the DC gain is taken care of only in the step optimization case, due to the Lagrange constraint.

Comparing the root-loci in Figures 2 and 4, one can note that while the root-locus of the PID controlled system (in blue) is closer to that of reference system (in red) for the impulse optimization, the complex poles of the PID controlled system (labeled $*$) are closer to those of the reference system for the step optimization, resulting in a more similar overshoot and settling time in that case.

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