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# Fractional Order PI Tuning for Integrating Plants with Time Delay ⋆

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**Abstract:** In the last decade, non-integer order controllers have received great attention, due to their capacity of achieving robustness of the controlled loops with respect to gain and parameter variations of the plant. However, despite the general interest, technical literature offers few widely accepted and easy tuning techniques for these new controllers. To overcome the lack of simple tuning rules, we use open-loop shaping ideas for tuning non-integer order PI controllers of integrating plants with time delay. We illustrate the potentiality and limitation of the proposed technique through extensive simulation. Simplicity and satisfaction of requirements are remarkable characteristics of the method.

*Keywords:* Fractional order controllers, fractional order PI controllers, tuning, loop shaping, time delay.

## 1. INTRODUCTION

In recent years, an increasing number of papers deal with applications of fractional calculus to control engineering problems. For example, fractional order PID (FO-PID or $PI^{\lambda}D^{\mu}$) controllers generalize classical PID by introducing derivative and integral actions of noninteger orders $\lambda$ and $\mu$, respectively (Podlubny (1999)). Many studies, indeed, have shown that noninteger order controllers, commonly known as fractional order controllers (FOC), may not only increase closed-loop performance, enhance robustness, and offer more design degrees of freedom, but also impact on many industrial and mechatronic applications. Hence, FOC have the potentiality to replace the classical PID in a high percentage of control loops (Åström and Hägglund (1995)).

Namely, in comparison with the classical PID, the FO-PID are more effective for controlling plants described by FO models (Podlubny et al. (1997), Chen (2006)). In short, FOC must be preferred for controlling FO models. Even for controlling integer order (IO) plants, FO-PID provide better tuning flexibility than common PID give. Namely, they have several adjustable parameters for satisfying control requirements beyond the range of classical PID (Caponetto et al. (2004)). According to Ma and Hori (2004) and Ma and Hori (2007), applying FO-PID to motion control ensures robust controllers, that are realizable with reasonable approximations. To sum up, FO-PID can ensure a high robustness to gain and parameter variations of the plant, that is achievable with more complex IO controllers only (Chen (2006)).

Unfortunately, there exist few design and tuning techniques for FO-PID, but no established methodology or easy tuning rules, as it is for Ziegler-Nichols rules for standard PID. So, currently the control engineering literature attempts to generalize classical design. Many authors, indeed, develop simple tuning methods by starting with traditional setting procedures. For example, Caponetto et al. (2004) apply the classical frequency domain design to FO-PID and Narang et al. (2010) generalize the approach of Barbosa et al. (2004) for designing classical PID. Maione and Lino (2007) generalize the popular Symmetrical Optimum approach to FO-PI tuning for position servo systems. For implementing a two-inertia speed control, Ma and Hori (2004) preliminarily apply a classical tuning procedure and then improve the design with a fractional controller. Analogously, Barbosa et al. (2008) use the Ziegler-Nichols rule to the conventional controller of a velocity servo and then adjust the fractional order settings of a FO-PID.

To obtain the FO-PID settings, other authors apply optimization approaches or minimize integral of squared error (ISE) or integral of time-weighted absolute error (ITAE) performance indexes. For example, for controlling a dc-motor with an elastic shaft, Chen (2006) searches the best FO-PID parameters by using a routine that minimizes the ISE and the ITAE indexes and constrains the maximum torque. Analogously, for tuning the controller of a servo, Monje et al. (2004a) apply an iterative technique that minimizes a nonlinear function subject to some given nonlinear constraints. Also the approach followed by Bettou and Charef (2010), Charef et al. (1992) minimizes the ISE criterion to get the parameter settings. Finally, Cao and Cao (2006) apply the Particle Swarm Optimization (PSO) to the ITAE criterion to determine the tuning of a controller for a first-order with an integrator plant. Even if other authors put forward similar proposals, these optimization approaches do not possess the simplicity of IO-PID.

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In this paper, we propose a simple and systematic approach for tuning FO-PI of dc-motor position servosystems. As many authors, e.g. Barbosa et al. (2004), Monje et al. (2004a), Monje et al. (2004b), we refer to loop shaping procedure for achieving stated frequency domain specifications and ensuring robustness to gain changes. However, to tune a FO-PI, we avoid complex non linear minimization algorithms, GA (genetic algorithms) or PSO based techniques. Namely, we develop simple closed formulas, directly relating performance specifications to the fractional order $\nu$ and to the remaining controller parameters.

We model the plant by an integrator, a first-order lag, and a time delay. However, we start by considering the plant with no delay. We first request good tracking performance in a significant frequency range and robust stability to parameter changes, with an almost constant phase margin in a sufficiently wide frequency range around the gain crossover frequency. These requirements are satisfied by appropriately shaping the open-loop frequency response. By the proposed method, we easily obtain the tuned values of parameters $K_P$ and $K_I$ (or $K_I$ and $T_C$), and the order $\nu$, that satisfy the required design specifications.

Then, we analyze how to extend the tuning method to take into account the effect of time delays in the control loop. Delays may be intrinsic to the controlled plant or originated by the propagation of signals in the loop. Therefore, the plant model takes into account a pure time delay $L_E$. Clearly, this parameter affects the phase of the open-loop transfer function, so that the phase margin specification requires updating the former tuning rules.

The paper is organized as follows. Section 2 reviews some fundamental notions about fractional calculus and FOC. Section 3 introduces the proposed tuning approach and extends it when the plant has an inner deadtime. Section 4 provides simulation results. Finally, Section 5 concludes the paper.

## 2. FUNDAMENTALS OF FRACTIONAL SYSTEMS AND CONTROLLERS

Historically, factional calculus is dated back to three centuries ago, namely to a famous exchange of letters between Leibniz and de L'Hôpital. Main mathematical studies originated from Riemann-Liouville definition of the fractional integral operator. Nowadays, the mathematical literature offers many different definitions of fractional derivatives and integrals. Another well-known one is by Grünwald-Letnikov. We refer to the Caputo's definition (Caputo (1967)) of the fractional derivative of order $\nu$ of a function $f(t)$:

$$ {}_0D_t^{\nu} f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-\tau)^{n-\nu-1} f(\tau)\, d\tau \qquad (1)$$

where $n-1 < \nu < n$, $n$ is an integer and $\Gamma(\cdot)$ is the Euler gamma function. Namely, the Caputo's definition is preferred here because its Laplace transform is given by:

$$\mathscr{L}\left\{{}_0D_t^{\nu} f(t)\right\} = s^{\nu} F(s) - \Sigma_{k=0}^{n-1} f^k(0)\, s^{\nu-k-1} \qquad (2)$$

where the commonly used initial conditions appear. On the contrary, the Riemann-Liouville definition leads to initial conditions that do not have obvious physical interpretation. In (2), for $f^{(k)}(0) = 0$, $k = 0, 1, \ldots, (n-1)$, it holds: $\mathscr{L}\left\{{}_0D_t^{\nu} f(t)\right\} = s^{\nu} f(s)$. Even if (2) makes evident the meaning and potentialities of fractional order operators in applications, some difficulties arise because $s^{\nu}$ is irrational.

More specifically, to realize a FOC, we need to approximate the differintegral irrational operator $s^{\nu}$. There exist many rational approximations (see references in Maione (2009)). Here, we refer to an efficient one that can be obtained after truncating a continued fraction expansion (see Maione (2008)):

$$s^{\nu} \approx \frac{\alpha_N(\nu, s)}{\beta_N(\nu, s)} \qquad (3)$$

where the denominator and numerator are both $N$-degree polynomials, with $N \geq 1$, whose coefficients depend on $\nu$. More precisely:

$$\alpha_N(\nu, s) = \alpha_{N0}(\nu)s^N + \alpha_{N1}(\nu)s^{N-1} + \ldots + \alpha_{NN}(\nu) \qquad (4)$$

$$\beta_N(\nu, s) = \beta_{N0}(\nu)s^N + \beta_{N1}(\nu)s^{N-1} + \ldots + \beta_{NN}(\nu) \qquad (5)$$

More specifically, the coefficients of $\alpha_N(\nu, s)$ are determined by:

$$\alpha_{Nj}(\nu) = (-1)^j\, B(N, j)\, (\nu + j + 1)_{(N-j)}\, (\nu - N)_{(j)} \qquad (6)$$

where $B(N, j) = \frac{N!}{j!(N-j)!}$, for $j = 0, ..., N$, are binomial coefficients and:

$$(\nu + j + 1)_{(N-j)} = (\nu + j + 1)(\nu + j + 2) \cdots (\nu + N) \qquad (7)$$

$$(\nu - N)_{(j)} = (\nu - N)(\nu - N + 1) \cdots (\nu - N + j - 1) \qquad (8)$$

define the the Pochammer functions, with $(\nu+N+1)_{(0)} = 1$ and $(\nu - N)_{(0)} =1$. In addition, it holds: $\alpha_{Nj}(\nu) = \beta_{N,N-j}(\nu)$.

## 3. THE LOOP SHAPING TUNING METHOD

We refer to a classical unitary feedback control system, where the plant is a dc-servomotor with transfer function

$$G_p(s) = \frac{K_E}{s\,(1 + T_E\, s)} \qquad (9)$$

and the fractional-order $PI^{\nu}$ controller takes the form

$$G_c(s) = K_P + \frac{K_I}{s^{\nu}} = \frac{K_I}{s^{\nu}}\,(1 + T_C\, s^{\nu}) \qquad (10)$$

with $T_C = K_P/K_I$ and the fractional order $0 < \nu < 1$. The open-loop frequency response given by $G(s) = G_c(s)\, G_p(s)$ is then:

$$G(j\omega) = \frac{K_E\, K_I\, \{1 + \omega^{\nu}\, T_C\, [\cos(\hat{\theta}) + j\sin(\hat{\theta})]\}}{\omega^{(1+\nu)}\, \{\cos(\vartheta) + j\sin(\vartheta)\}\, (1 + j\,\omega\, T_E)} \qquad (11)$$

where $\vartheta = 0.5\,(1+\nu)\,\pi$ and $\hat{\theta} = 0.5\,\nu\,\pi$. Moreover, introducing the non-dimensional frequency $u = \omega\, T_E$ leads to:

$$G(ju) = \frac{K_E\, K_I\, \left\{1 + \left(\frac{u}{T_E}\right)^{\nu}\, T_C\, [\cos(\hat{\theta}) + j\sin(\hat{\theta})]\right\}}{\left(\frac{u}{T_E}\right)^{(1+\nu)}\, [\cos(\vartheta) + j\sin(\vartheta)]\, (1 + ju)}. \qquad (12)$$

If we consider the closed-loop transfer function

$$F(ju) = \frac{1}{1 + G^{-1}(ju)} \qquad (13)$$

then applying the requirement $|F(ju)| \equiv 1$ states a perfect input-output tracking, namely $y(ju) \equiv r(ju)$. Of course, physical systems can only approximate this condition, in a limited bandwidth $u_B = \omega_B\, T_E$.

To obtain a stable performance despite changes in drive parameters, we ensure stability margins by appropriately shaping the open-loop frequency response around the crossover frequency. To this aim, we take advantage of the fractional integrator, which shows a "flat" phase diagram in a wide frequency interval and a magnitude diagram with fractional slope of $-20\,\nu$ dB/decade. More specifically, the values of $T_C$ and $\nu$ are selected to obtain a desired phase margin $PM_s$, which is held constant in a wide interval around the 0-dB crossover frequency $u_C$.

The tuning procedure first considers the demand of a bandwidth $u_B$ ensuring a good tracking response of the servosystem. Once we have fixed $u_B$, at the same time we get a good estimation of the 0-dB crossover frequency $u_C$ as a value belonging to the range $\left[\frac{u_B}{1.3}, \frac{u_B}{1.7}\right]$ (see Lurie and Enright (2000), Maciejowski (1989)). Hence, if we assume, for instance, $u_C = \frac{u_B}{1.7}$, then we have the value of the frequency around which to guarantee a specified phase margin, say $PM_s$, in a wide range of frequencies around $u_C$. With this consideration in mind, we begin to guarantee the requested $PM_s$. If we consider the phase of the open-loop transfer function (12), we can write, with $S = \sin(\hat{\theta})$ and $C = \cos(\hat{\theta})$:

$$\angle G(ju) = \tan^{-1}\left(\frac{\left(\frac{u}{T_E}\right)^{\nu} T_C\, S}{1+\left(\frac{u}{T_E}\right)^{\nu} T_C\, C}\right) - \tan^{-1}(u) - \vartheta. \tag{14}$$

Then, by using $u = u_C$, the definition of phase margin gives:

$$\begin{aligned} PM &= \pi + \arg\{G(ju_C)\} \\ &= \varphi_1(u_C) - \varphi_2(u_C) + 0.5\,(1-\nu)\,\pi. \end{aligned} \tag{15}$$

where $\varphi_1(u)$ and $\varphi_2(u)$ are the first and second arguments in (14), respectively. Now, if $\varphi_1(u_C) = \varphi_2(u_C)$, then

$$PM = PM_s = 0.5\,(1-\nu)\,\pi. \tag{16}$$

If there exists an appropriate parameter value $T_C = \overline{T}_C$ leading to $\varphi_1(u_C) = \varphi_2(u_C)$, then (16) directly relates the fractional order, $\nu$, and the phase margin, $PM$. For $T_C = \overline{T}_C$ it holds:

$$u_C = \frac{\left(\frac{u_C}{T_E}\right)^{\nu} \overline{T}_C\, S}{1+\left(\frac{u_C}{T_E}\right)^{\nu} \overline{T}_C\, C}. \tag{17}$$

Equation (17) yields:

$$\overline{T}_C = \frac{u_C}{\left(\frac{u_C}{T_E}\right)^{\nu} (S - u_C\, C)} \tag{18}$$

Now, by putting $u_C = \frac{u_B}{1.7}$ in (19), we can express $\overline{T}_C$ in terms of $u_B$ and $\nu$ as follows:

$$\overline{T}_C = \overline{a}\, u_B^{(1-\nu)}\, T_E^{\nu} = \overline{b}\, u_C^{(1-\nu)}\, T_E^{\nu} \tag{19}$$

where:

$$\overline{a} = \frac{(1.7)^{\nu}}{(1.7\,S - u_B\, C)} \tag{20}$$

$$\overline{b} = \frac{1}{S - u_C\, C} \tag{21}$$

Note that $\overline{a} > 0$ and $\overline{b} > 0$ must hold true, to guarantee $\overline{T}_C > 0$. Therefore, values of $u_B$ and of $\nu$ are limited by this constraint.

At this point, to determine $K_I$, let us consider the open-loop transfer function for $u = u_C$. The definition of gain crossover frequency gives $G(ju_C) = 1\,e^{-j\varphi_C}$, where $\varphi_C = \arg\{G(ju_C)\}$. Hence, $|G^{-1}(ju_C)|^2 = 1$. Using $u = \omega\, T_E$ yields:

$$|G^{-1}(ju_C)|^2 = A(u_C)\, B(u_C) \tag{22}$$

with

$$A(u_C) = \frac{1}{K_E^2\, K_I^2}\left(\frac{u_C}{T_E}\right)^{2(1+\nu)} \quad \text{and} \tag{23}$$

$$B(u_C) = \frac{1+u_C^2}{1+\overline{b}^2\, u_C^2 + 2\,\overline{b}\, u_C\, C}. \tag{24}$$

Therefore, the obvious equality $A(u_C)\, B(u_C) = 1$ leads to:

$$K_I = \frac{1}{K_E}\left(\frac{u_C}{T_E}\right)^{(1+\nu)} \sqrt{B(u_C)}. \tag{25}$$

Table 1 specifies the values of the parameters that can be obtained with fractional orders between 0.3 and 0.6, referring to the parameters of the example discussed in section 4 (i.e. $K_E = 0.9779$, $T_E = 0.0798$ s, and $u_B = 0.7$).

Table 1. Parameters of the $PI^{\nu}$-controller for the plant in section 4 as functions of the fractional order $\nu$, with $u_B = 0.7$

| $\nu$ | $PM_s$ | $\overline{a}$ | $\overline{b}$ | $\overline{K}_P$ | $\overline{K}_I$ |
|---|---|---|---|---|---|
| 0.3 | 63° | 7.9185 | 11.4803 | 4.7858 | 1.6563 |
| 0.4 | 54° | 2.8561 | 3.9268 | 3.6964 | 4.4071 |
| 0.5 | 45° | 1.8439 | 2.4042 | 3.0727 | 7.0506 |
| 0.6 | 36° | 1.4264 | 1.7637 | 2.6856 | 9.8982 |

Note that, for $\nu < 0.3$, $\overline{a} < 0$ and $\overline{b} < 0$, so that the controller can not be designed. For $\nu > 0.6$, the phase margin takes values that usually are considered too low for a robustness specification.

### *3.1 Plant with time delay*

Now we extend the proposed tuning method to plant models with deadtime, that may be intrinsic or induced by the propagation of signals in the loop. Hence, consider the following plant transfer function (in non-dimensional frequency):

$$G_p(ju) = \frac{K_E\, T_E}{j\,u\,(1+j\,u)}\, e^{-j\,\frac{L_E}{T_E}\,u}. \tag{26}$$

Note that, if $PM_s$ is specified, then the stability condition requires $L_E < DM$, where $DM = \frac{PM_s\, T_E}{u_C}$ is the delay margin, i.e. the maximum allowed time delay corresponding to $PM_s$. For $\nu = 0.3$, 0.4, 0.5, 0.6, we obtain $DM = 0.2131$, 0.1827, 0.1522, 0.1218 s, respectively.

Therefore, as to how we establish the system specification, we note that the deadtime does not influence the amplitude of $G(ju)$, whereas it affects the specification on the stability margin. Namely, the new argument of $G(ju)$ is:

$$\angle G(ju) = \tan^{-1}\left(\frac{\left(\frac{u}{T_E}\right)^{\nu} T_C\, S}{1+\left(\frac{u}{T_E}\right)^{\nu} T_C\, C}\right) - \tan^{-1}(u) - \vartheta - \frac{L_E\, u}{T_E} \quad (27)$$

and leads to a new phase margin in the gain crossover frequency $u_C$:

$$\begin{aligned} PM &= \pi + \arg\{G(ju_C)\} \\ &= \varphi_1(u_C) - \varphi_2(u_C) - L_E\,\frac{u_C}{T_E} + 0.5\,(1-\nu)\,\pi. \end{aligned} \quad (28)$$

If we specify the same phase margin $PM_s = 0.5\,(1-\nu)\,\pi$, then we find the value $T_C = \overline{T}_C$ leading to $\varphi_1(u_C) - L_E\,\frac{u_C}{T_E} = \varphi_2(u_C)$, where $\varphi_1(u)$ and $\varphi_2(u)$ are the first and second phase components in (27). With this value it follows:

$$u_C = \tan(\alpha - \beta) \quad (29)$$

with

$$\alpha = \arctan\left(\frac{\left(\frac{u_C}{T_E}\right)^{\nu} \overline{T}_C\, S}{1+\left(\frac{u_C}{T_E}\right)^{\nu} \overline{T}_C\, C}\right),\ \beta = L_E\,\frac{u_C}{T_E}. \quad (30)$$

Then, we obtain:

$$\overline{T}_C = \frac{u_C + \tau}{\left(\frac{u_C}{T_E}\right)^{\nu} \left[S - u_C\, C - \tau\,(C + u_C\, S)\right]} \quad (31)$$

with $\tau = \tan\left(L_E\,\frac{u_C}{T_E}\right)$. Equation (31) takes the same form of (19) by putting:

$$\overline{a} = \frac{(1.7)^{\nu}\,(u_B + 1.7\,\tau)}{u_B\left[1.7\,S - u_B\,C - \tau\,(1.7\,C + u_B\,S)\right]} \quad (32)$$

$$\overline{b} = \frac{u_C + \tau}{u_C\left[S - u_C\,C - \tau\,(C + u_C\,S)\right]}. \quad (33)$$

Hence, with the previous procedure and the new values (32) or (33) for $\overline{a}$ or $\overline{b}$, (25) allows us to determine $\overline{K}_I$. However, in this case, there exists a maximum allowed delay ensuring $\overline{a} > 0$ and $\overline{b} > 0$, i.e. $\overline{T}_C > 0$. This limit value is:

$$L_{max} = \left(\frac{T_E}{u_C}\right)\arctan\left(\frac{S - u_C\,C}{C + u_C\,S}\right) \quad (34)$$

and depends on $u_C$, then $u_B$, for an assigned value of the fractional order $\nu$. Then, it must hold $L_E < L_{max}$.

Table 2. Maximum allowed delay $L_{max}$ and values of the tuned parameters, with $u_B = 0.7$ and $L_E = 0.0191$ s

| $\nu$ | $L_{max}$ [s] | $PM_s$ | $\overline{a}$ | $\overline{b}$ | $\overline{K}_P$ | $\overline{K}_I$ |
|---|---|---|---|---|---|---|
| 0.3 | 0.0156 | 63° | - | - | - | - |
| 0.4 | 0.0461 | 54° | 5.9838 | 8.2270 | 4.5618 | 2.5960 |
| 0.5 | 0.0765 | 45° | 2.9981 | 3.9091 | 3.7920 | 5.3514 |
| 0.6 | 0.1070 | 36° | 2.1074 | 2.6057 | 3.3143 | 8.2683 |

Table 2 reports the values of $L_{max}$ corresponding to $u_B = 0.7$ and different values of the fractional order $\nu$. Moreover, the same table specifies the values of the new tuned parameters, that are necessary to obtain the phase margin $PM_s$ corresponding to some value of $\nu$, if the delay is $L_E = 0.0191$ s. For $\nu \leq 0.3$, $\overline{a} < 0$ and $\overline{b} < 0$ because the delay is greater than the maximum allowed value,

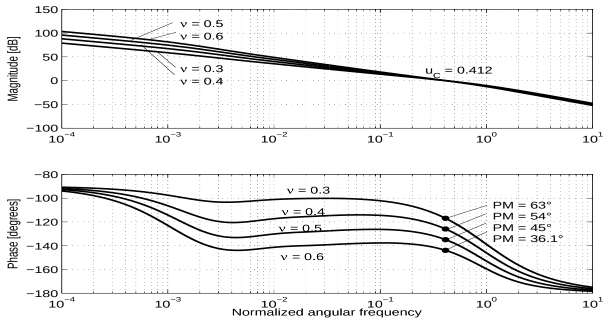


Fig. 1. Bode diagrams of the open-loop compensated transfer function $G(ju)$ for $0.3 \leq \nu \leq 0.6$

i.e. $L_E > L_{max}$. Then, tuning can not be made, even if $L_E < DM = 0.2131$ s. Table 2 does not show the values for $\nu \geq 0.7$, because the corresponding phase margin is usually considered low to be assumed as a robustness specification. A good value of $PM_s$ is, indeed, 35 or higher (Skogestad and Postlethwaite (2005)).

## 4. SIMULATION TESTS

To confirm the effectiveness of the tuning method, we test robustness to parameter variation or disturbance rejection. The simulation tests consider two different types of perturbations: an external step disturbance and a change in the time constant of the motor. The first type typically occurs when a motor drives a robotic arm carrying a load, which represents a constant resistance against the arm movement. The second type of perturbation is due to the different inertia values that can be associated to different positions of a robotic arm. We considered three cases, according to the maximum percentage amount of change (10%, 20%, and 30%) with respect to the nominal value of $T_E$. The nominal values of the plant model are: $K_E = 0.9779$, $T_E = 0.0798$ s, and $L_E = 0.0191$ s when a delay is considered.

### *4.1 Validation with no disturbance and no delay*

Firstly, we consider the reference step response without disturbances or delays. Table 1 gives the FOC parameters and Fig. 1 shows the open-loop frequency response after compensation by the FOC. Note how the phase margins are equal to the specified ones. Moreover, if we use a lower value of $u_B$, for example $u_B = 0.1$, $u_C$ is shifted to the left and falls in a more centered position into the range where the phase diagram is flat. Obviously, lower values for $u_B$ correspond to higher rise times.

Fig. 2 shows the step response of the controlled plant model. Table 3 indicates the performance indexes of the step responses for $0.3 \leq \nu \leq 0.6$ so that $\overline{a} > 0$ and $\overline{b} > 0$, and hence $\overline{T}_C > 0$. The overshoot ($OS\%$) and the settling time ($t_s$) to 2% of the steady-state value, increase with increasing $\nu$. The rise time ($t_r$) from 10% to 90% of the final value decreases with $\nu$. The results of $OS\%$ and $t_r$ are motivated by the corresponding decrease in the specified phase margin $PM_s = 0.5\,(1-\nu)\,\pi$.

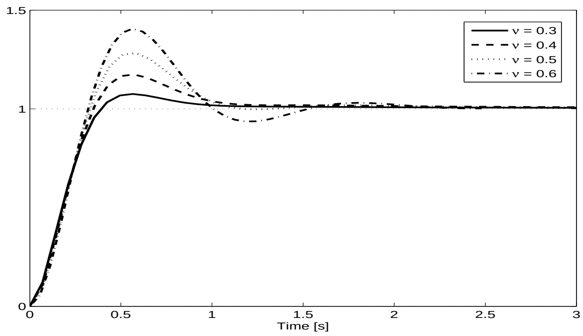


Fig. 2. Step response of the $PI^{\nu}$-controlled system with no disturbance or delay

Table 3. Performance indexes of the step response without disturbances or delays: $OS\%$ (percentage overshoot), $t_r$ (rise time), $t_s$ (settling time)

| $\nu$ | $OS\%$ | $t_r$ [s] | $t_s$ [s] |
|---|---|---|---|
| 0.3 | 7.54 | 0.2666 | 0.9710 |
| 0.4 | 17.39 | 0.2432 | 1.2101 |
| 0.5 | 28.27 | 0.2265 | 1.0514 |
| 0.6 | 40.58 | 0.2198 | 2.0270 |

### *4.2 Validation with external disturbance*

Now we analyze the disturbance rejection. We first test the tuning method by considering an external step disturbance (see Fig. 3).

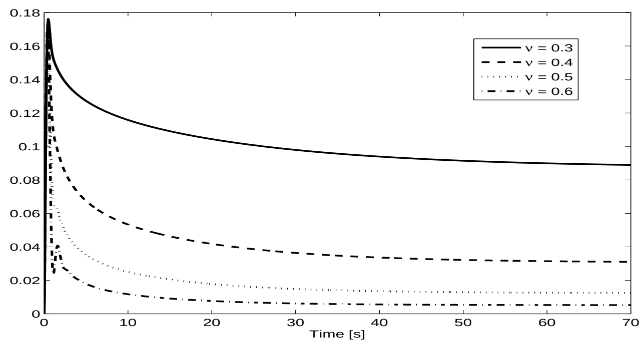


Fig. 3. Response to an external step disturbance

As it is expected, the steady-state values of the response decrease when $\nu$ increases. The values are 0.0873, 0.0306, 0.0124, and 0.0052, for $\nu = 0.3$, 0.4, 0.5, and 0.6, respectively. The peak values are 0.176, 0.172, 0.171, and 0.172. Finally, the settling times to 2% of the steady-state value decrease and are 67.8284, 40.6971, 26.6526, and 16.7576 s.

### *4.3 Validation by parameter variation*

We also consider a second type of perturbation due to changes of $T_E$ with respect to the nominal value, that can be determined by changes in the moment of inertia of the servomotor. We considered 10%, 20%, and 30% variations ($T_{E1} = 0.0878$ s, $T_{E2} = 0.0958$ s, $T_{E3} = 0.1037$ s). Then, we applied the same FO-PI controller tuned for the nominal plant. Figure 4 shows how the phase margin is not much affected. Variations of 20% and 30%

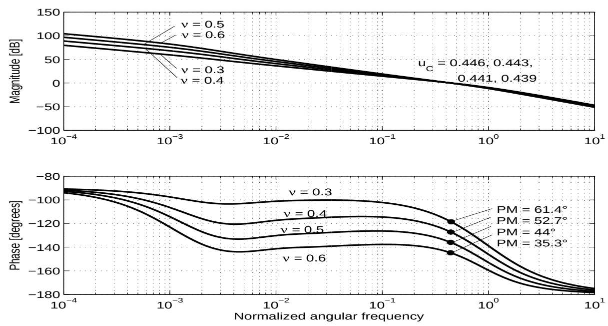


Fig. 4. Open-loop frequency response: 10% variation in $T_E$

Table 4. Performance indexes for changes of $T_E$

| Amount of variation | $\nu$ | $u_C$ | $PM$ | $OS\%$ | $t_r$ [s] | $t_s$ [s] |
|---|---|---|---|---|---|---|
| | 0.3 | 0.446 | 61.4° | 8.69 | 0.2699 | 0.9677 |
| 10% | 0.4 | 0.443 | 52.7° | 18.58 | 0.2478 | 1.1496 |
| | 0.5 | 0.441 | 44° | 29.47 | 0.2359 | 1.0598 |
| | 0.6 | 0.439 | 35.3° | 41.56 | 0.2275 | 2.0995 |
| | 0.3 | 0.479 | 60° | 9.84 | 0.2728 | 0.9693 |
| 20% | 0.4 | 0.474 | 51.5° | 19.72 | 0.2574 | 1.1120 |
| | 0.5 | 0.469 | 43° | 30.62 | 0.2411 | 1.9465 |
| | 0.6 | 0.465 | 34.5° | 42.49 | 0.2349 | 2.1683 |
| | 0.3 | 0.511 | 58.6° | 10.99 | 0.2808 | 0.9747 |
| 30% | 0.4 | 0.503 | 50.3° | 20.50 | 0.2660 | 1.0985 |
| | 0.5 | 0.497 | 42.1° | 31.67 | 0.2471 | 2.0404 |
| | 0.6 | 0.490 | 33.7° | 43.40 | 0.2418 | 2.2342 |

in $T_E$ give similar results and lead us to conclude that the controller design is robust with respect to variations of this parameter. On the other hand, the simulated step responses for the considered variations in $T_E$, further confirm the conclusion. Table 4 summarizes the associated performance indexes and the values obtained for the gain crossover frequency and the phase margin.

### *4.4 Validation with delay*

We consider a time delay $L_E = 0.0191$ s. Table 2 gives the FOC tuned parameters. Positive values of the parameters are obtained for $\nu \geq 0.4$. Figure 5 shows the step response affected by the considered deadtime and table 5 synthesizes the obtained performance.

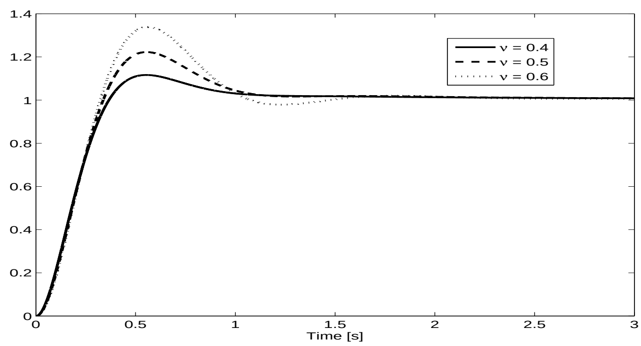


Fig. 5. Step response of the $PI^{\nu}$-controlled system with a delay $L_E = 0.0191$ s

Note that if we tune the FOC by taking into account the delay $L_E$, the $K_P$ gain does not vary too much with respect

Table 5. Performance indexes of the step response with a delay $L_E = 0.0191$ s

| $\nu$ | $OS\%$ | $t_r$ [s] | $t_s$ [s] |
|---|---|---|---|
| 0.4 | 11.65 | 0.2463 | 1.2502 |
| 0.5 | 22.27 | 0.2297 | 1.1683 |
| 0.6 | 33.88 | 0.2187 | 1.8373 |

to the value that is obtained for a plant without delay. See the values of $\overline{K}_P$ for $\nu = 0.4, 0.5, 0.6$ in tables 1 and 2 for a comparison. On the contrary, the $K_I$ gain considerably increases (see the same tables) and this motivates the corresponding increase in overshoot of the step response, even if the stability margin specification is verified in the same way.

## 5. CONCLUSION

This paper proposes a tuning method of the parameters in fractional-order PI controllers for plants modeled by a first order lag with an integrator and a time delay. The controlled plant shows good closed-loop performance and a high degree of robustness since the variations in step response are negligible also in presence of parameter variations, disturbances, and delay.

Hence the proposed loop shaping technique can be easily applied to tune FOC for common plants like dc-servomotors. The tuning method can consider time delays and obtains the required fractional robustness requirement by maintaining the phase margin in a wide frequency range. The delay, however, must be less than a value which depends on the plant time constant $T_E$, the specified closed-loop bandwidth $u_B$, and the fractional order $\nu$ of the integral action.

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