IFAC Conference on Advances in PID Control
PID'12
Brescia (Italy), March 28-30, 2012 FrPS.6

Multi criteria $\mathcal{H}_\infty$ optimal PID controllers from an undergraduate perspective

Bengt Lennartson *

* *Automation Research Group, Department of Signals and Systems, Chalmers University of Technology, Göteborg, Sweden. (e-mail: bengt.lennartson@chalmers.se)*

Abstract: A simple design method for robust PID controllers is presented. It is based on a multi criteria $\mathcal{H}_\infty$ optimal control formulation, which is shown to be easily solved by a few lines of MATLAB code. This optimal solution for PID controllers including low pass filtering, is complemented by a simple paper and pen solution that can be used to obtain nearly optimal solutions. The presented approach is shown to give significantly better results compared to ordinary text book solutions based on frequency domain loop shaping. The paper also includes a discussion on how to best formulate PID controllers for design, and how additional filtering may easily improve high frequency robustness.

Keywords: PID control, optimization, lead lag design, $\mathcal{H}_\infty$ control, performance, robustness, education

1. INTRODUCTION

In basic text books on feedback control, cf. Franklin et al. (2006); Ogata (2002), general design of PID controllers is mainly based on loop shaping in the frequency domain. This is complemented by tuning rules, often Ziegler Nichol's rules, Ziegler and Nichols (1942), and root locus design for special plant models. In loop shaping the controller is typically determined to obtain a specific gain cross over frequency and phase margin, to match desired demands on performance and robustness. Most often the low pass filter on the derivative action is neglected in the design. In e.g. Kristiansson and Lennartson (2006a) it is shown that to avoid too much sensitivity to sensor noise, the low pass filter will influence the stability margin significantly, and therefore must be a part of the design.

A systematic design method including this filtering of the derivative action is not very common in basic text books. But in e.g. Glad and Ljung (2006) it is observed that a PD controller including low pass filter is equivalent to a lead filter, while a PI controller can be interpreted as a lag filter with infinite lag ratio. To reduce the complexity of this PIPD design including four parameters, the maximum phase lift of the PD controller is recommended to be placed at the desired gain cross over frequency, the standard recommendation for lead filter design. The zero in the PI controller is typically chosen a decade below this frequency, which is the standard recommendation for lag filter design to avoid too much influence on the stability margins. This classical lead/lag based design method, where the two remaining parameters are determined to achieve a specific gain cross over frequency and phase margin, is evaluated systematically in this paper. The resulting controller is compared with a general PID controller formulated in Bode form, where also complex zeros can be allowed, as well as additional low pass filtering to further improve high frequency robustness.

To make a fair evaluation of any type of feedback control system it is important to consider the coupling between low-frequency, mid-frequency and high-frequency performance and robustness. For instance, improvement of load disturbance compensation implies normally either reduced pass-band robustness (stability margins) or reduced HF robustness due to increased lead action in the controller, Persson and Åström (1992). Based on these observations, a set of $\mathcal{H}_\infty$ performance and robustness criteria are used in this paper for general and objective evaluation of PID controllers, see also Lennartson and Kristiansson (1997); Kristiansson and Lennartson (2006a).

The evaluation procedure is formulated as a constrained optimization problem, and for a given controller structure there is an optimal solution to this multi criteria $\mathcal{H}_\infty$ optimal control problem. Based on this design procedure, optimal PI and PID controllers have recently been extensively evaluated. This includes a set of simple tuning procedures, Kristiansson and Lennartson (2006b) and the introduction of a new Robust Internal Model Control (RIMC) strategy, Lennartson and Kristiansson (2009). Alternative optimization strategies for PI and PID controllers have also been presented by e.g. Panagopoulos et al. (2002); Larsson and Hägglund (2011).

The presentation in this paper is made from a basic education and engineering perspective, where the multi criteria $\mathcal{H}_\infty$ optimal control problem is explicitly formulated as a short MATLAB program. Based on the generation of optimal PID controllers for some typical plant models in this paper, recommendations on suitable values for gain cross over frequency and phase margin are also obtained. Furthermore, for desired values of these parameters, it is shown how a simple paper and pen PID design solution can be formulated. This solution, which is based on a Bode plot of the plant model and two graphs, generates a PID controller that can be used to estimate suitable intervals for the control parameters in the search for an optimal solution.

In an introductory course on feedback control, this PID design strategy has been used successfully for a number of years. Based on the multi criteria optimization method, this optimal solution is also compared with the ones that can be achieved by including the standard simplifications for PIPD design presented above. These simplifications, which are presented to most students globally for the related lead/lag and PIPD design based on frequency loop shaping, are shown to give significant deterioration of load performance compensation. Some reasons for this performance loss are also discussed in the paper.

2. MULTIPLE CRITERIA $\mathcal{H}_\infty$ OPTIMIZATION

A set of important $\mathcal{H}_\infty$ criteria was introduced in Lennartson and Kristiansson (1997); Kristiansson and Lennartson (2006a) for design and evaluation of robust PID controllers. These criteria are briefly presented in this section from an undergraduate perspective. First, consider a system with a plant $G(s)$, where the load disturbance v is added to the control signal u and the sensor noise w is added (with minus sign) to the plant output y. This plant is controlled by a one-degree of freedom controller $K(s)$, resulting in a loop transfer $L(s) = G(s)K(s)$. Furthermore, the controller is assumed to include integral action, which means that it can be formulated as $K(s) = \bar{K}(s)K_i/s$ where $\bar{K}(0) = 1$.

2.1 Low frequency range

A fundamental goal of a feedback system is to compensate low frequency (LF) load disturbances. The transfer function from the disturbance v to the controlled output y is

$$G_{vy}(s) = \frac{G(s)}{1 + G(s)K(s)}$$

When the controller includes integral action we obtain in the LF range $1+L(j\omega) \approx L(j\omega))$ and $|G_{vy}(j\omega)| \approx \omega/K_i$. Obviously, the compensation of LF disturbances is improved when the integral gain K_i in the controller increases. Hence, a simple LF performance criterion to minimize is

$$\bar{J}_v = 1/K_i$$

To include a broader frequency range concerning load disturbances, we also consider the more complex $\mathcal{H}_\infty$ criterion

$$J_v = \max_\omega \frac{1}{\omega}|G_{vy}(j\omega)| = ||\frac{1}{s}G_{vy}(s)||_\infty \tag{1}$$

The frequency weight $1/\omega$ is included to still emphasize more on LF load disturbances. It is also motivated by the LF asymptotic behavior of $|G_{vy}(j\omega)|$ given above.

In MATLAB this criterion is easily computed, assuming that a model `Gvy` is available in terms of a transfer function or a state space model. Then the $\mathcal{H}_\infty$ norm is computed as `norm(Gvy/s,inf)`, where `s=tf('s')`.

2.2 High frequency range

In the high frequency (HF) range, it is important to avoid too much sensor noise in the control signal. Hence, consider the transfer function from the sensor noise $w(t)$ to the control signal $u(t)$

$$G_{wu}(s) = \frac{K(s)}{1 + G(s)K(s)}$$

The plant $G(s)$ is assumed to have low gain in the HF range ($\lim_{s\to\infty} G(s) = 0$), which means that $G_{uw}(j\omega) \approx K(j\omega)$ in this range. Furthermore, assuming that $K(s)$ is proper but not strictly proper, we make the additional approximation to consider the controller gain at $\omega = \infty$. By the notation $K_\infty = \lim_{s\to\infty} K(s)$ we then achieve $G_{wu}(j\omega) \approx K_\infty$. A reasonable HF criterion is therefore to consider the HF controller gain

$$\bar{J}_u = K_\infty$$

as a control activity measure. To guarantee that possible peaks in $K(s)$ in the HF range are not neglected when the HF properties of the control gain are considered, the parameter K_∞ may be replaced by the criterion

$$J_u = \max_\omega |G_{wu}(j\omega)| = ||G_{wu}(s)||_\infty \tag{2}$$

For controllers with higher gain in the HF range (the critical ones), these two measures give most often the same result. So far we have focused on sensor noise, but also note that the initial control amplitude, after a unit step in the reference signal, becomes K_∞, i. e. $u(0) = K_\infty$ when $r(t)$ = unit step. Thus, $\bar{J}_u = K_\infty$ and J_u are both relevant measures and need to be constrained to avoid too large control activity and sensitivity to sensor noise in the HF range.

When the controller has a roll off with an HF behavior $K(s) \approx K_\infty/s$, it is also of interest to consider the HF criterion

$$J_{HF} = \max_\omega \omega|G_{wu}(j\omega)| = ||sG_{wu}(s)||_\infty \tag{3}$$

Note that $J_{HF} \approx \max_\omega \omega K_\infty/\omega = K_\infty$, while J_u then measures the peak gain of G_{wu}, which typically occurs slightly above the mid frequency range. Hence, J_u may then be considered as a mid to high frequency measure, while J_{HF} is the HF criterion.

2.3 Mid-frequency range

In the pass band, robustness is achieved by ensuring good stability margins. Generally, the loop transfer $G(j\omega)K(j\omega)$ must be kept at an acceptable distance from the critical point (-1,0) in the Nyquist plot. To ensure this, different measures have been introduced like the classical phase margin φ_m and the gain margin G_m. More recently, the shortest distance to the point (-1,0), $\min_\omega |1 + G(j\omega)K(j\omega)|$, has been introduced as a stability measure. Consider the sensitivity function

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and its maximum gain

$$M_S = \max_\omega |S(j\omega)| = ||S(s)||_\infty \tag{4}$$

Obviously, this $\mathcal{H}_\infty$ criterion is the inverse of the shortest distance to the point (-1,0) in the Nyquist plot, and hence a lower value of M_S means a larger stability margin. For unstable plants, including those with integral action, it is also important to consider the complementary sensitivity function

$$T(s) = 1 - S(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$

and its maximum gain

$$M_T = \max_\omega T(j\omega)| = ||T(s)||_\infty \tag{5}$$

A restriction on M_T also controls the damping of the system, without reducing M_S too much. In this paper we will include the demands $M_S \leq 1.7$ and $M_T \leq 1.3$.

2.4 Controller design by multi criteria optimization

In all controller design, independent of method, the user has to adjust a set of tuning parameters ρ. This set may include the parameters in a PID controller or weighting function parameters in e.g. an LQG optimization criterion. The user has to adapt or tune these parameters to obtain a desired closed loop behavior, in our case measured by the $\mathcal{H}_\infty$ criteria suggested in this section.

An objective method to evaluate different design methods is to minimize one criterion with respect to the tuning parameters ρ, while constraints are introduced on the other criteria. In this paper the LF performance is minimized, while constraints are included on the mid and high frequency range criteria, i.e. the controllers are designed by solving the following constrained optimization problem

$$\min_{\rho} J_v(\rho)$$
$$M_S(\rho) \leq 1.7 \quad M_T(\rho) \leq 1.3 \quad J_u(\rho) \leq C_u \tag{6}$$

where the constant C_u depends on the plant model $G(s)$. It is chosen to give a reasonable control activity, while only marginally deteriorate the LF performance compared to a high gain solution. The HF criterion J_{HF} is considered as an optional measure for controllers with additional low pass filtering and roll off.

By this constrained multi criteria optimization procedure, different controllers may be designed and evaluated under equal conditions. For a paper and pen solution, including a simple calculator and Bode plots, the following alternative criteria can be applied.

$$\min_{\rho} \bar{J}_v(\rho) \quad \varphi_m(\rho) \geq 45^\circ \quad G_m(\rho) \geq 3 \quad \bar{J}_u(\rho) \leq C_u$$

Then a manual iterative search can be performed to achieve at least a semi optimal solution.

The expression *optimal controller* is from now on used for a controller, which is optimized according to (6) with all available controller parameters included in the tuning vector ρ. In this paper the routine `fmincon` from Matlab's Optimization Toolbox is used for the optimization.

3. PID CONTROLLERS

In this section different formulations of PID controllers will be discussed, and as a basis for further tuning, it is shown how a desired phase margin φ_m and gain cross over frequency ω_c can be achieved for the different controller formulations. The intention is mainly to efficiently compensate load disturbances. When good servo properties are required, the controller can always be augmented by a filter in the feed-forward path.

The traditional PID controller, with a low-pass filter on the derivative part, is often formulated as

$$K_{PID}(s) = K_p(1 + \frac{1}{sT_i} + \frac{sT_d}{1 + sT_f}) \tag{7}$$

3.1 PIPD controller

To get a better understanding of the PID controller in the frequency domain the controller can alternatively be formulated as a PI and PD controller connected in series, from now called a PIPD controller

$$K_{PIPD}(s) = K_p \frac{1 + s\tau_i}{s\tau_i} \frac{1 + s\tau_d}{1 + s\tau_d/b} \tag{8}$$

Note that the PD controller written in this form can also be considered as a lead filter with a lead ratio b, where the maximum phase lift $\varphi_{max} = \arcsin((b-1)/(b+1))$ occurs at the mid frequency $\omega_m = \sqrt{b}/\tau_d$. In the literature, see e.g. Franklin et al. (2006), it is recommended to choose the lead filter, here the PD controller, such that the gain cross over frequency ω_c is equal to or at least in the region of the mid frequency $\omega_m = \sqrt{b}/\tau_d$.

At the same time it is recommended to choose the integral time constant τ_i large enough to not disturb the phase too much at ω_c. A standard suggestion is to choose $1/\tau_i$ a decade lower than ω_c. This leads to a slow compensation of load disturbances. Hence, we moderate that suggestion by evaluating $1/\tau_i = 0.2\omega_c$, which results in a phase lag of approximately 11° at ω_c.

This classical PIPD design strategy will be evaluated in this paper. The PD controller is then designed such that a desired phase margin φ_m and cross over frequency ω_c are achieved, by placing the maximum phase lift at ω_c. The phase lag in the PI controller is compensated by adding extra phase lift (larger φ_{max}) at $\omega_m = \omega_c$ in the PD controller.

3.2 PID controller in Bode form

In e.g. Kristiansson and Lennartson (2006b) it has been shown that a PID controller, optimized with the filter included and all parameters free, often implies complex zeros in the controller. This has been verified for a large number of plants with poles strictly on the negative real axis. Hence, a suitable alternative parametrization of the PID controller is

$$K_{PID}(s) = K_i \frac{1 + 2\zeta\tau s + (\tau s)^2}{s(1 + s\tau/\beta)} \tag{9}$$

with the four parameters the *integral gain* K_i, the *high-frequency gain* $K_\infty = K_{PID}(\infty) = K_i\tau\beta$, the *zero damping* ζ and the *natural frequency* $1/\tau$. The notation β is short for $K_\infty/(K_i\tau)$, but may also be used as a design parameter, an alternative to K_∞.

Since this representation of the PID controller is written in Bode form, the parameters have a clear relation to the frequency function $K_{PID}(j\omega)$. It means that they have a natural physical meaning, especially K_i and K_∞ that define the low and high frequency behavior of the controller. For $\zeta \geq 1$ it is related to the PIPD formulation as $\tau_i = c\tau$ and $\tau_d = \tau/c$, where $c = \zeta + \sqrt{\zeta^2 - 1}$.

To achieve a desired phase margin φ_m and cross over frequency ω_c for this controller, we first observe generally that $|K_{PID}(j\omega_c)||G(j\omega_c)| = 1$ and $\angle K_{PID}(j\omega_c) = -180^\circ + \varphi_m - \angle G(j\omega_c)$. The gain and phase expressions for (9) then result in the following nonlinear relations

$$\angle K_{PID}(j\omega_c) = -90^\circ + \arccos \frac{1 - x^2}{\sqrt{(1 - x^2)^2 + (2\zeta x)^2}} - \arctan(x/\beta)$$

$$K_\infty |G(j\omega_c)| = \frac{x\beta\sqrt{1 + (x/\beta)^2}}{\sqrt{(1 - x^2)^2 + (2\zeta x)^2}}$$

where $x = \omega_c\tau$. Corresponding relations are illustrated in Fig. 1 for different values of β for the specific choice $\zeta = 1$. By selecting a desired K_∞ the second plot gives a suitable β, which by the first plot then determines τ, and finally $K_i = K_\infty/(\tau\beta)$.

When it comes to implementation it is straightforward to translate these design parameters in Bode form to the traditional ones in (7).

A PI controller can be considered as a special case of the PID controller (9) with $\beta = 1$ (low high-frequency gain) and $\zeta = 1$ (double zero).

$$K_{PI}(s) = K_i \frac{(1 + \tau s)^2}{s(1 + \tau s)} = K_i \frac{1 + \tau s}{s} \tag{10}$$

Desired values of the phase margin φ_m and the cross over frequency ω_c are then directly obtained by selecting $\tau =$

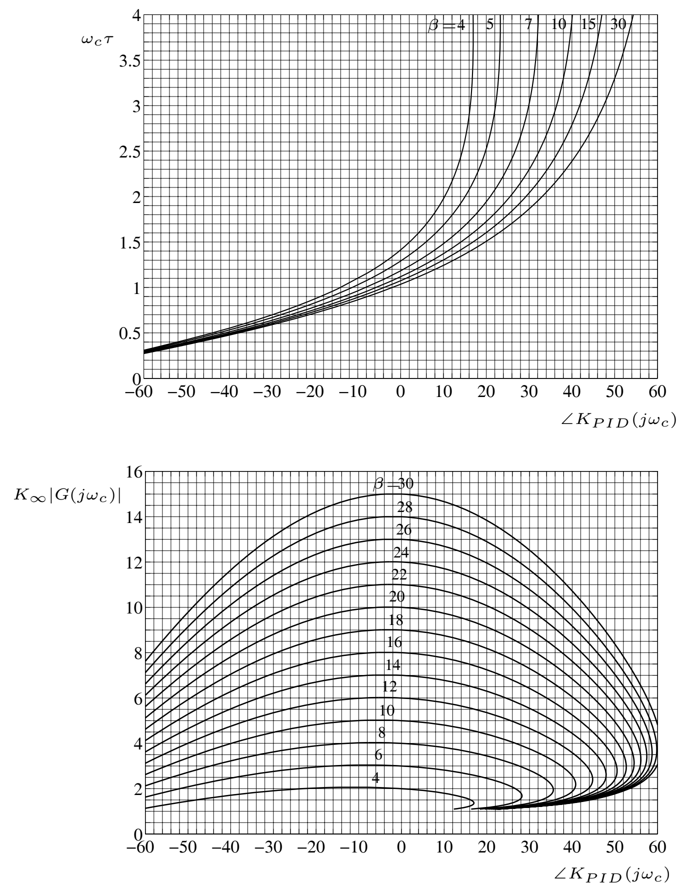

Fig. 1. $\omega_c\tau$ and $K_\infty|G(j\omega_c)|$ as functions of the desired PID controller phase $\angle K_{PID}(j\omega_c)$ for $\zeta = 1$ and different values of β.

$\tan(\varphi_m - 90^\circ - \angle G(j\omega_c))/\omega_c$ and $K_i = \omega_c/(|G(j\omega_c)| \sqrt{1+(\omega_c\tau)^2})$.

4. OPTIMIZATION AND EVALUATION

The PID controller (9) will now be optimized by solving the constrained optimization problem (6). Since J_u is most often equal to $\bar{J}_u = K_\infty = K_i\tau\beta$ for the reasonable control activities we are interested in, it is enough to specify a desired HF control gain K_∞, which then determines $\beta = K_\infty/(K_i\tau)$. The remaining parameters, K_i, τ, and ζ, are optimized by the following MATLAB program, based on the routine `fmincon` and applied to $G(s) = \frac{1}{(1+s)(1+0.5s)(1+0.25s)}$.

```
s=tf('s'); G=1/(1+s)/(1+0.5*s)/(1+0.25*s);
MS_max=1.7; MT_max=1.3; Kinf=15;
min_x=[3 0.4 0.6];    % Lower and upper limits
max_x=[5 0.9 1];      % on Ki, tau, zeta
x=(min_x+max_x)/2;    % Initial value = mean value
options = optimset('Algorithm','active-set');
x=fmincon(@(x) objfun(x,s,G,Kinf),x,[],[],[],[],...
   min_x,max_x,@(x) confun(x,G,MS_max,MT_max,Kinf),
   options);
Ki=x(1); tau=x(2); zeta=x(3); beta=Kinf/(Ki*tau);
K=tf(Ki*[tau^2 2*zeta*tau 1],[tau/beta 1 0]);

function Jv=objfun(x,s,G,Kinf)
   Ki=x(1); tau=x(2); zeta=x(3); beta=Kinf/(Ki*tau);
   Fd=tf(Ki*[tau^2 2*zeta*tau 1],[tau/beta 1]);
   Jv=norm(feedback(G/s,Fd),inf);
return

function [C,Ceq]=confun(x,G,MS_max,MT_max,Kinf)
   Ki=x(1); tau=x(2); zeta=x(3); beta=Kinf/(Ki*tau);
   K=tf(Ki*[tau^2 2*zeta*tau 1],[tau/beta 1 0]);
   MS=norm(feedback(1,G*K),inf);
   MT=norm(feedback(G*K,1),inf);
   stab=norm(feedback(G*K,1));
   if stab<inf, C=[MS-MS_max; MT-MT_max];
   else C=[1; 1]; end
   Ceq=[];
return
```

Control parameter intervals In this routine lower and upper limits on the three control parameters are defined, and J_v is computed by including the integral weighting $1/s$ in the loop. Generally, it is recommended to start with wider control parameter intervals, and then tighten when the optimal parameter region has been identified. Initial parameter intervals are preferably estimated by making a manual ω_c/φ_m design, based on the gain and phase of $G(j\omega_c)$ and the curves in Fig. 1. This design depends in its turn on a suitable choice of ω_c and φ_m, which will be further discussed below based on experience from typical plant models. If the control parameter intervals include a feasible solution, which satisfies the given constraints, the convergence of the optimization is very fast (<5 sec. for the given MATLAB example above on a standard LAPTOP-PC).

Plant models and their complexity Optimal controllers for the following plant models are evaluated in this paper.

$$G_1(s) = \frac{1}{(1+s)(1+0.5s)(1+0.25s)}$$
$$G_2(s) = \frac{1}{(1+s)^3}$$
$$G_3(s) = \frac{e^{-0.3s}}{(1+s)(1+0.5s)}$$
$$G_4(s) = \frac{1}{s(1+s)(1+0.2s)}$$
$$G_5(s) = \frac{(1+2s)}{s(1+0.2s+s^2)(1+0.02s)}$$
$$G_6(s) = \frac{e^{-s}}{(1+Ts)}$$

The complexity of a plant model can be characterized by its κ number, $\kappa = |G(j\omega_{180_G})|/G(0)$, cf. Hang et al. (1991), where ω_{180_G} is the frequency where the plant has a phase lag of -180°. From Table 2 it is clear that a minimum phase plant with one dominating pole has a κ close to zero, while Table 3 shows that a plant which is close to a pure time delay process has a $\kappa \approx 1$. For plants with integral action, the κ number is modified as $\kappa = \omega_{180_G}|G(j\omega_{180_G})|/(\lim_{\omega\to 0}\omega|G(j\omega)|)$.

Zero damping factor ζ For the plant models $G_1(s)$-$G_5(s)$, optimal $\mathcal{H}_\infty$ criteria based on (6), including optimal PID parameters, are presented in Table 1. The damping factor ζ in the PID controller (9) is clearly less than one, which motivates the introduction of the complex zeros in $K_{PID}(s)$. For the plant $G_5(s)$, which has a resonance with a damping factor 0.1, the optimal J_v is achieved for $\zeta = 0.28$, which unfortunately results in a negative overshoot of 40% in the load step response. This is avoided by introducing a min value of $\zeta = 0.5$ in the

Table 1. Optimal $\mathcal{H}_\infty$ criteria and PID parameters for different plant models.

Model	J_v	J_u	M_S	M_T	K_i	τ	ζ	β
$G_1(s)$	0.24	15	1.70	1.30	4.46	0.62	0.73	5.4
$G_2(s)$	0.57	15	1.70	1.30	1.97	1.17	0.69	6.5
$G_3(s)$	0.46	10	1.70	1.30	2.32	0.60	0.82	7.2
$G_4(s)$	0.74	20	1.70	1.30	1.38	1.35	0.77	10.7
$G_5(s)$	1.14	20	1.43	1.30	1.03	1.38	0.50	14.1

optimization, but at the cost of an increased J_v by 43%. This solution is shown in Table 1 and 2.

Gain cross over frequency ω_c From Table 2 and 3 it can also be concluded that the ratio ω_c/ω_{180G} is clearly less than one for the stable plants, typically around 0.6 for plants with lower κ numbers, and even lower ratios when κ increases. In Kristiansson and Lennartson (2006a) the following estimate was given

$$\omega_c/\omega_{180G} = 0.6 - 0.35\kappa \quad (11)$$

For plants with integral action this ratio is clearly higher; around one for the simple integral plant $G_4(s)$. A more systematic evaluation for different models gives the following rough estimate

$$\omega_c/\omega_{180G} = 1.4 - \kappa \quad (12)$$

for plants with integral action without resonances. These estimates are based on a reasonable choice of control activity J_u, see further details in Kristiansson and Lennartson (2006b).

The model $G_5(s)$ is highly resonant with a damping factor 0.1, and the kappa number gives no relevant information. But for this model another feature is observed, the phase curve is quite flat around -180° for nearly a decade, and for this model $\omega_c = 3.1\omega_{180G}$.

The general recommendation is therefore to relate the choice of ω_c to the shape of the plant phase curve around -180°, where a larger negative phase shift in this region results in a reduced ω_c. For stable non-oscillating plants the estimate (11) can be used, and (12) is useful for corresponding plants with integral action.

Phase margin φ_m The optimal phase margin φ_m is slightly above 45°, except for plants with high kappa numbers, see Table 3, where φ_m increases even up to 60°. The reason is that the phase of G is decreasing much faster than its gain, which implies a shorter distance to the point (-1,0) (larger M_S). However, since M_S is restricted, it is not allowed and to compensate for this, the loop gain has to be reduced, resulting in a large phase margin.

Controller phase $\angle K_{PID}(j\omega_c)$ In lead lag compensation the controller is expected to add a positive phase shift at ω_c. This is also often the case, see Table 2, but the optimization of J_v, which is approximately the same as optimizing $\bar{J}_v = 1/K_i$,

Table 2. Plant κ number and additional loop transfer properties for the optimal PID controllers presented in Table 1.

Model	κ	ω_c	ω_c/ω_{180G}	φ_m	$\angle K_{PID}(j\omega_c)$
$G_1(s)$	0.09	2.28	0.61	46.2°	11.0°
$G_2(s)$	0.13	1.11	0.64	45.9°	10.2°
$G_3(s)$	0.17	1.51	0.50	45.1°	-14.2°
$G_4(s)$	0.37	2.09	0.94	46.1°	43.3°
$G_5(s)$	4.21	3.88	3.08	49.5°	58.2°

Table 3. Load performance comparing optimal PID and PI controllers for $G_6(s)$, including additional loop transfer properties for the PID controllers. $J_u = 5G_6(0)$ for the PID and $J_u \approx G_6(0)$ for the PI controllers.

T	κ	J_v^{PID}/J_v^{PI}	ω_c/ω_{180G}	φ_m	$\angle K_{PID}(j\omega_c)$
1	0.44	0.69	0.38	50.3°	-47.1°
0.3	0.80	0.74	0.32	57.5°	-62.6°
0.1	0.96	0.84	0.27	59.5°	-71.6°
0.05	0.99	0.88	0.26	59.8°	-74.1°

means that a high K_i is desirable. For models with larger kappa numbers this results in a very large negative controller phase shift at ω_c, which is clearly demonstrated in Table 3.

PID or PIPD control Results from the optimization of the PIPD controller (8) are given in Table 4. Comparing with the optimal PID controllers in Table 1 where $\zeta < 1$, corresponding to complex zeros, the best PIPD controllers are obtained by choosing a double zero $\tau_i = \tau_d$. The performance deterioration in Table 4 is then varying between 10% and 40%, compared to the corresponding optimal PID controller.

For plants without integral action the product $\tau_i\omega_c$ is around one and the ratio $\omega_c/\omega_m \lesssim 0.5$ for the optimal PIPD controller. To force the integral time constant to be 5 times larger than $1/\omega_c$ is more costly, simply because we want to compensate the load disturbance down to zero as fast as possible. In other words we want the integral action to become active as fast as possible, and this occurs from the frequency $1/\tau_i$. Thus a smaller τ_i results in a faster complete compensation of a step load disturbance. Typical text book recommendations are $\tau_i = 10/\omega_c$, which of course gives additional performance deterioration.

The second demand to force ω_c at the maximum phase lift for the PD controller ($\omega_c = \omega_m$) results in even worse behavior. Indeed, for the plant model $G_3(s)$ it is not even possible to obtain a feasible solution demanding $\tau_i\omega_c = 5$ and $\omega_c = \omega_m$. However, the constraints $\tau_i\omega_c = 5$ and $\omega_c = 0.37\omega_m$ give a feasible solution that satisfy the stability demand $M_S \leq 1.7$.

The load disturbance step responses for the different PIPD controllers and the optimal PID controller are shown in Fig. 2 for $G_1(s)$ and $G_3(s)$. They confirm our conclusion that standard text book recommendations for PIPD loop shaping, more specifically $\tau_i\omega_c = 5 - 10$ and $\omega_c = \omega_m$, result in quite bad performance or does not even give a feasible solution. This can be compared to what is possible with an optimal PID controller

Table 4. Optimal criteria and parameters for PIPD controllers with specific equality constraints denoted in bold numbers, compared with optimal PID controllers. The same control activity J_u and stability constraints as in Table 1.

Model	J_v^{PIPD}/J_v^{PID}	τ_i	τ_i/τ_d	$\tau_i\omega_c$	ω_c/ω_m	b
$G_1(s)$	1.28	0.64	**1**	1.43	0.53	7.4
	2.34	2.04	5.34	**5**	0.46	4.2
	3.28	1.83	1.95	**5**	**1**	6.5
$G_3(s)$	1.12	0.55	**1**	0.79	0.26	9.4
	3.13	3.13	11.0	**5**	0.21	4.6
	5.23	3.60	5.21	**5**	**0.37**	6.7
$G_4(s)$	1.39	1.51	**1**	3.01	0.81	13.7
	1.59	2.38	2.04	**5**	0.78	9.9
	1.89	2.27	1.42	**5**	**1**	12.4

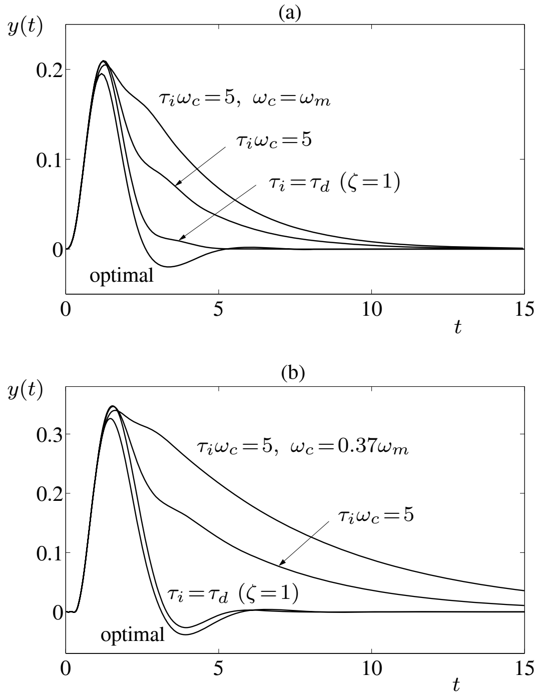

Fig. 2. Load disturbance step responses for (a) $G_1(s)$ and (b) $G_3(s)$, controlled by an optimal PID controller and different PIPD controllers according to Table 4.

without such restrictions, when reasonable demands on stability margins and control activity are required, cf. (6).

Additional low pass filtering in the PID controller To even further reduce the HF noise sensitivity and increase the HF robustness, a first order roll off in the controller can easily be included. Then the first order low pass filter in (9) is replaced by a second order low pass filter in the controller

$$K_{PIDf}(s) = K_i \frac{1 + 2\zeta s\tau + (s\tau)^2}{s(1 + 2\zeta_f s\tau/\beta + (s\tau/\beta)^2)}$$

The additional control parameter is the damping factor ζ_f. This extra filtering generates some additional phase lag that influences the stability margin. To compensate for this, still preserving the desired M_S and M_T constraints, the integral gain K_i has to be reduced, which increases J_v. From Table 5 we conclude that a reduction from $\zeta_f = 1$ to 0.5 reduces J_{HF} significantly but at the cost of somewhat additional performance loss on J_v. This trade off is up to the user to decide on. As always, improvement in one frequency range deteriorates the result in another range. In this case we see a clear but not dramatic relation between HF and LF properties.

Table 5. Optimal criteria for PIDf controllers, compared with optimal PID controllers. The same control activity J_u and stability constraints as in Table 1.

Model	$G_1(s)$		$G_3(s)$		$G_4(s)$	
ζ_f	1.0	0.5	1.0	0.5	1.0	0.5
J_v^{PIDf}/J_v^{PID}	1.04	1.14	1.02	1.04	1.06	1.31
J_{HF}/ω_c	251	83	315	113	321	100

5. CONCLUSIONS

A simple design method for four parameter PID controllers including low pass filtering is presented. It is based on multi criteria $\mathcal{H}_\infty$ optimization, and is shown to be easily implemented by a few lines of MATLAB code. The framework also includes a simple paper and pen solution that can be used for educational purposes. This manual approach also gives good initial values for the final nonlinear constrained optimization. The suggested method is compared with typical text book recommendations on frequency domain loop shaping. Such recommendations, which are introduced to simplify the four parameter design procedure, are shown to deteriorate the control performance significantly compared to a true optimal solution. What is interested from an educational point of view is also that both the paper and pen solution and the MATLAB optimization illustrate the important trade off between LF performance, HF control activity and stability margins. It is finally observed that the suggested PID design method has been successfully evaluated in an introductory course on feedback control for a number of years.

ACKNOWLEDGEMENTS

This paper is a continuation of ten years intensive and appreciated collaboration with Dr Birgitta Kristiansson on design of PID controllers.

REFERENCES

Franklin, G., Powell, J., and Emami-Naeini, A. (2006). *Feedback control of dynamic systems*. Pearson Prentice Hall.

Glad, T. and Ljung, L. (2006). *Reglerteknik - Grundläggande teori (Swedish text book)*. Studentlitteratur.

Hang, C., Åström, K., and Ho, W. (1991). Refinements of the Ziegler-Nichols tuning formula. In *IEE Proc D, Control Theory and Application*, volume 138(2), 111–118.

Kristiansson, B. and Lennartson, B. (2006a). Evaluation and simple tuning of PID controllers with high frequency robustness. *Journal of Process Control*, 16(2), 91–102.

Kristiansson, B. and Lennartson, B. (2006b). Simple and robust tuning of PI and PID controllers. *IEEE Control Systems Magazine*, 26(1), 55–69.

Larsson, P.O. and Hägglund, T. (2011). Control signal constraints and filter order selection for PI and PID controllers. In *2011 American Control Conference (ACC)*, 4994–4999. San Francisco, USA.

Lennartson, B. and Kristiansson, B. (1997). Pass band and high frequency robustness for PID control. In *Proc 36th Conf on Decision and Control*, volume 3, 2666–2671. San Diego, USA.

Lennartson, B. and Kristiansson, B. (2009). Evaluation and tuning of robust PID controllers. *IET Control Theory and Applications*, 3, 294–302.

Ogata, K. (2002). *Modern control engineering*. Prentice Hall.

Panagopoulos, H., Åström, K., and Hägglund, T. (2002). Design of PID controllers based on constrained optimisation. *IEE Proc Control theory and applications*, 149(1), 32–40.

Persson, P. and Åström, K. (1992). Dominant pole design - a unified view of PID controller tuning. In *Preprints 4th IFAC Symposium on adaptive Systems in Control and Signal Processing*, 127–132. Grenoble, France.

Ziegler, J. and Nichols, N. (1942). Optimum settings for automatic controllers. *Trans. ASME*, 64(11), 759–768.