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Balancing & Optimization for Order Reduction of Nonlinear Systems

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***Abstract*— This paper considers the problem of passing from a nonlinear time-invariant high-order system to a low-order approximation. At first, a review of some well-known methods for order reduction of nonlinear systems is presented and it is shown that the common feature of many model reduction methods is that they are obtained by applying a *projection* to system matrices. This resemblance led us to the idea of combining the linear balancing with an optimization procedure for nonlinear systems, resulting in the method of *balancing and optimization* to be presented here.**

I. Introduction

Typical nonlinear dynamic systems are modelled by means of a set of first-order coupled differential equations together with a set of algebraic output equations as follows:

$$\mathcal{S}_{nonlinear1}:\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))\\ \mathbf{y}(t)=\mathbf{h}(\mathbf{x}(t),\mathbf{u}(t))\end{cases}\qquad(1)$$

In this paper we will mostly deal with another representation of nonlinear systems as follows:

$$\mathcal{S}_{semilinear}:\begin{cases}\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)+\mathbf{F}\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t))\\ \mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)\end{cases}\qquad(2)$$

The problem that we will address is to simplify or approximate the original nonlinear system with another one with smaller number of state variables. There are however methods for order reduction of general nonlinear systems that we describe them in Section 2. In Section 3 we focus on the idea of dominant subspaces by reviewing some known model reduction methods for nonlinear systems and expanding an idea from linear to nonlinear systems. Section 4 shows the results of applying the idea of dominant subspaces together with system matrices optimization method and Section 5 contains concluding remarks.

II. Methods of Order Reduction for Nonlinear Systems

One of the most famous methods which is applicable for both linear and nonlinear systems is *singular perturbation* [4], [8]. This method is based on the assumption that the system equations can be separated in two parts so-called fast and slow modes. This method decreases the order of the model, first by ignoring the fast modes of the system, then it improves the quality of the approximation by considering *boundary layers* in reduced order system. In this method the concept of dominant subspace is bypassed by assuming that in modelling of some dynamic systems, there are some fast and slow modes and instead of just trimming the non-dominant part, its steady state effect is taken into account.

The Proper Orthogonal Decomposition is another known method that has been widely used to determine efficient bases to construct projection matrix [11]. In this method for a fixed input the state variables trajectories at certain instances of time are measured and saved in the matrix χ. If the singular values of this matrix decrease rapidly, this matrix could be approximated by a low-order matrix as it is shown in (3).

$$\chi=\mathbf{U}\mathbf{\Sigma}\mathbf{V}^{*}\approx\mathbf{U_k}\mathbf{\Sigma_k}\mathbf{V_k^{*}},\qquad \mathbf{k}\ll\mathbf{n}\qquad(3)$$

where U and V are unitary matrices and Σ is diagonal [2]. The first k leading columns of U (U_k) are used to construct the projection matrix [1], [3]. Dominant subspace in the sense of POD is the part of the state space which absorbs the most energy from specific inputs, so the POD method is an option for assessing a dominant subspace.

Another well known method is nonlinear balancing which is an extension of balancing for linear systems [7] in the sense that it is based on extended definition of balancing and Hankel singular functions [10], [9]. The main objective involved in balancing theory are the controllability and observability energy functions, which their computation requires solution of an optimal control problem at each point on the state space grid, which is the result of two nonlinear Lyapunov partial differential equation and Hamilton-Jacobi partial differential equation associated with an optimal control problem. This method similar to the concept of balancing for linear systems finds a coordinate transformation of the form $\mathbf{x}=\mathbf{\Psi}(\mathbf{z})$ that balances the system due to extended definition of balancing for nonlinear systems. As it is apparent on the one hand the procedure for nonlinear balancing presents computational difficulties, which restricts its application to very low order nonlinear systems. On the other hand it finds a very meaningful coordinate transformation as a point of view of dynamic systems which specifies the dominant subspace of state space by assuming the effect of input and state variables on output in the sense of energy.

Another method is system matrices optimization [6], [5] for specific structure of nonlinear systems shown in(4). In this method determination of dominant state state variables (dominant subspace) plays a significant role in the quality of model reduction. The first idea for pointing out dominant state variables is engineering impression, which could be helpful in many practical problems but for complex technical systems, choosing dominant state variables usually is not a straightforward task and some more advanced methods are required. Proceeding from the general system description (2) the reduced system is set up as follow:

$$\Sigma:\begin{cases}\dot{\tilde{\mathbf{x}}}(t)=\tilde{\mathbf{A}}\tilde{\mathbf{x}}(t)+\tilde{\mathbf{B}}\mathbf{u}(t)+\tilde{\mathbf{F}}\mathbf{g}(\mathbf{W}\tilde{\mathbf{x}}(t),\mathbf{u}(t))\\ \mathbf{y}(t)=\tilde{\mathbf{C}}\tilde{\mathbf{x}}(t)\end{cases}\qquad(4)$$

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Accordingly, the vector $\mathbf{g}$ of nonlinearities is taken over from the original system (2) into the reduced order system and no additional nonlinearities are introduced. The five matrices $\tilde{A}, \tilde{B}, \tilde{F}, \tilde{C}$ and W must subsequently determined such that they optimally fit the snapshots of the original system in the sense of Euclidian norm.

III. Dominant Subspace

Our original motivation for undertaking this work arouse from a hidden, but significant gap in current model reduction methods that we call it dominant subspace. The concept of dominant subspace is such important that even finding it for a system might yield a reduced model without any further effort. For instance the balanced and truncation method evaluates the reduced order system just by truncating the non-dominant subspace. In this method the Hankel singular values are gages that indicate the significance of each new state variable. Another example is the proper orthogonal decomposition (POD) method which forms the dominant subspace (by a linear transformation) and sorts the new state variables such that the higher state variables contain more information of the snapshots. In this method the singular values are indicators of importance of each new state variable. As it mentioned the quality of model reduction by *system matrices optimization* method is directly related to finding the dominant subspace, subsequently an automatic method for building dominant subspace seems indispensable. Our goal in this section is to present an automatic method for designating the dominant subspace for nonlinear systems.

A. Singular Value Decomposition (SVD)

One approach is to find a linear combination of the state variables of the original system such that the time histories of all n state variables of the original system be approximated from the first $\tilde{n}$ components of the new state variables in the least square sense. This problem is solved by using singular value decomposition of the snapshots matrix (similar to POD). Applying this method leads to the orthogonal matrices $\mathbf{U}$, $\mathbf{V}$ and a diagonal matrix $\mathbf{\Sigma}$ of the real nonnegative singular values as follow:

$$\chi = \mathbf{U\Sigma V}, \qquad \mathbf{\Sigma} = diag(\sigma_1, \sigma_2, \ldots, \sigma_n) \tag{5}$$

$$if \left\{ \begin{array}{l} (\sigma_1 > \ldots > \sigma_{\tilde{n}} \gg \sigma_{\tilde{n}+1} > \ldots > \sigma_n) \\ \mathbf{Z} = \mathbf{UX} \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \mathbf{Z_{dominant}} = \{\mathbf{z_1}, \ldots, \mathbf{z_{\tilde{n}}}\} \\ \mathbf{Z_{non-dominant}} = \{\mathbf{z_{\tilde{n}+1}}, \ldots, \mathbf{z_n}\} \end{array} \right. \tag{6}$$

The order of dominant subspace ($\tilde{n}$) is chosen so that the singular values related to state variables $\tilde{n}+1$ up to n are negligible in comparison to the first $\tilde{n}$ ones. The first $\tilde{n}$ columns of the matrix U build the projection matrix (linear combination).

B. Balancing the system based on snapshots

Since there are very powerful methods like balanced realization in dominance analysis of linear systems, a transformation matrix which guides us through dominant state variables could be computed based on the linearized system. As it is mentioned in Section 2.3 the nonlinear balancing method [9] has good theoretical background but its complicated computation restricts its application to find the dominant subspace. In this section we propose a method for approximating the coordinate transformation $\mathbf{\Psi}(\mathbf{z})$ (that balances the system) based on linearized system. There are two methods for obtaining the linearized system, first the analytical evaluation around one operating point, second numerical evaluation based on snapshots derived by simulating the system with the most typical inputs.

Theorem 1: The transformation matrix that balances the linearized system of a nonlinear system is equal to the linearized coordinate transformation $\mathbf{\Psi}(\mathbf{z})$ that balances the nonlinear system[9].

Based on Theorem1, if we find the transformation matrix of the linearized system and apply it to the nonlinear system, it approximately balances the system around the operating point. In our case the linearized system based on snapshots has some advantages in comparison to the analytical linearized system. The first point is that in *system matrices optimization* method the snapshots already exists, so it is much more convenient to compute the system matrices from the snapshots than analytically linearizing the original system. Second, linearization based on the snapshots often gives a "more global linearized system" which is close to various linearized systems in unlike operating points. It should be noted that the second advantage is based on this assumption that the linearized systems of the nonlinear system in different operating points are not very far from each other and we can obtain an "average" linear system by exploiting the snapshots.

After approximately finding the dominant subspace, the next step is to reduce the order to the dimension of the dominant subspace. In [12] the reduced order system has obtained by truncating the non-dominant subspace. Theorem1 shows the idea of that method.

Theorem 2: Consider a nonlinear system represented in (1). Without loss of generality, let (1) be differentiable at an equilibrium, say $[X_s, U_s]$, then the system obtained by linearizing (1) around this operating point can be written as:

$$\Sigma : \left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{Jx}(t) + \mathbf{Gu}(t), \\ \mathbf{y}(t) = \mathbf{Hx}(t) \end{array} \right.$$

where $\mathbf{J}$ and $\mathbf{G}$ are the system and input Jacobian, respectively. Following the procedure in Balanced & Truncation method for linear systems, a transformation matrix $\mathbf{P}$ can be constructed and be used to transform the linearized system into balanced coordinates as follows:

$$\Sigma : \left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t), \\ \mathbf{y}(t) = \mathbf{Cx}(t) \end{array} \right.$$

where $\mathbf{A} = \mathbf{P^{-1}JP}$, $\mathbf{B} = \mathbf{P^{-1}G}$ and $\mathbf{C} = \mathbf{HP}$. Since the new system is balanced, the balancing criterion can be used to partition the system matrices in two dominant and non-dominant parts as follow:

$$A = \left[\begin{array}{cc} A_r & A_{12} \\ A_{21} & A_{22} \end{array} \right], \; B = \left[\begin{array}{c} B_r \\ B_2 \end{array} \right]$$

$$P = \left[\begin{array}{cc} P_r & P_2 \end{array} \right], \; P^{-1} = \left[\begin{array}{c} P_r^{-1} \\ P_2^{-1} \end{array} \right]$$

If $\|X - X_s\|$ is sufficiently small and the reduced order of linearized system exists, then the nonlinear reduced order system is given as:

$$\Sigma : \begin{cases} \dot{\mathbf{z_r}}(t) = \mathbf{P_r^{-1} f}(\mathbf{P_r z_r}(t) + \mathbf{X_s} - \mathbf{P_r P_r^{-1} X_s}, \mathbf{u}(t)), \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{P_r z_r}(t) + \mathbf{X_s} - \mathbf{P_r P_r^{-1} X_s}, \mathbf{u}(t)) \end{cases}$$

As it mentioned before, finding dominant subspaces is just the first step in order reduction and usually for improving the reduced order system performance much endeavor is needed. Thus truncation is just the simplest way with less computation load to reach the reduced order system. In order to obtain more precise results a more powerful method is developed subsequently.

IV. Balancing & Optimization

With respect to the idea of dominant subspace we can take the method of system matrices optimization as a complement for the task of order reduction. In this paper we suggest a procedure for the order reduction of nonlinear systems by combining this two methods and we name it *Balancing & Optimization*. Following steps should be put into practice for accomplishing this method:

1) *Producing the snapshots* of the original system for typical inputs and save the numerical values of state variables and their derivatives, inputs, nonlinear part and outputs, respectively in matrices χ, $\dot{\chi}$, $\mathbf{\Psi}$, $\mathbf{\Gamma}$ and $\mathbf{\Upsilon}$. It should be noted that the inputs should be selected such that they stimulate all active modes of the system.

$$\begin{array}{rr} \chi = [\mathbf{x(t_1)x(t_2)\cdots x(t_N)}] & \dot{\chi} = [\mathbf{\dot{x}(t_1)\dot{x}(t_2)\cdots \dot{x}(t_N)}] \\ \mathbf{\Psi} = [\mathbf{u(t_1)u(t_2)\cdots u(t_N)}] & \mathbf{\Gamma} = [\mathbf{g(t_1)g(t_2)\cdots g(t_N)}] \\ & \mathbf{\Upsilon} = [\mathbf{y(t_1)y(t_2)\cdots y(t_N)}] \end{array}$$

2) *Linearizing the nonlinear system* which could be carried out by using snapshots or Jacobian matrices. Sometimes the first one, due to the advantages mentioned in Subsection 3.2, is preferred. The matrices $\mathbf{A_{lin}}, \mathbf{B_{lin}}$ and $\mathbf{C_{lin}}$ in (8) is computed by solving optimization problem (7).

$$\min_{M} \| \dot{\chi} - \underbrace{[A_{lin} \quad B_{lin}]}_{\mathbf{M}} \left[\begin{array}{c} \chi \\ \Psi \end{array} \right] \|, \qquad \min_{\mathbf{C_{lin}}} \| \mathbf{\Upsilon} - \mathbf{C_{lin}} \chi \| \tag{7}$$

After linearization, the original nonlinear system is approximated with a linear system as shown in (8).

$$\begin{array}{l} \Sigma : \begin{cases} \mathbf{\dot{x}}(t) = \mathbf{Ax}(t) + \mathbf{Bu}(t) + \mathbf{Fg}(\mathbf{x}, \mathbf{u}) \\ \mathbf{y}(t) = \mathbf{Cx}(t) \end{cases} \\ \simeq \quad \Sigma_{lin} : \begin{cases} \mathbf{\dot{x}}(t) = \mathbf{A_{lin}x}(t) + \mathbf{B_{lin}u}(t) \\ \mathbf{y}(t) = \mathbf{C_{lin}x}(t) \end{cases} \end{array} \tag{8}$$

3) *Finding the dominant subspace* by approximately balancing the nonlinear system. This task is carried out with the same procedure as in Theorem1. The transformation matrix $\mathbf{P}$ and the Hankel singular values of the linearized system (8) are initial materials to construct the dominant subspace. After applying the transformation matrix $\mathbf{P}$ to the nonlinear system, the first new $\tilde{n}$ state variables related to bigger Hankel singular values span the dominant subspace. Thus the first $\tilde{n}$ rows of matrix $\mathbf{P^{-1}}$ build the matrix $\mathbf{P_{\tilde{n}}^{-1}}$ which is used to construct the dominant state variables and their snapshots as follow:

$$\mathbf{x_{do}} = \mathbf{P_{\tilde{n}}^{-1} x}, \quad \chi_{\mathbf{do}} = \mathbf{P_{\tilde{n}}^{-1}} \chi, \quad \dot{\chi}_{\mathbf{do}} = \mathbf{P_{\tilde{n}}^{-1}} \dot{\chi} \tag{9}$$

4) *Applying system matrices optimization method* in order to find the best system matrices that fit the snapshots of the dominant subspace by solving optimization problems shown in (10). The snapshots of the dominant state variables can be computed by applying the transformation matrix to the snapshots of the original system (9) and the result is a reduced order system like (4).

$$\begin{array}{c} \min_{\mathbf{\tilde{M}}} \| \dot{\chi}_{\mathbf{do}} - \underbrace{[\tilde{A} \quad \tilde{B} \quad \tilde{F}]}_{\mathbf{\tilde{M}}} \left[\begin{array}{c} \chi_{do} \\ \Psi \\ \Gamma \end{array} \right] \|, \\ \min_{\mathbf{\tilde{C}}} \| \mathbf{\Upsilon} - \mathbf{\tilde{C}} \chi_{\mathbf{do}} \|, \; \min_{\mathbf{W}} \| \chi - \mathbf{W} \chi_{\mathbf{do}} \| \end{array} \tag{10}$$

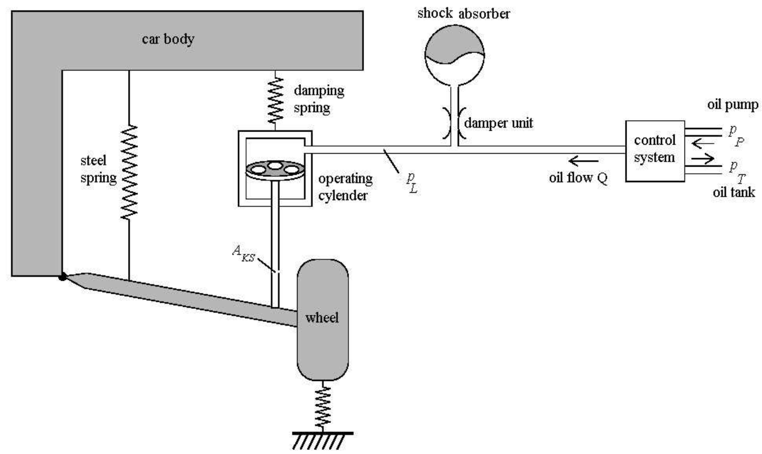

Fig. 1. construction of hydropneumatic vehicle suspension

V. Illustrative Example

In this part we apply our new method to the model of an active hydropneumatic suspension [5]. This device increases comfortableness and safety by significantly reducing the incongruous movements of the car body compared to a traditional passive spring shock-absorber system. The inputs of this system are the in and outflow of oil in the hydropneumatic system which should be regulated by the controller using measurement data from sensors. In order to design and simulate such a controlled system a modelling of the system is necessary. The simulation can be accelerated through order reduction and furthermore it simplifies all control algorithms which directly use the model, such as state feedback observer combinations or model based feed forward controllers. Fig.(1) shows the mechanical construction of the suspension for a single wheel and the mass of the related part of the car body. We define the three first state variables (with scaling) as outputs of the system, so the matrix $\mathbf{C}$ in original model has the following value:

$$\mathbf{C} = \left(\begin{array}{cccccccccc} 50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 50 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Due to the complexity of the nonlinear part, we don't discuss the details of the model in here, further information is available in [5].

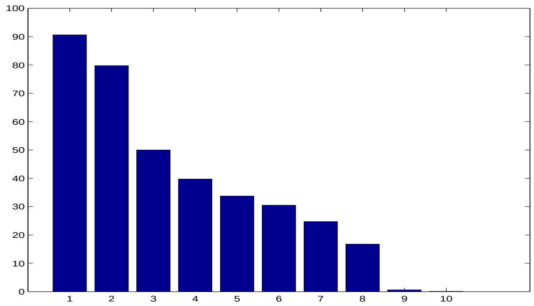

Fig. 2. Hankel Singular Values $\sigma_1, \sigma_2, \ldots, \sigma_{10}$

A. Carrying Out the Order Reduction

In preparation, we simulate the original system with typical inputs. For the input u_1 positive and negative step functions with height 0.1 m and for the input u_2 of the servo valve, positive and negative square and triangle functions are designated. The list of typical inputs are shown below:

$$\mathbf{u_1}(t) = \begin{bmatrix} 0 \\ u_{square}(t) \end{bmatrix},\ \mathbf{u_2}(t) = \begin{bmatrix} 0 \\ u_{triangle}(t) \end{bmatrix},$$

$$\mathbf{u_3}(t) = -\mathbf{u_1}(t),\ \mathbf{u_4}(t) = -\mathbf{u_2}(t),\ \mathbf{u_5}(t) = \begin{bmatrix} 0.1\sigma(t) \\ 0 \end{bmatrix},$$

$$\mathbf{u_6}(t) = -\mathbf{u_5}(t),\ \mathbf{u_7}(t) = \begin{bmatrix} 0.1\sigma(t) \\ u_{square}(t) \end{bmatrix},$$

$$\mathbf{u_8}(t) = \begin{bmatrix} 0.1\sigma(t) \\ u_{square}(t) \end{bmatrix}, \mathbf{u_9}(t) = -\mathbf{u_8}(t),\ \mathbf{u_{10}}(t) = -\mathbf{u_7}(t)$$

with $u_{square} = \sigma(t) - \sigma(t-0.2)$ and $u_{triangle} = \max\{0, \min\{5t, 2-5t\}\}$ where $\sigma(t)$ is the unit step function. In our simulation we use $\mathbf{u}_5$ for comparing the results of three order reduction methods which are nonlinear balanced and truncation, system matrices optimization and balancing & optimization. In the next step we use the linearizing method exploiting the snapshots and by assuming the decreasing slope and values of Hankel singular values, the order of reduced system is estimated.

B. Results

After linearizing the system using the snapshots, the matrix transformation $\mathbf{P}$ is evaluated using balancing techniques of linear systems and the result is as follow:

$$\mathbf{P} = 10^3 \begin{pmatrix} -0.0005 & 0.0007 & -0.0000 & -0.0003 & 0.0001 & -0.0000 & -0.0000 & -0.0000 & -0.0001 & 0.0000 \\ -0.0489 & -0.0347 & -0.0119 & 0.0199 & 0.0183 & 0.0017 & 0.0032 & 0.0008 & 0.0067 & 0.0041 \\ 0.0000 & 0.0000 & 0.0003 & 0.0001 & 0.0001 & -0.0001 & -0.0002 & 0.0008 & 0.0067 & 0.0041 \\ -0.0001 & 0.0030 & 0.0009 & -0.0030 & 0.0008 & 0.0002 & 0.0021 & 0.0019 & -0.0015 & -0.0002 \\ -0.0001 & 0.0001 & 0.0002 & 0.0003 & 0.0002 & -0.0000 & -0.0001 & -0.0002 & 0.0001 & 0.0001 \\ -0.0095 & -0.0135 & 0.0029 & 0.0088 & -0.0139 & -0.0006 & 0.0009 & 0.0014 & -0.0017 & -0.0145 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0001 & -0.0001 & -0.0001 & -0.0000 & 0.0000 \\ -0.0001 & -0.0010 & 0.0003 & 0.0018 & -0.0009 & -0.0002 & -0.0001 & -0.0001 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0001 & -0.0001 & 0.0002 & 0.0051 & 0.0100 \\ -0.0006 & -0.0001 & 0.0040 & -0.0040 & -0.0025 & 0.0177 & -0.0184 & 0.0389 & 1.1351 & 0.2221 \end{pmatrix}$$

The next step is to balance the system and specifying the dominant subspace. The Hankel singular values of the linearized system are shown in Fig.2, as it is trivial from the bar graph we can divide the state variables in three partitions. The first two state variables are the most important ones, from three to eight are in second stage and state variables nine and ten are the less important ones. It seems

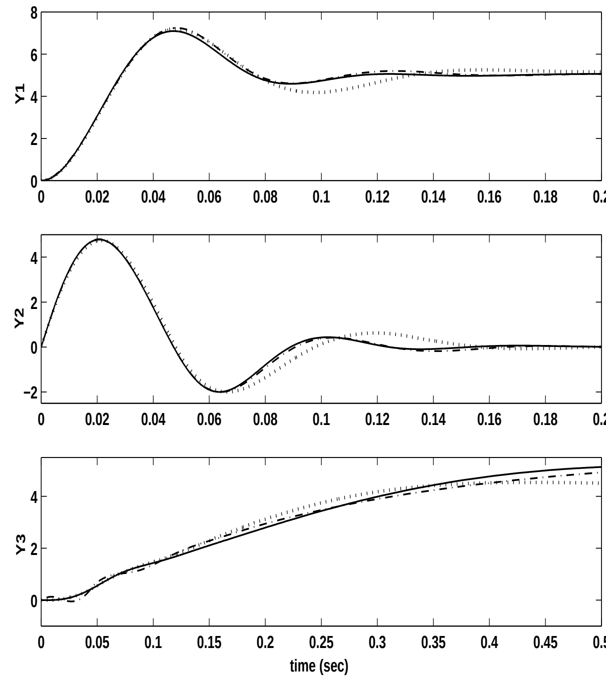

Fig. 3. Time curves of output variables of the original system (solid), reduced order system of 7_{th} order (system matrices optimization, dotted) and 5_{th} order (balanced & optimization, dash-dotted) excited by $\mathbf{u_5}(t)$.

that reduction down to order eight is quite reasonable but to orders less than eight requires trial and error. Regarding our simulation, the balanced and truncation for nonlinear systems and system matrices optimization both fail in orders less than seven (yield to unstable systems), but with our new method we can reduce the order to five. The results are shown in Fig.(3).

VI. Conclusion

Throughout this paper, we have exhibited many choices facing those seeking to reduce the order of nonlinear systems and we improved the system matrices optimization method by exploiting the idea of balancing in linear systems and introducing the new idea of dominant subspace.The technical example shows that our new method works reasonably and provides understandable results which is much better than other mentioned methods. In particular the example illustrates that *balancing & optimization* method remedies the drawbacks of system matrices optimization method by assuming the effect of output and introducing a procedure for pointing out the dominant subspace.

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