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Can Any Reduced Order Model Be Obtained Via Projection?

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Abstract— **The paper investigates the properties of general reduced order models obtained by projection of a high order system. It answers questions such as are any two models of different orders related by a projection? Is it possible to obtain the same reduced order model using different projections? How to find, if it exists, a projection that relates the two models? Etc. It is shown that answers to those questions can be obtained by investigating the properties of a certain matrix pencil. In case the system is square the problem becomes that of a generalized eigenvalue, and in non-square systems the key tool is the Kronecker Canonical Form.**

I. Introduction

MOST of the order reduction methods include the following two steps. The first one is a state transformation into a state space realization in which the state variables can be ranked according to some measure of importance. The second step is truncation of the least important state variable. The two operations together constitute a projection into a lower dimension, and are therefore called Projection Order Reduction (POR). Their outcome is a Projection Reduced Order Model (PROM).

The various POR methods differ in the criterion that is used for ranking the state variables. Probably the simplest method is partial fraction expansion, which is equivalent in the state space to transforming the system into a diagonal realization. The method is widely used for lightly damped systems, where it is known as 'modal truncation' [2], [8]. The most popular POR method is Truncated Balanced Realization [11], with all of its extensions [2], [4], [10], [13], [17]. In the basic formulation of the method [11], the system in the new realization has controllability and observability gramians that are diagonal and equal. State variables that correspond to larger diagonal elements are more controllable and observable, and are therefore retained in the reduced order model. In other variations the goal is to balance other matrices, yet the structure of transformation followed by truncation still remains. Another POR method is Component Cost Analysis [14] where the contribution of each state variable to a certain cost is investigated.

While in the methods that have been described so far the projection is an intentional part of a heuristic algorithm, in other cases it arises naturally. For example, Wilson [15], and later Hyland and Bernstein [7], have solved the optimal L_2 model reduction problem by direct optimization, without imposing any structure. It turned out that it is given in terms of a projection into a lower order subspace and therefore is sometimes referred to as the "optimal projection". That projection structure was used in [18] to derive an efficient algorithm for a sub-optimal L_2 reduced order model.

Despite the widespread use of POR, very little, if any, works dealt with its intrinsic properties. This paper is concerned with problems, which apply to POR in its general form. For example, given two models of different orders, is it always possible to obtain the one with the lower dimension by POR of the other? Is it possible to obtain the same reduced order model using different projections? Etc. The problem was first addresses in [5], [6], with partial answers to those questions, for square systems. The current paper extends those results into more general framework, in particular the case of non-square systems. It is shown that the main tool of the analysis is the Kronecker Canonical Form of a certain matrix pencil. The structure of that pencil reveals if a reduced order model can be a PROM and if the projection leading to it unique.

II. Order reduction via Projection

The model order reduction problem for linear systems is usually defined as follows. Given the n-th order linear, time invariant, system $G(s)$, find an r-th order ($r<n$) system $G_r(s)$, with the same number of inputs and outputs, which is an approximation of it. Most of the existing order reduction methods are defined in state space. They start with a state space realization of $G(s)$

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1a}$$

$$y(t) = Cx(t) + Du(t) \tag{1b}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and look for a reduced order model

$$\dot{x}_r(t) = A_r x_r(t) + Bu(t) \tag{2a}$$

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$$y_r(t) = C_r x(t) + D_r u(t) \qquad (2b)$$

where $x_r \in R^r$. The common, two steps, structure consists of a state transformation into a more insightful realization, followed by truncation of the 'less important' state variables. The various existing methods differ by the transformation that is used and the criterion for ranking the state variables. Let the $n \times n$ nonsingular matrix T, and its inverse, be partitioned as

$$T = \begin{bmatrix} R & \overline{R} \end{bmatrix} \;\; , \;\; T^{-1} = \begin{bmatrix} L \\ \overline{L} \end{bmatrix} \qquad (3)$$

with $R \in R^{n \times r}$, $L \in R^{r \times n}$. The state transformation $x=Tx'$ leads to the following realization.

$$\begin{bmatrix} \dot{x}'_1 \\ \dot{x}'_2 \end{bmatrix} = \begin{bmatrix} LAR & LA\overline{R} \\ \overline{L}AR & \overline{L}A\overline{R} \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} + \begin{bmatrix} LB \\ \overline{L}B \end{bmatrix} u(t) \qquad (4a)$$

$$y(t) = \begin{bmatrix} CR & C\overline{R} \end{bmatrix} x'(t) + Du(t) \qquad (4b)$$

Suppose that, using any criterion, x'_1 is more important than x'_2. Then it is assumed that $x'_2 \approx 0$ and the reduced order approximation of the system (1) is

$$\dot{x}_r(t) = LARx_r(t) + LBu(t) \qquad (5a)$$

$$y(t) = CRx_r(t) + Du(t) \qquad (5b)$$

It is evident from (5) that the direct transmission term $Du(t)$ plays no role in this order reduction procedure. It will be therefore assumed from now on, for convenience, that D=0. Since L and R are sub-blocks of T and its inverse they satisfy

$$LR = I_r \qquad (6)$$

The model (5), with any L and R satisfying (6) is known as Projection Reduced Order Model (PROM). To see the origin of this name, define the matrix

$$P = LR \, , \qquad (7)$$

and the pseudo full order state vector

$$\hat{x}(t) = Rx_r(t) = Px(t) \, , \qquad (8)$$

which is x_r expressed in the coordinates of the n-th order space of x. From (6) it follows immediately that P^2=P, hence P is a projection matrix. Multiplying eq. (5a) by R, the reduced order model can be written as

$$\dot{\hat{x}}(t) = P(A\hat{x}(t) + Bu(t)) \qquad (9a)$$

$$y(t) = C\hat{x}(t) \qquad (9b)$$

Hence P projects obliquely the time derivative of the state vector into a certain subspace. The PROM is therefore a minimal realization of the system (PA,PB,C). The identity $P\hat{x}(t) = \hat{x}(t)$ implies that the reduced order model is also a minimal realization of (PAP,PB,CP). The latter form is sometimes preferred since it resembles the familiar similarity transformation.

Despite their wide use, PROM's have very few generic properties. They do not preserve stability or instability, relative degree, and even minimality or non-minimality. The following easily proven results discuss the invariance properties of under state transformations.

Property 1: The projection P that relates (A,B,C) and (A_r,B_r,C_r) is invariant under state transformation of the reduced order realization.

Property 2: The projection P that relates (A,B,C) and (A_r,B_r,C_r) changes under state transformation of the full order realization into TPT^{-1}.

Assuming that (A_r,B_r,C_r) is minimal, Property 1 means that P is a projection into all minimal realizations of $G_r(s)$. Property 2 means that if there exists a projection relation between certain (A,B,C) and (A_r,B_r,C_r), there exists a projection relation between any pair of minimal realizations of $G(s)$ and $G_r(s)$. However the specific projection is not preserved. Hence for systems, rather than realizations, the only relevant question is whether a projective relation exists.

So far we have discussed the properties of two models, with different order, which are known to be related by a projection. We would like now to consider the inverse problem, i.e. finding the projection that relates two given models. In particular, the existence and uniqueness of such projection. The questions can be phrased as follows: Given the system $G(s)$, is any r-th order $G_r(s)$, with the same dimensions, a PROM of it? And if so, can $G_r(s)$ be reached from a single realization of $G(s)$ by more than one projection?

Remark 2.1: Throughout this paper we consider only real projections, so expressions like "a projection does not exist" should read as "a *real* projection does not exist", etc. This distinction is important especially in square systems where a certain (real parameter!) $G_r(s)$ may be obtained only by a complex projection matrix P.

The simplest possible case, second to first order, can be analyzed in using a direct approach and elementary algebraic and geometrical operations [5]. This line of analysis, however, does not seem promising for higher order systems. In the next section the same problems are addressed for the general case.

III. Existence and Uniqueness

As was shown in section II, the existence of a projection is independent of specific realizations. Let (A,B,C) and (A_r,B_r,C_r) be any realizations of $G(s)$ and $G_r(s)$ respectively. Then for $G_r(s)$ to be a PROM of $G(s)$ the following relationships must hold.

$$LR = I_r \; , \; LAR = A_r \; , \; LB = B_r \; , \; CR = C_r \qquad (10a\text{-}d)$$

Considering L and R as the unknowns, equations (10) are a set of $(2r+m+p)r$ equations with $2nr$ unknowns. The number of equations and unknown will be the same for

$$r^*=n-(m+p)/2 \tag{11}$$

There are three possible cases.

1. $r < r^*$, i.e. more unknowns than equations. Seemingly that means that the equations have infinitely many solutions, hence generically any $G_r(s)$ is a PROM of a given $G(s)$, and can be obtained via infinitely many projections. It will be shown later that this is indeed the case for square systems, but not necessarily true for non-square systems.
2. $r = r^*$, i.e. same number of equations and unknowns. Seemingly that means that the equations have a finite number of solutions, possibly zero. Not every $G_r(s)$ is a PROM of a given $G(s)$. Those who are, can be obtained by a finite number of projections. Again this is generically true only for square systems.
3. $r > r^*$, i.e. more equations than unknowns. That means that generically the equations have no solution. For a given $G(s)$, the class of models $G_r(s)$ that are PROM has measure zero.

As an example for case 3 (which is possible only in MIMO system), let

$$G(s)=\begin{bmatrix} \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+1)^2} \end{bmatrix} \quad ; \quad G_r(s)=\begin{bmatrix} \dfrac{b_1}{s+a} & \dfrac{b_2}{s+a} \end{bmatrix}$$

It can be easily shown that the a projection relationship between these second and first order systems exists only for (a, b_1, b_2) satisfying

$$(b_1+b_2)^2-ab_1-b_2=0$$

Hence the PROMs constitute a two-dimensional surface in the three dimensional space of 2×1 first order models. Case 3 implies that in some cases the class of reduced order models that can be obtained by projection is very narrow. In heuristic methods the question is whether it is justified to look only at that narrow class. Even more surprising is the fact that the optimal L_2 reduced model belongs to that class.

Eqs. (10) have a clear structure. (10c,d) are linear, while (10a,b) are bilinear, i.e. contain only cross product terms between the two groups of unknowns (L and R). The linear equations can be replaced by

$$L=B_rB^+ + XB_\perp \quad , \quad R=C^+C_r+C_\perp Y \tag{12a,b}$$

where $B^+\in R^{m\times n}$ is a left inverse of B, and $B_\perp\in R^{(n-m)\times n}$ is a basis for the left null-space of B. Similarly, $C^+\in R^{n\times p}$ is a right inverse of C, and $C_\perp\in R^{n\times(n-p)}$ is a basis for the left null-space of C. $X\in R^{r\times(n-m)}$ and $Y\in \mathrm{R}^{(n-p)\times r}$ are the new unknown matrices. Substituting them into (10a,b) and rearranging, the equivalent equations are

$$\begin{bmatrix} X & I_r \end{bmatrix}\begin{bmatrix} B_\perp C_\perp & B_\perp C^+C_r \\ B_rB^+C_\perp & B_rB^+C^+C_r-I_r \end{bmatrix}\begin{bmatrix} Y \\ I_r \end{bmatrix}=0 \tag{13a}$$

$$\begin{bmatrix} X & I_r \end{bmatrix}\begin{bmatrix} B_\perp AC_\perp & B_\perp AC^+C_r \\ B_rB^+AC_\perp & B_rB^+AC^+C_r-A_r \end{bmatrix}\begin{bmatrix} Y \\ I_r \end{bmatrix}=0 \tag{13b}$$

Defining $\widetilde{X}=\begin{bmatrix} X & I_r \end{bmatrix}$, $\widetilde{Y}=[Y^T \; I_r]^T$, these equations can be written as

$$\widetilde{X}H_i\,\widetilde{Y}=0 \qquad i=1,2 \tag{14}$$

where H_1, $H_2\in R^{(n-m+r)\times(n-p+r)}$ are the coefficient matrices. Eq. (14) implies that

$$\widetilde{X}(\lambda H_1-H_2)\widetilde{Y}=0 \qquad \forall\,\lambda\in C \tag{15}$$

The solution will therefore be obtained by investigating the properties of the linear matrix pencil λH_1-H_2. We therefore recall some facts regarding linear matrix pencils taken from [1], [9], and [12].

For every $h\times q$ pencil λE-A there exist square and nonsingular S and V such that

$$F(\lambda)=S(\lambda H_1-H_2)V=\begin{bmatrix} blockdiag\{F_i(\lambda)\} & 0 \\ 0 & 0 \end{bmatrix} \tag{16}$$

where the linear pencils $F_i(\lambda)$, i=1,…,K, which are unique, assume one of four possible structures

$$\boldsymbol{type1}=\begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \lambda & -1 \end{bmatrix}, \quad \boldsymbol{type2}=\begin{bmatrix} \lambda & & \\ -1 & \ddots & \\ & \ddots & \lambda \\ & & -1 \end{bmatrix}$$

$$\boldsymbol{type3}=\bar{\mathrm{J}}\lambda-\mathrm{I}, \quad \text{or} \quad \boldsymbol{type4}=\lambda\mathrm{I}-\bar{\mathrm{F}}.$$

$\bar{J}$ is a nilpotent Jordan matrix and $\bar{F}$ is in Jordan form. This is known as the Kronecker Canonical Form (KCF). When $h\neq q$ almost all matrix pencils λE-A have the same KFC depending only on h and q. This is the Generic Kronecker Structure (GKS). A precise definition of that notion can be found in [1], but we will continue with the intuitive definition stated above. The explicit structure of the GKS will be discussed in Section V. The normal rank of λE-A is defined as

$$nrk\;(\lambda E-A)=q-r_0=h-l_0 \tag{17}$$

where r_0 and l_0 are the number of ***type1*** and ***type2*** elements respectively. In accordance with considering the generic case, we make the following assumption regarding λH_1-H_2.

Assumption 3.1: The matrix pencil λH_1-H_2 has full normal rank, i.e. nrk(λH_1-H_2)=min(h,q). Equivalently, there exists a scalar λ_0 such that $\lambda_0 H_1$-H_2 has full rank.

Assumption 3.1 implies that there exists a scalar λ_0 such that $\lambda_0 H_1$-H_2 has full rank. Notice the pencil is only a tool for solving (13). Hence it can be assumed, without loss of generality, that H_1 has full rank. Because if it is not, it can be replaced by $\lambda_0 H_1$-H_2, which is then labeled as H_1.

The sparse form of $F(\lambda)$ is the key to the derivations throughout this paper. We will use the notation $F_{Ik,Jq}(\lambda)$ to denote the sub-matrix obtained by selecting only the rows and columns belonging to the sets $I_k=\{i_1, \ldots, i_k\}$ and $J_q=\{j_1, \ldots, j_q\}$ respectively. In particular, $F_{Ik,Jq}(\lambda)=0$ means that $F_{ij}(\lambda)=0 \ \forall\ i\in I_k, j\in J_q$

Lemma 3.2: Let $I_r=\{i_1, \ldots, i_r\}$ and $J_r=\{j_1, \ldots, j_r\}$ be two sets such that $F_{Ir,Jr}(\lambda)=0$ and let

$$\hat{S}_r=\begin{bmatrix}\hat{S}_{1\,r\times(n-m)} & \hat{S}_{2\,r\times r}\end{bmatrix} \quad , \quad \hat{V}_r=\begin{bmatrix}\hat{V}_{1\,(n-m)\times r} \\ \hat{V}_{2\,r\times r}\end{bmatrix} \tag{18}$$

consist of rows of S and columns of V belonging to I_r and J_r respectively. If $\hat{S}_2$ and $\hat{V}_2$ are nonsingular, then

$$X=\hat{S}_2^{-1}\hat{S}_1 \quad , \quad Y=\hat{V}_1\hat{V}_2^{-1} \tag{19a,b}$$

is a solution of (13).

Proof: It follows from (15) that $F_{ij}(\lambda)=0$ implies $s_iH_1v_j=s_iH_2v_j=0$, hence $\hat{S}_rH_1\hat{V}_r=\hat{S}_rH_2\hat{V}_r=0$. The structure of $\tilde{X}$ is recovered by pre-multiplying by $\hat{S}_2^{-1}$. Similar operation gives Y.

The solution given by (19) is independent of pre-multiplying $\hat{S}_r$ and post-multiplying $\hat{V}_r$ by nonsingular matrices. However, in general it is not unique, as explained in the next corollary.

Corollary 3.3: Let I_r and J_r satisfy Lemma 3.1 (including the invertiblity of $\hat{S}_2$ and $\hat{V}_2$). If there exists sets I_{r1} and J_{r2}, with $\max(r_1,r_2)>r$, such that $I_r\subseteq I_{r1}$, $J_r\subseteq J_{r2}$ and $F_{Ir1,Jr2}(\lambda)=0$, then (13) has infinitely many solutions.

Proof: Assume first that $r_1=r+1$, $r_2=r$. Then there exists a solution based on $\hat{S}_r,\hat{V}_r$ and one can construct $\hat{S}_{r+1}$, i.e. a matrix with the properties of $\hat{S}_r$, by adding one more row. Adding the r+1 row, multiplied by any constant, to any of the other rows, yields a new "$\hat{S}_r$", while $\hat{S}_2$ will become singular only for at most one value of that constant. In general, create $\hat{S}_{r1}$ and $\hat{V}_{r2}$ of all rows and columns in I_{r1} and J_{r2}. Then $U_1\hat{S}_{r1}$ and $\hat{V}_{r2}U_2$ can be used instead of $\hat{S}_r$ and $\hat{V}_r$ for almost all full rank $U_1\in R^{r\times r2}$, $U_2\in R^{r1\times r}$.

The next lemma, whose proof is immediate from the construction of the KCF, characterizes the rows and columns that satisfy Lemma 3.2.

Lemma 3.4: Let S_k and V_k denote the sets of rows of S and the columns of V, respectively, which correspond to the block $F_k(\lambda)$, where S_{K+1} and V_{K+1} correspond to zero rows and columns, if exist. Then $s_iH_1v_j=s_iH_2v_j=0$ if one of the following holds

1) $s_i\in S_k\ ,\ v_j\in V_l,\ k\neq l$.

2) s_i is the i_k row of S_k, v_j is the j_k column of V_k and $F_k(i_k,j_k)=0$.

At this point we distinguish between square, i.e. same number of inputs and outputs, and non-square systems, as their analyses take somewhat different routes.

IV. PROM'S OF SQUARE SYSTEMS

This case has already been discussed in [5], [6] and its results will therefore be given in a concise manner. When the system is square, i.e. $m=p$, the pencil λH_1-H_2 is square as well and $H_i\in R^{q\times q}$ where $q=n+r-m$. The problem reduces then to that of generalized eigenvalues and eigenvectors. Assumption 3.1 means in this case that the matrix pencil is regular [9], and the discussion following it leads to assuming that H_1 is nonsingular. (in terms of Section III the GKS in this case consists of one ***type4*** element). Consider the generalized eigenvalue problem

$$(\lambda H_1-H_2)v=0 \tag{20}$$

which has N distinct (in the geometrical sense) eigenvalues, each of multiplicity N_k. S and V are square, nonsingular, real matrices such that

$$F(\lambda)=S(\lambda H_1-H_2)V=diag\{\lambda I-J_1,\cdots,\lambda I-J_N\} \tag{21}$$

and J_k are real Jordan block. The analysis then uses a microscopic form of Lemma 3.4, where J_k, the subblocks of the **type4** element play the same role as the sub-pencils F_k.

The following Theorem, which is the main result of this section, gives the maximum value of r, and the number of projections relating the full and the educed order models.

Theorem 4.1:

a) Generically, any model with order $r<r^*$ ($r^*=n$-m) is a PROM of the full order system, and it can be obtained by infinitely many projections.

b) No model of order $r>r^*$ that leads to the pencil (21) is a PROM. A PROM of order $r>r^*$ leads to a singular pencil.

c) If $r=r^*$, then

1) All the solutions of (13) are of the form (19).

2) The maximum number of real solutions of (13) (equivalently, of (10)) is

$$\begin{pmatrix} n-m \\ 2(n-m)\end{pmatrix}=\frac{(2(n-m))!}{(n-m)!(n-m)!}$$

3) If n-m is odd and all the eigenvalues of (20) are complex then (13) has no solution.

Proof: see [5].

The following result sheds new light on Theorem 4.3.

Lemma 4.2 [16]: The generalized eigenvalues of (H_2,H_1) are zeros of $G(s)$-$G_r(s)$.

Two topics get a clear answer using this result. First, the pencil $F(\lambda)$ is singular when each value of s is a zero of $G(s)$-$G_r(s)$, namely when $\det(G(s)\text{-}G_r(s))\equiv 0$. In SISO systems this means that $G(s)$ and $G_r(s)$ are two realizations of the same system, hence $G(s)$ is necessarily nonminimal.

In MIMO systems the situation may occur with minimal $G(s)$ as well. The second topic that gets an immediate answer is the question in the title of this paper. In a square system with $r=n-m$, a reduced model cannot be obtained via projection if r is odd and all the zeros of $G(s)-G_r(s)$ are complex.

V. PROM'S OF NON-SQUARE SYSTEMS

In this section the generic situation in a non-square system is considered. The dimensions of H_i are $h\times q$, where $h=n+r-m$, $q=n+r-p$. To simplify the notation, we assume that $m>p$ (the system has more inputs than outputs) which means that $q > h$. Clearly all the results apply to the other case by interchanging 'columns' and 'rows' everywhere. Assumption 3.1 implies in this case that the KCF does not include ***type2*** and ***type3*** elements, as well as zero rows. This is automatically included in the GKS.

Define the integers $\alpha=[h/(q-h)]$ and $\beta=h \bmod (q-h)$, i.e.

$$h = (q-h)\alpha + \beta \tag{22}$$

For $q<2h$ The GKS has the following structure [1].

$$F(\lambda) = diag\{\underbrace{L_\alpha,\dots,L_\alpha}_{q-h-\beta\ blocks},\underbrace{L_{\alpha+1},\dots L_{\alpha+1}}_{\beta\ blocks}\} \tag{23}$$

where L_k as a **type1** block with dimensions $k\times(k+1)$. In the (highly unlikely in reality) case where $q>2h$ (equivalently $q-h>h$), $\alpha=0$, $\beta=h$, and F(λ) assumes the following structure

$$F(\lambda) = \begin{array}{cc} \begin{bmatrix} L_1 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & L_1 & 0 & \cdots & 0 \end{bmatrix} \\ \quad 2h\ columnss \qquad q-2h\ columns \end{array} \tag{24}$$

In both cases, permutation of rows and columns gives the following structure.

$$F'(\lambda) = \lambda[I_h \quad 0_{h\times(q-h)}] - [0_{h\times(q-h)} \quad I_h] \tag{25}$$

The diagonals overlap in case the pencil is in form (23). As in the square system case we ask what is the maximum order of a model that guarantees it to be a PROM. The problem is more complicated than in the square case since there is no single simple formula that relates the maximal zero sub-pencil to the dimensions h and q. Adding to that the fact that those dimensions depend on r itself only increases the complexity.

The strategy for extracting the maximum zero sub-pencil is to use as many rows as possible that belong to same block. This is because while choosing a single row into the set I_r eliminates two columns from belonging to the set J_r, choosing l rows from the same block disqualifies only $l+1$ columns. The dependence on the block structure is thus evident. Suppose the rows of I_r come from M separate blocks, each one contributing l_i rows. In order to have enough columns for a square matrix of zeros, the following must hold

$$r = \sum_{i=1}^{M} l_i \le q - \sum_{i=1}^{M}(l_i+1) = q - r - M \tag{26}$$

Recalling that $q=n-p+r$ we have

$$r \le n - p - M \tag{27}$$

This is not a closed form formula since M is a function of r, but it allows easy computation of the maximum r. Results for some typical cases are shown in Table 1. r^* in the last column is the number obtained in Section III by simply counting the number of equations. Expressing r^* in (11) in terms of h and q, it immediately follows that

$$r^* = (q+h)/4 \tag{28}$$

TABLE I
THE MAXIMUM ORDER OF A GENERIC PROM

n	m	p	r_{max}	r*
100	80	80	20	20
100	80	40	37	40
100	80	20	40	50
100	80	10	45	55
100	80	5	47	57
100	80	1	49	59
100	50	50	50	50
100	50	25	62	62
100	50	10	67	70
100	50	5	71	72
100	50	1	74	74

As can be seen the actual maximum possible value of r is in general less than that predicted by the number of equations. This is unlike the square system case where the two values were in accordance. The discrepancy cannot be explained by the requirement of a real solution since the GKS is completely structural. The only possible explanation is that for values of r between r_{max} and r^*, the left-hand sides of the set of equations (10) have inherent nonlinear dependence that causes the set to contain contradictions.

The relationship between r_{max} and r^* can be explored for a special case. Assume that h is a whole multiple of $q-h$, so that $\alpha=h/(q-h)$, $\beta=0$. The GKS has then $q-h$ identical $\alpha\times(\alpha+1)$ blocks. Assume also that that r consists of M full blocks, hence $M=r/\alpha=r(q-h)/h$. Then

$$r \le q - r - M = q - r - r(q-h)/h\,, \tag{29}$$

leading to

$$r_{max} = qh/(q+h) \tag{30}$$

Comparing the values in (28) and (30) we have

$$\frac{q+h}{4} - \frac{qh}{q+h} = \frac{(q-h)^2}{4(q+h)} > 0 \quad \forall q \ne h \tag{31}$$

Hence the actual maximum r is smaller than r^*. Other special cases yield the same result.

As a final remark we note that the GKS leads to the smallest possible zero sub-pencil, and any deviation from it tends to increase the zeros area. This is what happens with reduced order models, which are PROM, but having a dimension greater than $r_{\max}$. As an example consider a fourth order system with three inputs and two outputs, i.e. n=4, m=3, p=2. In this case $r_{\max}$ =1. The GKS for second order models is

$$F(\lambda)=\begin{bmatrix}\lambda & -1 & 0 & 0\\ 0 & \lambda & -1 & 0\\ 0 & 0 & \lambda & -1\end{bmatrix}, \qquad (32)$$

and clearly there is no 2×2 sub-pencil of zeros. A second order PROM will have the following KCF.

$$F(\lambda)=\left[\begin{array}{cc|cc}\lambda & -1 & 0 & 0\\ \hline 0 & 0 & \lambda-f_{11} & -f_{12}\\ 0 & 0 & -f_{21} & \lambda-f_{22}\end{array}\right] \qquad (33)$$

where the f_{ij} represent a real Jordan block of two real, a complex pair or two identical eigenvalues. In any event, the 2×2 sub-pencil of zeros exists in the bottom left corner. In this case the normal rank of the pencil is 3, but it has two eigenvalues. Such a situation can be easily detected by calculating the eigenvalues of the *square* pair (H_1U, H_2U) for two arbitrary values of a $q\times h$ matrix U. The eigenvalues of (H_1, H_2), if exist, are eigenvalues in both cases. For r=3 the GKS is

$$F(\lambda)=\begin{bmatrix}\lambda & -1 & 0 & 0 & 0\\ 0 & \lambda & -1 & 0 & 0\\ 0 & 0 & \lambda & -1 & 0\\ 0 & 0 & 0 & \lambda & -1\end{bmatrix} \qquad (34)$$

and a third order PROM will result in

$$F(\lambda)=\left[\begin{array}{cc|cc|c}\lambda & 0 & 0 & 0 & 0\\ -1 & \lambda & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ \hline 0 & 0 & \lambda & -1 & 0\end{array}\right] \qquad (35)$$

VI. CONCLUSION

The properties of the projection relationship between two models of different orders have been investigated. An algorithm to calculate all the projections that relate the two models, as well as conditions for the existence and uniqueness, have been presented. In the case of a generic square system, any reduced model whose order is less than n-m can be obtained by infinitely many projections, any reduced model whose order is greater than n-m cannot be obtained by projection, and the case of exactly n-m depends on the eigenvalue structure of a certain pencil. In the non-square case the maximum value of projectable models depends on the Generic Kronecker Structure of the matrix pencil. In general, larger difference between the number of inputs and outputs leads to a smaller maximal order than is predicted by counting equations and unknowns.

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