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On the asymptotic properties of the Hessian in discrete-time linear quadratic control

Osvaldo J. Rojas and Graham C. Goodwin

Abstract— **This paper studies the asymptotic properties of the Hessian in discrete-time linear quadratic optimal control. We show that the singular values of the Hessian converge, in a well defined sense, to the principal gains in the frequency domain of an associated normalized system transfer function. We treat the stable and unstable case for multi-input multi-output linear systems. Potential applications of the ideas presented here include fast and/or robust algorithms for constrained model predictive control of discrete-time linear systems.**

I. INTRODUCTION

This paper is aimed at contributing to the development of fast and/or robust algorithms for constrained model predictive control (MPC) of discrete-time linear systems. This class of algorithms have become widely used in process control applications [7], [8], [9]. Usually the problem is posed as a quadratic program [1], [10]. There has also been recent interest in, so called, explicit forms of model predictive control [2], [16]. The latter algorithms offer the potential for greater computational efficiency in high speed applications. Other approximate algorithms aimed at high speed applications are described in [6], [4]. A special type of algorithms in this class uses a singular value decomposition (SVD) of the associated Hessian to suggest directions in which the system input has a "high gain" to the corresponding cost function [11], [13], [15]. Similar ideas have been suggested for constrained continuous-time linear systems [5].

The current paper contributes to this circle of ideas by further exploring the SVD structure of discrete-time linear quadratic optimal control problems. Earlier work (see for example [13]) was restricted to *stable single-input single-output* systems. Here we extend the result to stable and/or *unstable multi-input multi-output* systems. The key result in the paper is to establish an asymptotic equivalence between the singular values of the Hessian and the principal gains in the frequency domain of an associated normalized system transfer function. This result allows one to apply frequency domain intuition to the control of constrained linear systems. Potential applications include:

- improved algorithms for high dimension multi-input multi-output systems of the type described in [12];
- new methods for treating ill-conditioning problems in the constrained-control of unstable systems (see Section III below)
- utilization of frequency domain insights to develop model predictive control algorithms exhibiting robustness to unconstrained uncertainty. In particular, it seems heuristically reasonable that one should restrict the optimization to the set of frequencies where the relative model error is small. This would allow one to link traditional robust control approaches of unconstrained linear systems to the constrained case (Details of this idea are currently being developed but the core building block is the kind of frequency domain link established in the current paper).

The layout of the remainder of the paper is as follows: in Section II we describe the finite horizon linear quadratic optimal control problem which is central to MPC algorithms. In section III we address the case of unstable systems. Section IV examines the limiting properties of the singular values of the Hessian when the prediction horizon is taken to infinity. The result is substantiated with an example based on a MIMO plant having both stable and unstable modes in Section V. Finally, we draw conclusions in Section VI.

II. RECEDING HORIZON QUADRATIC OPTIMAL CONTROL

Linear quadratic optimal receding horizon control solves, at each time step, the following optimization problem

$$\mathscr{P}_N(x): \quad \{u_k\}^{\text{OPT}} \triangleq \arg\min J_N(x,\{u_k\}) \tag{1}$$

where

$$J_N(x,\{u_k\}) \triangleq \sum_{k=0}^{N-1}\left[x_k^T Q\, x_k + u_k^T R\, u_k\right] + x_N^T P\, x_N \tag{2}$$

subject to

$$\begin{aligned} x_0 &= x \\ x_{k+1} &= Ax_k + Bu_k \\ u_k &\in \mathbb{U} \\ x_k &\in \mathbb{X} \\ x_N &\in \mathbb{X}_f \subset \mathbb{X} \end{aligned} \tag{3}$$

for $k = 0, 1, \dots, N-1$, $N \in \mathbb{Z}^+$. $x_k \in \mathbb{R}^n$ is the state vector and $u_k \in \mathbb{R}^m$ is the input vector. x is the current measured (or observed) state. $Q = Q^T \geq 0$, $R = R^T > 0$ and $P = P^T \geq 0$. The pair (A, B) is assumed stabilisable and the pair $(A, Q^{\frac{1}{2}})$ detectable. $\mathbb{U}$ and $\mathbb{X}$ are convex input and state constraints respectively that contain the origin. $\mathbb{X}_f$ is a final constraint set. Once the optimal control sequence $\{u_k\}^{\text{OPT}}$ is obtained, only the first element u_0^{OPT} is applied

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O. J. Rojas and G. C. Goodwin are with the School of Electrical Engineering & Computer Science, The University of Newcastle, Callaghan NSW 2308, Australia. orojas@ee.newcastle.edu.au

to the plant and the procedure is repeated when a new state measurement x becomes available. More details about receding horizon control can be found in [8], [7], [3].

III. PROBLEM FORMULATION FOR UNSTABLE SYSTEMS

Consider the following normalized system based on the objective function defined in (2),

$$\begin{aligned} \mathcal{S}_n: \quad & x_{k+1} = A\,x_k + BR^{-\frac{1}{2}}\,u'_k \\ & y_k = Q^{\frac{1}{2}} x_k \end{aligned} \tag{4}$$

Assume the eigenvalues of A are not on the unit disc. Then, we can separate stable and unstable modes using a suitable state space transformation and write $\mathcal{S}_n$ as follows,

$$\begin{aligned} \mathcal{S}_d: \quad & \begin{bmatrix} x^s_{k+1} \\ x^u_{k+1} \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x^s_k \\ x^u_k \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} u'_k \\ & y_k = \begin{bmatrix} C_s & C_u \end{bmatrix} \begin{bmatrix} x^s_k \\ x^u_k \end{bmatrix} \end{aligned} \tag{5}$$

where the superscripts indicate stable and unstable parts. One way of implementing the optimization problem $\mathscr{P}_N(x)$ in (1) is to include the above model implicitly when building the matrices for the associated quadratic program (QP). However, it is well known that this approach leads to serious numerical problems, especially when long prediction horizons are used [14], [7]. The reason for this is that the solution to the system of equations (5) contains exploding terms of the type A^k_u. Ways of avoiding this difficulty include pre-stabilizing the predictions [14], or considering the system equations (5) as explicit equality constraints in the optimization. In this Section we propose yet another alternative. This new implementation of $\mathscr{P}_N(x)$ is also instrumental to extend our earlier results on the characterization of the singular values of the Hessian of $J_N(x, \{u_k\})$ when $N \to \infty$ [11] to unstable systems. This will be further analysed in Section IV. In this contribution, we propose to solve only the stable modes of (5) in *forward time i.e.*,

$$y^s_k \triangleq C_s x^s_k = C_s A^k_s x^s_0 + \sum_{j=0}^{k-1} C_s A_s^{k-1-j} B_s u'_j \tag{6}$$

and solve the unstable modes in *reverse time*, starting from the unstable state x^u_N at time $k = N$. That is,

$$y^u_k \triangleq C_u x^u_k = C_u A_u^{-(N-k)} x^u_N - \sum_{j=k}^{N-1} C_u A_u^{k-1-j} B_u u'_j \tag{7}$$

In doing so, we ensure that the associated Markov parameters, for both stable and unstable modes, are convergent. This prevents the appearance of any numerical issue. We note that the final unstable state x^u_N and input sequence $\{u'_j\}$ are related to the unstable state initial conditions via the following equality

$$A_u^{-N} x^u_N - \sum_{j=0}^{N-1} A_u^{-j-1} B_u u'_j = x^u_0 \tag{8}$$

Define the vectors

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} y_1^T & y_2^T & \dots & y_N^T \end{bmatrix}^T \\ \mathbf{y}_s &= \begin{bmatrix} y_1^{s\,T} & y_2^{s\,T} & \dots & y_N^{s\,T} \end{bmatrix}^T \\ \mathbf{y}_u &= \begin{bmatrix} y_1^{u\,T} & y_2^{u\,T} & \dots & y_N^{u\,T} \end{bmatrix}^T \\ \mathbf{u} &= \begin{bmatrix} (u'_0)^T & (u'_1)^T & \dots & (u'_{N-1})^T \end{bmatrix}^T \end{aligned} \tag{9}$$

From (5) we note that $\mathbf{y} = \mathbf{y}_s + \mathbf{y}_u$. Therefore, we have

$$\begin{aligned} & \mathbf{y} = \underbrace{(\Gamma_s + \Gamma_u)}_{\Gamma} \mathbf{u} + \Omega_u x^u_N + \Omega_s x^s_0 \\ & \text{subject to} \\ & L\mathbf{u} + A_u^{-N} x^u_N = x^u_0 \end{aligned} \tag{10}$$

where

$$\Gamma_s \triangleq \begin{bmatrix} C_sB_s & 0 & \cdots & 0 \\ C_sA_sB_s & C_sB_s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_sA_s^{N-1}B_s & C_sA_s^{N-2}B_s & \cdots & C_sB_s \end{bmatrix} \tag{11}$$

$$\Gamma_u \triangleq - \begin{bmatrix} 0 & C_uA_u^{-1}B_u & C_uA_u^{-2}B_u & \dots & C_uA_u^{-(N-1)}B_u \\ & 0 & C_uA_u^{-1}B_u & \dots & C_uA_u^{-(N-2)}B_u \\ & & \ddots & \ddots & \vdots \\ & \mathbf{0} & & 0 & C_uA_u^{-1}B_u \\ & & & & 0 \end{bmatrix} \tag{12}$$

and

$$\Omega_s = \begin{bmatrix} C_sA_s \\ C_sA_s^2 \\ \vdots \\ C_sA_s^N \end{bmatrix} \quad \Omega_u = \begin{bmatrix} C_uA_u^{-(N-1)} \\ C_uA_u^{-(N-2)} \\ \vdots \\ C_uA_u^{-1} \\ C_u \end{bmatrix} \tag{13}$$

The matrix that defines the equality constraint in (10) is given by

$$L = \begin{bmatrix} -A_u^{-1}B_u & -A_u^{-2}B_u & \dots & -A_u^{-N}B_u \end{bmatrix} \tag{14}$$

Observe that the matrix formulation of the output predictions $\mathbf{y}$ of the normalized system $\mathcal{S}_d$ in (10) contains only terms of the form A^k_s and A^{-k}_u. The proposed formulation requires the inclusion of only n_u equality constraints, where n_u is the number of unstable states in $\mathcal{S}_d$. However, this can be regarded as being a marginal increase in the complexity of $\mathscr{P}_N(x)$. We can now express the objective function in (2) using the matrix notation introduced in (10) *i.e.*,

$$\begin{aligned} & J_N(x, \{u_k\}) = J'_N(x^s_0, x^u_N, \mathbf{u}) \triangleq y_0^T y_0 \\ & \qquad + \|\Gamma\mathbf{u} + \Omega_u x^u_N + \Omega_s x^s_0\|_2^2 + \|\mathbf{u}\|_2^2 \\ & \text{subject to} \\ & L\mathbf{u} + A_u^{-N} x^u_N = x^u_0 \end{aligned} \tag{15}$$

Note that we have considered $P = Q$ in (2). However, this is not restrictive since, as we shall see in Section IV, we are interested in the case $N \to \infty$. After some matrix manipulations, we can write

$$J'_N(x^s_0, \mathbf{z}) = \mathbf{z}^T H \mathbf{z} + 2\mathbf{z}^T F x^s_0 + c_0 \tag{16}$$

where $\mathbf{z} = \left[\mathbf{u}^T \ (x_N^u)^T\right]^T$ and

$$H = \begin{bmatrix} \Gamma^T\Gamma + I & \Gamma^T\Omega_u \\ \Omega_u^T\Gamma & \Omega_u^T\Omega_u \end{bmatrix}, \quad F = \begin{bmatrix} \Gamma^T\Omega_s \\ \Omega_u^T\Omega_s \end{bmatrix} \tag{17}$$

In addition, c_0 in (16) is an appropriate constant value. In the next Section we will analyse in detail the singular value structure of the Hessian in (17) when the prediction horizon N is taken to infinity.

IV. ASYMPTOTIC SINGULAR VALUE STRUCTURE OF THE HESSIAN

The importance of a singular value decomposition (SVD) of the Hessian H is that it provides a set of orthogonal basis vectors (the singular vectors) with a specific associated gain (the singular values) to the performance index $J_N'(x_0^s, \mathbf{z})$. This decomposition can be exploited to develop sub-optimal solutions to $\mathscr{P}_N(x)$ [13], [15]. We shall show that when $N \to \infty$ the singular values of H become even more insightful for they converge, in a well defined sense, to the principal gains in the frequency domain of the normalized system $\mathcal{S}_d$ in (5). We first establish the following Lemma.

Lemma 4.1: Let the prediction horizon N be infinite. If the infinite horizon optimal control problem $\mathscr{P}_\infty(x)$ in (1) is feasible, then the optimal solution to $\mathscr{P}_\infty(x)$ is equal to the solution to the following optimization problem,

$$\begin{aligned} \mathscr{P}_\infty'(x): \quad & \mathbf{z}^{\text{OPT}} \triangleq \arg\min J_N'(x_0^s, \mathbf{z}) \\ & \text{subject to} \\ & x_0 = x \ , \ u_k \in \mathbb{U} \ , \ x_k \in \mathbb{X} \\ & x_\infty^u = 0 \end{aligned} \tag{18}$$

Proof: Feasibility of the optimization problem $\mathscr{P}_\infty(x)$ implies that there exists a stabilizing control sequence $\{u_k\}$ – equivalently, a control sequence $\{u_k'\}$ – such that $x_k \to 0$ when $k \to \infty$. In particular, $x_k^u \to 0$. Also, note that the equality constraint (15) becomes redundant. The result follows. ■

We observe that the Hessian of the equivalent optimization problem $\mathscr{P}_\infty'(x)$ is given by

$$H' = \Gamma^T\Gamma + I \tag{19}$$

We next analyse the singular value structure of H' when the prediction horizon N is taken to infinity. We make explicit the dependence on N by using the notation H_N'. We express the SVD of H_N' in terms of the SVD of the matrix Γ. In particular, consider

$$\Gamma = U\Sigma V^T \tag{20}$$

where $U \in \mathbb{R}^{Nn\times Nn}$ and $V \in \mathbb{R}^{Nm\times Nm}$ are orthogonal matrices. In addition, we have

$$\Sigma = \begin{bmatrix} S \\ 0 \end{bmatrix} \tag{21}$$

where S is a diagonal matrix containing the singular values of Γ *i.e.*, $S = \operatorname{diag}(\sigma_1, \ldots, \sigma_{Nm})$. Using (20) we obtain the following SVD of H_N',

$$H_N' = V(S^2 + I_{Nm})V^T \tag{22}$$

where I_{Nm} is the $Nm \times Nm$ identity matrix.

From the definitions of Γ_s and Γ_u in (11) and (12) respectively, we observe that Γ has the structure

$$\Gamma = \begin{bmatrix} h_0 & h_{-1} & \dots & h_{-(N-1)} \\ h_1 & h_0 & \dots & h_{-(N-2)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & \dots & h_0 \end{bmatrix} \tag{23}$$

where

$$h_k = \begin{cases} -C_u A_u^k B_u & \text{for } k = -1, -2, \ldots \\ C_s A_s^k B_s & \text{for } k = 0, 1, 2, \ldots \end{cases} \tag{24}$$

We then have the following Lemma,

Lemma 4.2: Let $\overline{G}(z)$ be the two-sided Z-transform of the infinite sequence $\{h_k : k = -\infty, \ldots, \infty\}$ in (24). Then $\overline{G}(z)$ is given by

$$\overline{G}(z) = z\ G(z) \ , \tag{25}$$

where $G(z)$ is the transfer function of the normalized system in (5). Moreover, the region of convergence of $\overline{G}(z)$ is given by

$$\max\{|\lambda_i(A_s)|\} < |z| < \min\{|\lambda_i(A_u)|\} \ ,$$

where $\lambda_i(\cdot)$ is the set of eigenvalues of the corresponding matrix.

Proof: The two-sided Z-transform of $\{h_k\}$ is given by

$$\overline{G}(z) \triangleq \sum_{k=-\infty}^{\infty} h_k\, z^{-k} = \sum_{k=-\infty}^{-1} h_k\, z^{-k} + \sum_{k=0}^{\infty} h_k\, z^{-k}$$

Substituting the values of $\{h_k\}$ in (24), we obtain

$$\overline{G}(z) = \underbrace{-\sum_{k=-\infty}^{-1} C_u A_u^k B_u\, z^{-k}}_{a} + \underbrace{\sum_{k=0}^{\infty} C_s A_s^k B_s\, z^{-k}}_{b} \ . \tag{26}$$

Le us first analyse the 'b' term above. This is a geometric series that converges if and only if

$$|z| > \max\{|\lambda_i(A_s)|\} \tag{27}$$

In addition, note that

$$G_s(z) \triangleq C_s(zI - A_s)^{-1}B_s = \sum_{k=1}^{\infty} C_s A_s^{k-1} B_s\, z^{-k} \tag{28}$$

Comparing the above expression to 'b' in (26), we see that

$$b = zG_s(z) \tag{29}$$

For the 'a' term in (26) we let $j = -k$. Hence,

$$a = -\sum_{j=1}^{\infty} C_u A_u^{-j} B_u\, z^j \tag{30}$$

Now, we can write

$$a = -\sum_{j=1}^{\infty} C_u (A_u^{-1})^{j-1} A_u^{-1} B_u (z^{-1})^{-j} \qquad (31)$$

The above is a geometric series that converges if and only if

$$|z| < \frac{1}{\max\{|\lambda_i(A_u^{-1})|\}} = \min\{|\lambda_i(A_u)|\} \quad .$$

Comparing the form of the terms in (31) to that of the terms in (28) we conclude that

$$a = -C_u (z^{-1} I - A_u^{-1})^{-1} A_u^{-1} B_u \qquad (32)$$

from which

$$\begin{aligned} a &= z\, C_u (A_u^{-1} z - I)^{-1} A_u^{-1} B_u \\ &= z\, C_u (zI - A_u)^{-1} B_u \\ &= z\, G_u(z) \end{aligned} \qquad (33)$$

Therefore,

$$\begin{aligned} \overline{G}(z) = a + b &= z\, G_s(z) + z\, G_u(z) \\ &= z\, G(z) \ . \end{aligned} \qquad (34)$$

for all $z \in \mathbb{C}$ such that

$$\max\{|\lambda_i(A_s)|\} < |z| < \min\{|\lambda_i(A_u)|\}$$

This concludes the Proof. ∎

The region of convergence of the two-sided Z Transform $\overline{G}(z)$ includes the unit circle and this allows us to refer to the frequency response of $\overline{G}(z)$ by taking $z = e^{jw}$.

The fact that the sequence $\{h_k\}$ contains only decaying terms ensures that given any $\varepsilon > 0$ there exists $k_0 > 0$ such that

$$\left| \left\| \overline{G}(e^{jw}) \right\|_F^2 - \left\| \sum_{k=-k_0}^{k_0} h_k e^{-jwk} \right\|_F^2 \right| < \varepsilon \,, \quad \forall w \in [-\pi, \pi] \qquad (35)$$

where $\|\cdot\|_F$ is the Frobenius norm. The above is equivalent to saying that for $k > k_0$ and $k < -k_0$ the terms of the matrix sequence $\{h_k\}$ are negligible. As a result, the autocorrelation of the matrix sequence $\{h_k\}_{k=-\infty}^{\infty}$ can be approximated as follows,

$$\begin{aligned} &\Phi_l \triangleq \sum_{k=-\infty}^{\infty} h_k^T h_{k+l} \approx \sum_{k=-k_0}^{k_0-l} h_k^T h_{k+l} \ , \qquad 0 \le l \le 2k_0 \\ &\Phi_l \approx 0 \ , \qquad l > 2k_0 \ , \\ &\Phi_{-l} = \Phi_l^T \end{aligned} \qquad (36)$$

Using the definition of the matrix Φ_l above, we see that $\Gamma^T \Gamma$ can be written as

$$\Pi_N \triangleq \Gamma^T \Gamma = \left[\begin{array}{ccccccc} \multicolumn{3}{c}{X1} & | & \multicolumn{3}{c}{\mathbf{0}} \\ \hline \Phi_{2k_0} & \dots & \Phi_0 & \dots & \Phi_{-2k_0} & & \mathbf{0} \\ & \ddots & & \ddots & & \ddots & \\ \mathbf{0} & & \Phi_{2k_0} & \dots & \Phi_0 & \dots & \Phi_{-2k_0} \\ \hline \multicolumn{2}{c}{\mathbf{0}} & | & \multicolumn{4}{c}{X2} \end{array} \right] \qquad (37)$$

provided $N \geq (4k_0 + 1)$. $X1$ and $X2$ are appropriate submatrices. We then have the following key result relating the singular values of Γ to the principal gains in the frequency domain of the transfer function $G(e^{jw})$.

Theorem 4.3: Consider

1) the matrix Γ in (23) and its corresponding singular value decomposition (20);
2) the singular value decomposition of the transfer function $G(e^{jw})$ *i.e.*,

$$G(e^{jw}) = \widehat{U}(e^{jw}) \widehat{\Sigma}(w) \widehat{V}^H(e^{jw}) \qquad (38)$$

where $\widehat{U} \in \mathbb{C}^{n \times n}$, $\widehat{V} \in \mathbb{C}^{m \times m}$, $\widehat{\Sigma}(w) = \mathrm{diag}\{\hat{\sigma}_1(w), \dots, \hat{\sigma}_q(w)\}$, $q \triangleq \min(n, m)$. In addition, $\widehat{U}^H \widehat{U} = \widehat{U} \widehat{U}^H = I_n$, $\widehat{V}^H \widehat{V} = \widehat{V} \widehat{V}^H = I_m$;
3) the $Nm \times m$ matrix

$$E_{N,w} \triangleq \left[\mathbf{e}_{N,w}^1 \quad \cdots \quad \mathbf{e}_{N,w}^m \right] \triangleq \frac{1}{\sqrt{N}} \overline{E}_{N,w} \,, \qquad (39)$$

where

$$\overline{E}_{N,w} \triangleq \begin{bmatrix} \widehat{V}(e^{jw}) \\ e^{jw} \ \widehat{V}(e^{jw}) \\ \vdots \\ e^{j(N-1)w} \ \widehat{V}(e^{jw}) \end{bmatrix}$$

and

$$w = \frac{2\pi}{N} p \,, \quad p \in \{0, \dots, N-1\} \ . \qquad (40)$$

Let $k_0 > 0$ be such that (35) holds. Let

$$w_0 \triangleq \frac{2\pi}{N_0} p_0 \ \in \ [-\pi, \pi] \qquad (41)$$

for arbitrary $N_0 \geq (4k_0 + 1)$ and $p_0 \in \{0, \dots, N_0 - 1\}$. Then, for every index $i \in \{1, \dots, q\}$ there exists at least one singular value σ_j of Γ, with $j \in \{1, \dots, Nm\}$ such that when $\frac{N}{N_0} \to \infty$,

$$\sigma_j = \hat{\sigma}_i(w_0) \qquad (42)$$

Proof: Let

$$\mathbf{\Phi}(e^{jw}) = \sum_{l=-\infty}^{\infty} \Phi_l \, e^{-jwl} \approx \sum_{l=-2k_0}^{2k_0} \Phi_l \, e^{-jwl}$$

with $w \in [-\pi, \pi]$, be the Discrete Time Fourier Transform of the autocorrelation matrix in (36). The existence of

$\mathbf{\Phi}(e^{jw})$ is guaranteed by Lemma 4.2. In particular, note that

$$\mathbf{\Phi}(e^{jw}) = \overline{G}(e^{jw})^H \overline{G}(e^{jw})$$

and by means of Lemma 4.2

$$\mathbf{\Phi}(e^{jw}) = G(e^{jw})^H e^{-jw} e^{jw} G(e^{jw}) = G(e^{jw})^H G(e^{jw})$$

Using (38) we can write

$$\mathbf{\Phi}(e^{jw})\widehat{V}(e^{jw}) = \widehat{V}(e^{jw})\widehat{\Sigma}(w)^T\widehat{\Sigma}(w) \tag{43}$$

Next, consider $N_0 > 4k_0 + 1$, $0 \le p_0 \le N_0 - 1$ and $w_0 = \frac{2\pi}{N_0}p_0$. By direct calculation, we can write

$$\Pi_{N_0} E_{N_0,\, w_0} = \frac{1}{\sqrt{N_0}} \begin{bmatrix} W_1 \\ \overline{E}_{N_0,\, w_0}[2k_0m+1:(N_0-2k_0)m\ ,\ :]\ \mathbf{\Phi}(e^{jw_0})\widehat{V}(e^{jw_0}) \\ W_2 \end{bmatrix} \tag{44}$$

where $\overline{E}_{N_0,\, w_0}[2k_0m+1 : (N_0-2k_0)m\ ,\ :]$ represents the section of the matrix $\overline{E}_{N_0,\, w_0}$ that starts at the $(2k_0m+1)$-th row and finishes at the $((N_0-2k_0)m)$-th row. In addition, W_1 and W_2 are appropriate sub-matrices of dimension $2k_0m \times m$. Replacing (43) in (44) we have

$$\Pi_{N_0} E_{N_0,\, w_0} = \begin{bmatrix} \frac{1}{\sqrt{N_0}} W_1 \\ E_{N_0,\, w_0}[2k_0m+1:(N_0-2k_0)m\ ,\ :]\ \widehat{\Sigma}(w_0)^T\widehat{\Sigma}(w_0) \\ \frac{1}{\sqrt{N_0}} W_2 \end{bmatrix}$$

which can be considered column-wise as follows,

$$\Pi_{N_0}\, \mathbf{e}^i_{N,w_0} = \begin{bmatrix} \frac{1}{\sqrt{N_0}}\mathbf{c}_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \hat{\sigma}_i^2(w_0)\, \mathbf{e}^i_{N,w_0}[2k_0m+1 : (N_0-2k_0)m\,] \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{\sqrt{N_0}}\mathbf{c}_2 \end{bmatrix}$$

for $i = 1, \ldots, q$. Subtracting $\hat{\sigma}_i^2(w_0)\, \mathbf{e}^i_{N,w_0}$ from both sides of the above equation yields

$$\Pi_{N_0}\mathbf{e}^i_{N,w_0} - \hat{\sigma}_i^2(w_0)\, \mathbf{e}^i_{N,w_0} = \mathbf{d}_{N_0,\, w_0} \tag{45}$$

where

$$\mathbf{d}_{N_0,\, w_0} \triangleq \frac{1}{\sqrt{N_0}} \begin{bmatrix} \mathbf{d_1} \\ \mathbf{0}_{(N_0-4k_0)m} \\ \mathbf{d_2} \end{bmatrix} \tag{46}$$

Here $\mathbf{0}_{(N_0-4k_0)m}$ is a column vector with zero entries and length $(N_0-4k_0)m$. It can be easily shown that the norms of both vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are bounded. They are determined by the entries of sub-matrices X_1 and X_2 in (37) (which are independent of the prediction horizon N_0), the fixed value $\hat{\sigma}_i^2(w_0)$ and the entries of vector $\mathbf{e}^i_{N_0,w_0}$ which are bounded. As a result, we can find $T_{w_0} > 0$ such that

$$\|\mathbf{d}_{N_0,\, w_0}\|_2 = \frac{1}{\sqrt{N_0}}\sqrt{\|\mathbf{d_1}\|_2^2 + \|\mathbf{d_2}\|_2^2} \le \frac{1}{\sqrt{N_0}}\, T_{w_0} \quad . \tag{47}$$

We now select $N = LN_0$ and $p = L\,p_0$, for some $L \in \mathbb{Z}^+$, such that

$$w = \frac{2\pi p}{N} = \frac{2\pi p_0}{N_0} = w_0 \tag{48}$$

Therefore, for every $\varepsilon_0 > 0$ we can select L such that

$$\|\mathbf{d}_{N,\, w_0}\|_2 \le \frac{1}{\sqrt{N}}\, T_{w_0} = \frac{1}{\sqrt{L\,N_0}}\, T_{w_0} < \varepsilon_0 \quad . \tag{49}$$

The above is satisfied for $L \in \mathbb{Z}^+$ such that

$$L > \frac{T^2_{w_0}}{N_0\,\varepsilon_0^2} \tag{50}$$

We conclude that, for every $w_0 = \frac{2\pi}{N_0}p_0$ and $\varepsilon_0 > 0$, there exists $N = LN_0$ with $L \in \mathbb{Z}^+$ satisfying (50) such that

$$\|\Pi_N \mathbf{e}^i_{N,w_0} - \hat{\sigma}_i^2(w_0)\mathbf{e}^i_{N,w_0}\|_2 < \varepsilon_0 \tag{51}$$

for $i = 1, \ldots, q$. Therefore

$$\lim_{\frac{N}{N_0}\to\infty} \|\Pi_N \mathbf{e}^i_{N,w_0} - \hat{\sigma}_i^2(w_0)\mathbf{e}^i_{N,w_0}\|_2 = 0 \tag{52}$$

Properties of the vector norm $\|\cdot\|_2$ ensure that, in the limit when $\frac{N}{N_0} \to \infty$,

$$\Pi_N \mathbf{e}^i_{N,w_0} = \hat{\sigma}_i^2(w_0)\mathbf{e}^i_{N,w_0} \tag{53}$$

That is, in the limit, $\hat{\sigma}_i^2(w_0)$ is an eigenvalue of Π_N and $\mathbf{e}^i_{N,w_0}$ is the corresponding eigenvector. However, by definition, the singular values of the matrix Γ are equal to the square root of the eigenvalues of Π_N and the corresponding eigenvector of Π_N is equal to the right singular vector of Γ. This completes the Proof. ∎

The importance of the above Theorem is that it establishes a well defined equivalence between the singular values of the matrix Γ and the principal gains in the frequency domain of the transfer function $G(e^{jw})$ related to the normalized system (5). This result, in turn, provides an asymptotic characterization of the singular value structure of the Hessian H'_N in (19).

V. Numerical Example

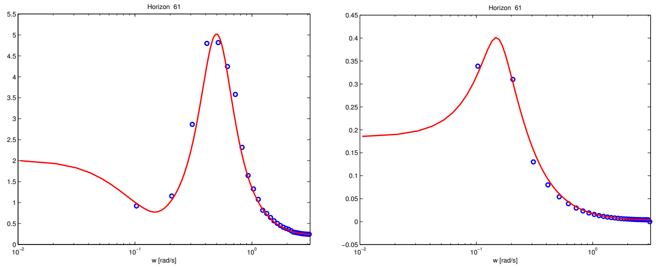
We illustrate the result of Theorem 4.3 with the following example. Let $\mathcal{S}_d$ in (5) be defined via the matrices

$$A_s = \begin{bmatrix} 1.442 & -0.64 \\ 1 & 0 \end{bmatrix}\ ,\ B_s = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\ ,\quad C_s = \begin{bmatrix} 0.721 & -0.64 \\ -0.36 & 0.32 \end{bmatrix} \tag{54}$$

and

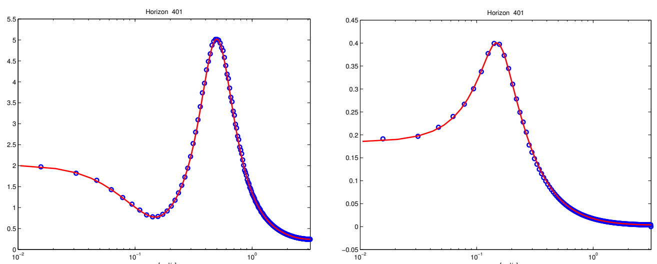
$$A_u = 2\ ,\ B_u = \begin{bmatrix} 0 & 1 \end{bmatrix}\ ,\ C_u = \begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix} \tag{55}$$

We then compute the singular values of Π_N in (37) for two different prediction horizon N and we compare them with the principal gains squared of the system $\mathcal{S}_d$. The results are presented in Figure 1 for $N = 61$ and in Figure 2 for $N = 401$. We observe that, as the prediction horizon N is increased, the singular values of Π_N converge to the continuous line representing the square of the principal gains of $\mathcal{S}_d$ as predicted by Theorem 4.3.



(a) $\hat{\sigma}_1(w)^2$ (b) $\hat{\sigma}_2(w)^2$

Fig. 1. Singular values of matrix Π_N (circles) with $N = 61$. The continuous lines indicate the two principal gains squared of $G(e^{jw})$.



(a) $\hat{\sigma}_1(w)^2$ (b) $\hat{\sigma}_2(w)^2$

Fig. 2. Singular values of matrix Π_N (circles) with $N = 401$. The continuous lines indicate the two principal gains squared of $G(e^{jw})$.

VI. Conclusions

This paper has investigated the asymptotic properties of the Hessian in discrete time linear quadratic optimal control. We have shown that the singular values of the Hessian converge, in a well defined sense, to the principal gains squared of the associated normalized system transfer function.

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