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Improved performance of the controlled Kuramoto-Sivashinsky equation via actuator and controller switching

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Index Terms— **Distributed parameter systems, Actuator scheduling, Nonlinear control, Supervisory system.**

Abstract— **The objective of this manuscript is to examine the performance improvement of a class of nonlinear transport processes subject to spatiotemporally varying disturbances through the employment of a comprehensive and systematic actuator activation policy. To do so, it is assumed that multiple groups of actuators are available with only one such actuator group being active over a time interval of fixed length while the remaining actuator groups are kept dormant. Using enhanced controllability and performance improvement measures, the candidate actuator groups are first placed in locations that individually provide certain robustness with respect to an appropriately defined "worst" spatial distribution of disturbances. Once the multiple actuator groups are in place, a switching scheme is developed that dictates the switching of a different actuator group at different time intervals and the corresponding control signal supplied to it while being active. Embedded in the decision policy is the activation of actuators that lie spatially closer to the spatiotemporal disturbances, thereby improving the control authority of the actuators and enhancing the ability of the system to minimize the effects of the above class of disturbances.**

I. MATHEMATICAL FORMULATION

We consider the 1-D controlled Kuramoto-Sivashinsky equation in the bounded interval $\Omega = [-\pi, \pi]$

$$\begin{aligned}&\frac{\partial U(\xi,t)}{\partial t} + \nu \frac{\partial^4 U(\xi,t)}{\partial \xi^4} + \frac{\partial^2 U(\xi,t)}{\partial \xi^2} \\ &+ U(\xi,t)\frac{\partial U(\xi,t)}{\partial \xi} = \sum_{i=1}^{m} b_i(\xi)u_i(t) + d(\xi)w(t),\end{aligned} \tag{1}$$

along with the periodic boundary conditions

$$\frac{\partial^j U}{\partial \xi^j}(-\pi,t) = \frac{\partial^j U}{\partial \xi^j}(\pi,t), \quad j = 0,1,2,3, \tag{2}$$

and the initial condition $U(\xi,0) = U_o(\xi)$. The distribution $U(\xi,t)$ denotes the state of the PDE, $\xi \in [-\pi,\pi]$ is the spatial coordinate and ν denotes the *instability parameter*. The control signals $u_i \in \mathbb{R}$ describe the temporal components of the external excitation and the functions $b_i(\xi)$ are the actuator distribution functions that describe the spatial influence of the actuating devices. The spatial function $d(\xi)$ denotes the distribution of process disturbances and $w(t)$ its temporal component.

In order to bring the above system to a form that is conducive to controller synthesis, actuator placement, actuator switching, and enables the derivation of an accurate finite dimensional approximation, we place it in an abstract formulation written as an evolution equation in the appropriate Hilbert space. Following the formulation in [11], [13], we consider the state space as the space of square integrable periodic functions with zero mean, defined via

$$H = \dot{L}^2(\Omega) = \left(\phi \in L^2(\Omega), \int_{-\pi}^{\pi} \phi(\xi)\, d\xi = 0\right), \tag{3}$$

with inner product $\langle \cdot,\cdot \rangle$ and corresponding induced norm $|\cdot|$. Associated with the above, are the two interpolating spaces [3] V and V' given by

$$V = \dot{H}^2_p(\Omega), \qquad V' = H^{-2}(\Omega). \tag{4}$$

One then has the Gelf'and triple space

$$V \hookrightarrow H \hookrightarrow V', \tag{5}$$

with both embeddings dense and continuous, [1], [12]. Here V is a reflexive Banach space with norm denoted by $\|\cdot\|_V$, and V' denotes the conjugate dual of V (i.e. the space of continuous conjugate linear functionals on V) and let $\|\cdot\|_*$ denote the usual norm on V'. In particular, it is assumed that $|\varphi| \le c_V \|\varphi\|_V$ for some positive constant c_V. The notation $\langle \cdot,\cdot \rangle$ will also be used to denote the duality pairing between V' and V induced by the continuous and dense embeddings given in (5) above.

Define the operator $A : V \to V'$ by

$$\langle A\phi, \psi \rangle = \int_{-\pi}^{\pi} \phi''(\xi)\, \psi''(\xi)\, d\xi, \qquad \phi,\psi \in V, \tag{6}$$

with domain $D(A) = \dot{H}^4_p(\Omega)$. One can show [11] that A^{-1} is compact and symmetric operator, and thus it has a set of orthonormal eigenfunctions that form a basis in $\dot{L}^2(\Omega)$, [7]. Continuing, we define the linear operator $L : V \to V'$

$$\langle L\phi, \psi \rangle = -\int_{-\pi}^{\pi} \phi'(\xi)\, \psi'(\xi)\, d\xi. \tag{7}$$

We also define the bilinear form $\Gamma : V \times V \to V'$ via

$$\langle \Gamma(\phi,\psi), \chi \rangle = \gamma(\phi,\psi,\chi), \qquad \phi,\psi,\chi \in V, \tag{8}$$

where the trilinear form $\gamma(\phi,\psi,\chi) : V \times V \times V \to \mathbb{R}$ is

$$\gamma(\phi,\psi,\chi) = \int_{-\pi}^{\pi} \phi(\xi)\, \psi'(\xi)\, \chi(\xi)\, d\xi. \tag{9}$$

Finally, we define the *location-parameterized* input operators $B_i(\xi) : \mathbb{R} \to V'$, $i = 1,\ldots,m$ via

$$\langle B_i(\xi)u_i, \phi \rangle = \int_{-\pi}^{\pi} b_i(\xi)\phi(\xi)\, d\xi\, u_i(t), \qquad \phi \in V, \tag{10}$$

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and the disturbance operator

$$\langle Dw(t), \phi \rangle = \int_{-\pi}^{\pi} d(\xi)\phi(\xi)\, d\xi\, w(t), \qquad \phi \in V. \qquad (11)$$

The PDE (1) can then be written as an evolution equation

$$\begin{aligned} \frac{d}{dt}x(t) &= -\nu Ax(t) - Lx(t) + \Gamma(x(t),x(t)) \\ &\quad + \sum_{i=1}^{m} B_i(\xi)u_i(t) + Dw(t), \quad \text{in } V'. \end{aligned} \qquad (12)$$

Setting $\mathcal{A}x = -\nu Ax - Lx$, $\mathcal{B}(\Xi) = \begin{bmatrix} B_1(\xi) & \dots & B_m(\xi) \end{bmatrix}$ and $u(t) = \begin{bmatrix} u_1(t) & \dots & u_m(t) \end{bmatrix}^T$ we can re-write (12) as

$$\frac{d}{dt}x(t) = \mathcal{A}x(t) + \Gamma(x(t),x(t)) + \mathcal{B}(\Xi)u(t) + Dw(t), \quad (13)$$

in V'. A finite-dimensional approximation of the evolution system (13) realized through the slow eigenmodes of the differential operator $\mathcal{A}$ can be derived, as it naturally emerges from the time-scale decomposition of the operator's eigenspectrum, [8], [9], [13]. In the present study, we adhere to the approach presented in earlier work by Christofides and co-workers in [2], [10]. In summary, we consider the decomposition $H = H_s \oplus H_f$, in which H_s denotes the finite dimensional space spanned by the unstable/slow eigenspectrum $H_s = \text{span}\{\varphi_1, \dots, \varphi_n\}$ with the orthogonal complement $H_f = \text{span}\{\varphi_{n+1}, \varphi_{n+2}, \dots\}$. Associated with the above, we define the orthogonal projection operators $\mathcal{P}_s$ and $\mathcal{P}_f$ that yield the following decomposition for the state

$$x = \mathcal{P}_s x + \mathcal{P}_f x = x_s + x_f.$$

Application of the projection operators to the system (12) yields the following equivalent form

$$\begin{aligned} \frac{dx_s}{dt} &= \mathcal{A}_s x_s + \mathcal{P}_s\Gamma(x,x) + \sum_{i=1}^{m} \mathcal{P}_s B_i(\xi)u_i(t) + \mathcal{P}_s Dw(t), \\ \frac{dx_f}{dt} &= \mathcal{A}_f x_f + \mathcal{P}_f\Gamma(x,x) + \sum_{i=1}^{m} \mathcal{P}_f B_i(\xi)u_i(t) + \mathcal{P}_f Dw(t), \\ x_s(0) &= \mathcal{P}_s x(0), \qquad x_f(0) = \mathcal{P}_f x(0). \end{aligned} \qquad (14)$$

Following [10], $\mathcal{A}_s$ is an n-dimensional matrix with diagonal structure $\mathcal{A}_s = \text{diag}\{\lambda_i\}$. Furthermore, we have

$$\begin{aligned} \Gamma_s(x,x) &= \mathcal{P}_s\Gamma(x,x) = \Gamma_s(x_s + x_f, x_s + x_f) \\ \Gamma_f(x,x) &= \mathcal{P}_f\Gamma(x,x) = \Gamma_f(x_s + x_f, x_s + x_f), \end{aligned} \qquad (15)$$

being Lipschitz vector functions and the unbounded operator $\mathcal{A}_f$ is the infinitesimal generator of an exponentially stable C_0 semigroup. Please note that the eigenvalues and eigenfunctions associated with the slow/unstable subsystem are given by $\lambda_j = -\nu j^4 + j^2$, $\phi_j(\xi) = \frac{1}{\sqrt{\pi}}\sin(j\xi)$, $j = 1, 2, \dots, n$.

By neglecting the fast and stable infinite dimensional subsystem, one considers the state $\widetilde{x}_s$ associated with the resulting finite dimensional system given by

$$\begin{aligned} \frac{d\widetilde{x}_s}{dt} &= \mathcal{A}_s\widetilde{x}_s + \Gamma_s(\widetilde{x}_s, \widetilde{x}_s) + \sum_{i=1}^{m} \mathcal{P}_s B_i(\xi)u_i(t) + D_s w(t), \\ \widetilde{x}_s(0) &= \mathcal{P}_s x(0). \end{aligned} \qquad (16)$$

One may now define meaningful control and optimal actuator placement objectives: *The development of an integrated control and optimal actuator placement policy scheme that provides actuator location-parameterized controllers for the approximate system, while preserving stability and providing enhanced performance.*

We summarize below different approaches for such an integrated actuator placement and controller synthesis framework.

II. Integrated nonlinear controller synthesis and actuator placement

The goal here is to select, for a fixed actuator location, a nonlinear state feedback control signal that provides stability and satisfies certain performance requirements. Then, one parameterizes the controller with respect to the actuator locations, and using optimization criteria, minimizes a certain performance index with respect to the candidate actuator locations in order to arrive at the specific actuator locations that give the "best" performance. With that in mind, we embark on the development of a fixed-actuator location-parameterized nonlinear controller synthesis framework.

A. Robust nonlinear controller design

We now propose a nonlinear control policy which is somewhat different from the one considered in [10]. For ease of notation we set

$$B(\xi_i) \triangleq B_i(\xi), \qquad i = 1, 2, \dots, m,$$

to denote the input operator at location $\xi_i \in [-\pi, \pi]$, and thus (16) is re-written as

$$\begin{aligned} \frac{d\widetilde{x}_s}{dt} &= \mathcal{A}_s\widetilde{x}_s + \Gamma_s(\widetilde{x}_s, \widetilde{x}_s) + \sum_{i=1}^{m} \mathcal{P}_s B(\xi_i)u_i(t) + D_s w(t), \\ \widetilde{x}_s(0) &= \mathcal{P}_s x(0). \end{aligned}$$

Implicitly it is assumed that all actuating devices have identical output specifications differing only at the location ξ that they are placed at.

Consider now the finite dimensional system with fixed actuator locations $\Xi_0 = [\xi_1^0 \dots \xi_m^0] \in \mathbb{R}^m$. The proposed control law is of the following form:

$$u = -\mathcal{B}_s^{-1}(\Xi_0)\Gamma_s(\widetilde{x}_s, \widetilde{x}_s) - \mathcal{K}\widetilde{x}_s, \qquad (17)$$

where $\mathcal{B}_s(\Xi_0) = \mathcal{P}_s\mathcal{B}(\Xi_0) = \begin{bmatrix} \mathcal{P}_s B(\xi_1^0) & \dots & \mathcal{P}_s B(\xi_m^0) \end{bmatrix}$, $u = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix}^T$ and $\mathcal{K}$ a suitably chosen $m \times n$ feedback gain matrix.

With the above control law, the following stability results can be derived.

Lemma 2.1: Assume that for the PDE system (1) the number of unstable/slow modes is equal to the number of actuating devices (i.e. $n = m$) and the matrix $\mathcal{B}_s$ is invertible. The control law (16) ensures that the solution $U(\xi,t)$ of the resulting closed-loop is exponentially stable for any initial condition $U_o \in H$.

Proof: If the choice of the feedback gain $\mathcal{K}$ is such that $\mathcal{A}_s - \mathcal{B}_s(\Xi_0)\mathcal{K}$ is a Hurwitz matrix, then the stability of the closed-loop system readily follows from [2]. ■

Remark 2.1: The proposed controller (17) results in partial feedback linearization. The rationale for such a choice is so that the actuator locations appear explicitly (via $\mathcal{B}_s(\Xi_0)$) in the closed-loop system equations.

In the absence of disturbances, the above control law results in the following closed loop system

$$\frac{d\widetilde{x}_s}{dt} = \Big(\mathcal{A}_s - \mathcal{B}_s(\Xi_0)\mathcal{K}\Big)\widetilde{x}_s. \tag{18}$$

Once the m actuator locations Ξ_0 are decided, one computes the gain $\mathcal{K}$ in order to meet a prespecified set of closed-loop performance and robustness specifications at the controller design stage. Within the proposed context however, one assumes that process disturbances enter explicitly the unstable subsystem's equations and the closed-loop system becomes

$$\frac{d\widetilde{x}_s}{dt} = \Big(\mathcal{A}_s - \mathcal{B}_s(\Xi_0)\mathcal{K}\Big)\widetilde{x}_s + D_s w(t) \tag{19}$$

where D_s denotes the spatial component of the disturbance ($= \mathcal{P}_s D$) and $w(t)$ the temporal square integrable component. To further enhance the robustness of the controller gain $\mathcal{K}$ in (17), D_s is assumed to be the "worst" spatial disturbance and therefore it is expressed as the sum of the first n modes, which equivalently translates to a class of disturbances whose spatial component "excites" the first n unstable/slow modes.

Remark 2.2: As was similarly suggested in [4], a possible "worst" spatial distribution of process disturbances may be realized through the unit spatial step function with $d(\xi) = 1$. This describes disturbances entering uniformly at every single point in the spatial domain; hence there is no spatial bias. Alternatively, one may consider a disturbance that excites all the modes of the system and thus given by the sum $d(\xi) = \sum_{i=1}^{\infty} \phi_i(\xi)$. Furthermore, its projection $\mathcal{P}_s d(\xi) = \mathcal{P}_s \sum_{i=1}^{\infty} \phi_i(\xi)$ becomes the truncated sum $\sum_{i=1}^{n} \phi_i(\xi)$. Finally, its vector representation is explicitly given by

$$D_s = \begin{bmatrix} \int_{-\pi}^{\pi} \Big(\sum_{i=1}^{n} \phi_i(\xi)\Big)\phi_1(\xi)\,d\xi \\ \int_{-\pi}^{\pi} \Big(\sum_{i=1}^{n} \phi_i(\xi)\Big)\phi_2(\xi)\,d\xi \\ \vdots \\ \int_{-\pi}^{\pi} \Big(\sum_{i=1}^{n} \phi_i(\xi)\Big)\phi_n(\xi)\,d\xi \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{20}$$

For fixed actuator locations $\Xi_0 \in \mathbb{R}^m$, one considers an associated $\mathcal{H}^\infty$ Riccati Equation, and in that manner the control design reduces to that of minimizing the (RMS) L^2 gain γ of the closed loop transfer function from $w(t)$ to $\widetilde{x}_s(t)$. In particular, this requires the solution of

$$\begin{aligned} &\mathcal{A}_s^T P_\infty(\Xi_0) + P_\infty(\Xi_0)\mathcal{A}_s + Q \\ &- P_\infty(\Xi_0)\Big(\mathcal{B}_s(\Xi_0)\mathcal{B}_s^T(\Xi_0) - \frac{1}{\gamma^2} D_s D_s^T\Big)P_\infty(\Xi_0) = 0, \end{aligned} \tag{21}$$

for the smallest possible $\gamma > 0$ where $Q = Q^T > 0$. The resulting gain is then given by

$$\mathcal{K} = \mathcal{B}_s^T(\Xi_0)P_\infty(\Xi_0). \tag{22}$$

Recapitulating the main aspects of this subsection, one first decides on the "best" actuator locations Ξ_0 using certain criteria (primarily controllability-related as seen in the sequel). Once this is achieved, one then computes the robust gain $\mathcal{K}$ from (22) using (21) wherein robustness is meant with respect to the worst possible class of disturbances entering in the slow/unstable subsystem dynamic description considered earlier. As a consequence *both* a spatial and a temporal robustness of the control signal is ensured. Finally, the control law is given by (17). However, the feedback gain is only optimal with respect to an a priori chosen $\mathcal{B}_s(\Xi_0)$. Integrating the actuator placement with the optimal feedback design is pursued below.

B. Optimal actuator placement

By parameterizing the above controller (17) by the admissible actuator locations (*cf.* Remark 2.1), one may proceed to the location optimization. First, define $\Omega_c \subset \Omega$ as the set of *admissible candidate actuator locations* given by

$$\Omega_c = \left\{ \begin{array}{c} \xi \in \Omega : \int_{-\pi}^{\pi} b_i(\xi)\phi_j(\xi)\,d\xi \neq 0, \text{ for at} \\ \text{least all } j \geq n \text{ and all } i = 1,\ldots,m \end{array} \right\}, \tag{23}$$

where $\phi_j(\xi)$ are the eigenmodes associated with the slow/unstable dynamics. Associated with the above, we define

$$\Theta_c = \left\{ \begin{array}{l} \Xi = [\ \xi_1 \quad \ldots \quad \xi_m\ ] : \xi_i \in \Omega_c \text{ and} \\ \mathcal{B}_s(\Xi) = [\ \mathcal{P}_s B(\xi_1) \quad \ldots \quad \mathcal{P}_s B(\xi_m)\ ] \text{ invertible} \end{array} \right\}$$

which denotes the set of admissible locations for a given group of actuators, whose actuating devices are placed at locations Ξ that yield approximate controllability and which ensure that $\mathcal{B}_s(\Xi)$ is invertible.

Remark 2.3: The search for optimal actuator locations $\Xi = [\xi_1 \ldots \xi_m]$ will be restricted to the candidate locations in the set Θ_c which guarantees approximate controllability, i.e. all m-dimensional vectors Ξ whose elements are in Ω_c. This condition is imposed for index values (at least) up to $j = n$, which would ensure that (at least) the first n modes are controllable by each of the m actuating devices.

Two different methods are summarized here that provide an integrated controller synthesis and actuator placement framework. The first one is an "open-loop" procedure in which the actuator placement problem is decoupled from the controller design one. In this case, one *first* selects the actuator locations so that the system is "more" controllable in the above sense, and *then* designs the feedback gain based on some performance and/or robustness criteria (e.g. $\mathcal{H}^2/\mathcal{H}^\infty$ as in (21)). The second method introduces a coupling between the actuator placement and the controller synthesis problem by considering sets of matrix inequalities that provide both a feedback gain and optimal actuator locations while enhancing the stability of the closed loop system.

1) Method 1: (open-loop slow modes): Choose the "optimal" actuator locations Ξ from the set Θ_c that maximize the bound $\alpha(\Xi)$ in

$$z^T W_c(\Xi) z \geq \alpha(\Xi) z^T z, \qquad z \in \mathbb{R}^n, \tag{24}$$

where $W_c(\Xi)$ denotes the Ξ-parameterized controllability Gramian of the finite dimensional subsystem defined via

$$\mathcal{A}_s W_c(\Xi) + W_c(\Xi)\mathcal{A}_s^T = -\mathcal{B}_s(\Xi)\mathcal{B}_s^T(\Xi), \quad \Xi \in \Theta_c. \tag{25}$$

In other words, the "best" actuator locations are given via

$$\begin{aligned} \Xi^{opt} &= \arg \sup_{\Xi\in\Theta_c} \alpha(\Xi) = \arg \sup_{\Xi\in\Omega_c} \lambda_{min}\Big(W_c(\Xi)\Big) \\ \text{or} \quad \Xi^{opt} &= \arg \sup_{\Xi\in\Theta_c} \text{trace}\Big[W_c(\Xi)\Big]. \end{aligned} \tag{26}$$

The above would make the unstable system "more" controllable where the measure of controllability is given by the positiveness of the Gramian. Once these actuator locations are found, one may then proceed to the design of the feedback gain which exhibits robustness with respect to the worst spatial distribution of the disturbances, as presented in the previous subsection via (21).

One may consider "closed-loop" techniques for the actuator placement, in which performance and/or stability considerations are utilized instead of only enhanced controllability.

2) Method 2a: (locations yielding robust stability): First, using solely stability criteria, the optimal location $\Xi \in \mathbb{R}^m$ is found as the one that provides quadratic stability.

Remark 2.4: Using solely stability criteria, one searches in the set Θ_c to find the actuator group locations Ξ_j that yield quadratic stability. This translates to finding both the locations Ξ_j (if more than one) within Θ_c and the feedback gain $\mathcal{K}$ that render the following inequalities

$$\Big(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}\Big)^T P + P\Big(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}\Big) < 0. \tag{27}$$

feasible. Please notice that this may produce a very conservative controller gain $\mathcal{K}$.

Remark 2.5: In the above optimization problem, one must compute the feedback gain $\mathcal{K}$, and can convexify the above LMI by setting $Y = \mathcal{K}Q$. Hence the system is quadratically stabilizable if and only if there exist $Q > 0$ and Y such that the LMI

$$\mathcal{A}_s Q + Q\mathcal{A}_s + \mathcal{B}_s(\Xi_j)Y + Y^T\mathcal{B}_s^T(\Xi_j) < 0, \tag{28}$$

is feasible. If this is achieved, then the quadratic function $V = \widetilde{x}^T Q^{-1}\widetilde{x}$ establishes quadratic stability under the linear state feedback control law: $u = YQ^{-1}\widetilde{x}_s$. If more than one set Ξ_j of actuator locations renders the above LMIs feasible, then one may adopt the approach in [5] and extend it to the case of placing a group of actuators. In this case the feedback gain $\mathcal{K}$ and the actuator group locations Ξ_j are found so that in addition to quadratic stabilizability and enhanced controllability, they also provide a certain robustness with respect to disturbances. In this case, one chooses the feedback gain $\mathcal{K}$ so that the (RMS) L^2 gain γ in the following polytopic LMIs is minimized for $\Xi_j \in \Theta_c$

$$\begin{bmatrix} \begin{pmatrix} (\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K})Q + D_s D_s^T \\ +Q(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K})^T \end{pmatrix} & QC_z^T \\ C_z Q & -\gamma^2 I \end{bmatrix} \leq 0, \tag{29}$$

where D_s, C_z denote the matrix distributions of the process disturbance and controlled output respectively in the closed-loop representation

$$\begin{aligned} \dot{\widetilde{x}}_s(t) &= \mathcal{A}_s x(t) + \mathcal{B}_s(\Xi_j)u + D_s w(t), \\ z(t) &= C_z \widetilde{x}(t). \end{aligned} \tag{30}$$

In essence, one minimizes the effects of the disturbance on the controlled output, i.e. minimize the $\mathcal{H}^\infty$ norm of the closed loop transfer function

$$T_{zw}(s;\Xi_j) \triangleq C_z\Big(Is - (\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K})\Big)^{-1} D_s.$$

Following [4] and Remark 2.2, to further enhance this robustness property of the feedback gain, one may choose $C_z \equiv I_n$ which ensures that the effects of the disturbance on every single component of $\widetilde{x}_s$ are minimal. Furthermore, the "worst" case of spatial distribution of the disturbance may be considered, which translates to setting $d(\xi) = \mathcal{P}_s 1_\Omega(\xi)$ and which means that the disturbance affects each of the states (or modes) of the slow subsystem. In summary, the LMIs (29) with D_s given by (20) and $C_z = I_n$ will ensure that the effects of the disturbance on every single state are minimized and that the "worst" possible disturbance distribution function is considered. In relation to the original PDE (1), it translates to having the disturbance enter at every single point in the spatial domain Ω with the property that it at least affects the slow eigenmodes, (i.e. $d(\xi) = \mathcal{P}_s \sum_{j=1}^\infty \phi_j(\xi)$).

By coupling together the actuator location with the robustness of the feedback controller, one then simply considers

$$\Xi^{opt} = \arg \inf_{\Xi\in\Theta_c} \gamma(\Xi) \tag{31}$$

where now (21) (or equivalently (29)) becomes

$$\begin{aligned} &\mathcal{A}_s^T P_\infty(\Xi) + P_\infty(\Xi)\mathcal{A}_s + I \\ &-P_\infty(\Xi)\Big(\mathcal{B}_s(\Xi)\mathcal{B}_s^T(\Xi) - \frac{1}{\gamma^2(\Xi)} D_s D_s^T\Big) P_\infty(\Xi) = 0. \end{aligned} \tag{32}$$

Equivalently, one optimizes the $\mathcal{H}^\infty$ norm of the transfer function $T_{zw}(s;\Xi)$ with respect to the candidate locations in the set Θ_c. Finally, the associated *truly* robust controller gain (worst spatial disturbance, smallest RMS gain) is

$$\mathcal{K} = \mathcal{B}_s(\Xi^{opt})P_\infty(\Xi^{opt}). \tag{33}$$

3) Method 2b: (locations yielding optimal performance): While the above method provided both the optimal actuator location and the feedback gain by considering robustness bounds on the $\mathcal{H}^\infty$ norm of the closed loop system, the method below considers an approach in which the feedback gain and the optimal locations are found by minimizing the $\mathcal{H}^2$ norm of the closed loop transfer function $T_{zw}(s;\Xi)$. Summarizing the results in [6], we consider the infinite horizon LQR/$\mathcal{H}^2$ cost functional

$$J = \int_{t_0}^{\infty} \widetilde{x}_s^T(\tau)Q\widetilde{x}_s(\tau) + u^T(\tau)Ru(\tau)\,d\tau. \tag{34}$$

Its optimal value for a given actuator location Ξ_0 is given by

$$J^{opt} = \widetilde{x}_s^T(t_0)P_2\widetilde{x}_s(t_0),$$

where P_2 is the solution to the associated LQR/$\mathcal{H}^2$ Algebraic Riccati equation. When one parameterizes the Riccati solution by the actuator locations Ξ and minimizes this location-parameterized optimal cost, one arrives at the optimal location. The optimal locations are therefore given by

$$\Xi^{opt} = \arg\min_{\Xi\in\Theta_c} \widetilde{x}_s^T(t_0)P_2(\Xi)\widetilde{x}_s(t_0), \tag{35}$$

where $P_2(\Xi)$ is the solution to the Ξ-parameterized Riccati equation

$$\mathcal{A}_s^T P_2(\Xi) + P_2(\Xi)\mathcal{A}_s - P_2(\Xi)\mathcal{B}_s(\Xi)R^{-1}\mathcal{B}_s^T(\Xi)P_2(\Xi) + Q = 0, \tag{36}$$

and the corresponding optimal gain is thus given by $\mathcal{K}(\Xi^{opt}) = -R^{-1}\mathcal{B}_s^T(\Xi^{opt})P_2(\Xi^{opt})$.

The dependence of (35) on initial conditions $\widetilde{x}_s(t_0)$ can be circumvented by assuming that $\widetilde{x}_s(t_0)$ is a random vector uniformly distributed on the unit sphere, and hence the measure is simply given by the trace of the matrix $P_2(\Xi)$,

$$\Xi^{opt} = \arg\min_{\Xi\in\Theta_c} \text{trace}\,\Big[P_2(\Xi)\Big].$$

However, using the trace of a location-parameterized Riccati solution tends to yield actuator locations that average all modes. In addition, the above optimization does not make any use of the spatial distribution of disturbances. This is considered below.

Remark 2.6: Similar to the extended LQR problem considered in (35) above, one may employ an $\mathcal{H}^2$ formulation, in which the $\mathcal{H}^2$ norm of the closed loop transfer function

$$T_{zw}(s;\Xi) = I\left(sI - \Big(\mathcal{A}_s - \mathcal{B}_s(\Xi)\mathcal{K}(\Xi)\Big)\right)^{-1} D_s$$

is minimized, and the criterion (35) becomes

$$\Xi^{opt} = \arg\min_{\Xi\in\Theta_c} \text{trace}\,\Big[D_s^T P_2(\Xi) D_s\Big].$$

In this case, the scheme takes into consideration the spatial distribution of disturbances, as in the first method via (31)-(33).

III. An integrated framework for control and actuator switching policies

The procedure described by (27) or (29) provides a method for *simultaneously* finding the *optimal* actuator locations and the *global* feedback gain $\mathcal{K}$ common to possibly more than one candidate actuator groups. More precisely, it finds those N group locations Ξ within a subset $\Theta_c^{lmi} \subset \Theta_c$ that render (29) feasible, where Θ_c^{lmi} is defined via

$$\Theta_c^{lmi} = \Big\{\Xi\in\Theta_c : \text{LMIs in (29) are feasible for } \textit{some}\ \gamma\Big\}.$$

Furthermore the method finds those locations within Θ_c^{lmi} that render the LMIs (29) feasible *and* simultaneously provide robustness with respect to the worst spatial disturbance distribution. The resulting nonlinear control law is then given by:

$$u = -\mathcal{B}_s^{-1}(\Xi^{opt})\Gamma_s(\widetilde{x}_s,\widetilde{x}_s) - \mathcal{K}\widetilde{x}_s. \tag{37}$$

The reason for considering many candidate group locations Ξ within Θ_c^{lmi} that all have a *common* feedback gain will become apparent later, where one can further simplify the proposed switching actuator policy. In summary, one keeps the *same* linear part of the above control signal (i.e. $-\mathcal{K}\widetilde{x}_s$ in (37)) that is common to *all* (say N) candidate actuator groups (regardless of the actuators used) within Θ_c^{lmi}. Every Δt time units, one switches to a different group of actuating devices (each of which is optimal in its own right if that were to be used) using a certain switching policy, in order to further enhance the performance of the closed loop system. The issues at hand are:

(i) the switching policy: *How does one decide at the beginning of a given interval $[t_k, t_k+\Delta t)$ which actuator group to activate, and which $(N-1)$ ones to keep dormant for the duration of that interval?*

(ii) the control policy: *Once a given set of actuators is chosen to be active over the interval $[t_k, t_k+\Delta t)$, what is the controller signal that must be supplied to this group of actuators?*

A. Actuator switching algorithm with a common feedback gain and a common Lyapunov function

1) Find N actuator groups that satisfy either (27) or (29)
2) For each of these locations Ξ_i ($i = 1,2,\ldots,N$), find the N solutions to the Lyapunov equations for $\Xi \in \Theta_c^{lmi}$
$$(\mathcal{A}_s - \mathcal{B}_s(\Xi)\mathcal{K})^T\Sigma_1(\Xi) + \Sigma_1(\Xi)(\mathcal{A}_s - \mathcal{B}_s(\Xi)\mathcal{K}) = -Q_{cl},$$
3) At the beginning of each interval $[t_k, t_k+\Delta t)$, form the N inner products $\widetilde{x}_s^T(t_k)\Sigma_1(\Xi)\widetilde{x}_s(t_k)$ and choose the actuator group to be activated over $[t_k, t_k+\Delta t)$ via
$$\arg\min_{\Xi\in\Theta_c^{lmi}} \widetilde{x}_s^T(t_k)\Sigma_1(\Xi)\widetilde{x}_s(t_k)$$
4) Repeat step 3 for the next interval $[t_{k+1}, t_{k+1}+\Delta t)$.

B. Actuator switching algorithm for a common Lyapunov function and different feedback gains

1) Find the N actuator groups and the N gains $\mathcal{K}(\Xi)$ associated with them that satisfy: $(j = 1, 2, \ldots, N)$

$$(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}(\Xi_j))^T P + P(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}(\Xi_j)) < 0,$$

2) For each of these locations Ξ_j, find the N solution to the Lyapunov equation

$$(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}(\Xi_j))^T \Sigma_2(\Xi_j) \\ + \Sigma_2(\Xi_j)(\mathcal{A}_s - \mathcal{B}_s(\Xi_j)\mathcal{K}(\Xi_j)) = -Q_{cl},\ j = 1, 2, \ldots, N$$

3) At the beginning of each interval $[t_k, t_k + \Delta t)$, form the N inner products $\widetilde{x}_s^T(t_k)\Sigma_2(\Xi)\widetilde{x}_s(t_k)$ and choose the actuator group to be activated over $[t_k, t_k + \Delta t)$ via

$$\arg \min_{\Xi \in \Theta_c^{lmi}} \widetilde{x}_s^T(t_k)\Sigma_2(\Xi)\widetilde{x}_s(t_k)$$

4) Repeat step 3 for the next interval $[t_{k+1}, t_{k+1} + \Delta t)$.

C. Actuator switching algorithm for different Lyapunov functions and different feedback gains

1) Find N actuator groups using any method from § 2.
2) For each of these actuator groups $(i = 1, 2, \ldots, N)$, find the N feedback gains $\mathcal{K}(\Xi)$ via the solution of

$$\mathcal{A}_s^T \Sigma_3(\Xi_i) + \Sigma_3(\Xi_i)\mathcal{A}_s + Q - \\ \Sigma_3(\Xi_i)\left(\mathcal{B}_s(\Xi_i)R^{-1}\mathcal{B}_s^T(\Xi_i) - \tfrac{1}{\gamma^2}D_s D_s^T\right)\Sigma_3(\Xi_i) = 0,$$

3) At the beginning of each interval $[t_k, t_k + \Delta t)$, form the N inner products $\widetilde{x}_s^T(t_k)\Sigma_3(\Xi)\widetilde{x}_s(t_k)$ and choose the actuator group to be activated over $[t_k, t_k + \Delta t)$ via

$$\arg \min_{\Xi \in \Theta_c} \widetilde{x}_s^T(t_k)\Sigma_3(\Xi)\widetilde{x}_s(t_k)$$

4) Repeat step 3 for the next interval $[t_{k+1}, t_{k+1} + \Delta t)$.

IV. Numerical Results

We considered (1) with the instability parameter $\nu = 0.2$ which results in two unstable modes and thus we considered $n = 2$ as the dimension of the slow/unstable subsystem. The initial condition of the state was $U_o(\xi) = \sin(\xi) + \xi^3 - \pi^2\xi$. A total of $N = 5$ actuator groups were assumed to be available for switching in the time interval $[t_0, t_f] = [0, 0.5]$. The disturbance term $d(\xi)w(t)$ was chosen as a moving spatiotemporal disturbance whose spatial distribution $d(\xi)$ changes within the spatial domain Ω. The evolution of the system norm for the open-loop case, the closed-loop with a fixed actuator and the closed-loop with a moving actuator are depicted in Figure 1. The performance improvement for the moving actuator case is overwhelming, and suggests that the proposed moving actuator policy can better compensate for spatiotemporally varying disturbances.

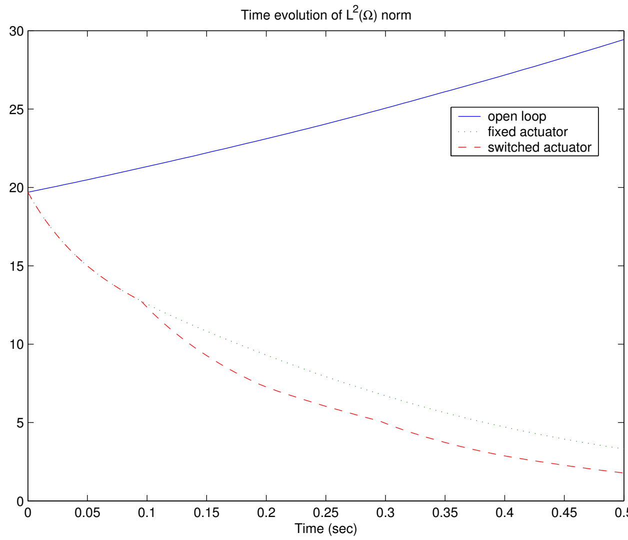

Fig. 1. Evolution of L_2 system norm.

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