Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

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Packet-Based Control

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***Abstract*— This paper studies a novel control method for networked control systems. This method is motivated by a more efficient use of the packet structure. Use of multipoint packets to reduce network traffic and computation time is considered. A solution is obtained by first transforming the sampled-data problem into a multirate sampling/control setting. Upon which, the associated, H_2-optimal, sampled-data controller is derived. The paper concludes with a performance comparison of this method to some more traditional ones.**

I. Introduction

The use of networks is becoming ubiquitous in control systems. A key advantage of controlling a system over a network is the absence of point to point wiring infrastructure. Thus, the implementation of complex systems is greatly simplified. A node connected to the network automatically shares information with all other nodes. Systems become more configurable and can easily be expanded and monitored. Furthermore, distributed control systems can be realized without imposing any extra demands on the system realization. Finally, new configurations become possible through wireless technology.

There are several challenges, however, that arise when a control system is networked. Most of these problems can be attributed to either the sharing of the communication medium or the extra complexity associated with data transmission. In traditional digital control systems the quality of performance asymptotically approaches the continuous time performance level as the sampling period goes to zero. This is not the case with networked control systems. In networked control systems discrete signals are encoded into a packet, sent across the network, and then decoded at the destination node. As a result, a tradeoff arises between a performance gain associated with an increase in sampling frequency and a performance degradation caused by encoding/decoding and network traffic induced delays.

Traditionally, the synthesis problem for a networked control system is dealt with by first transforming the problem into classical discrete or sampled data framework and then designing the controller to deal with the network issues. This paper considers the development of controllers that fully utilize the packet structure of network communication. These types of controllers will be referred to as packet-based controllers. Since a Networked Control System's performance degradation can mainly be attributed to communication delays, one measure of the efficiency of individual packet use can be taken as the resulting delay if a fixed amount of data is to be transmitted.

As was previously stated, delays are introduced into the system as a result of network traffic and encoding/decoding computations. In [1] the size and distribution of these delays was studied. It was found that in network protocols such as DeviceNet, ControlNet, and Ethernet, both computation and traffic related delays are significant. This implies that in order to fully utilize the packet communication structure one can either decrease the transmission frequency or the packet size or both. Different methods of reducing the transmission frequencies, such as the use of state estimators or deadbands, were studied in [2], [3].

To address this problem we consider the following communication protocol; store a finite number of output samples at the digital encoder and then transmit them in one packet, this will be referred to as the *output packet*. Moreover, instead of calculating one constant control input to be applied over the next control interval, divide the control interval into several subintervals and calculate a vector of control inputs whose elements will be consecutively applied over the next control interval, and send this vector in a single packet, this will be referred to as the *control packet*. Fig. 1 illustrates the time instances at which the output is sampled and the corresponding control inputs for the case where five output measurements are included in the *output packet* and the control interval is divided into five subintervals.

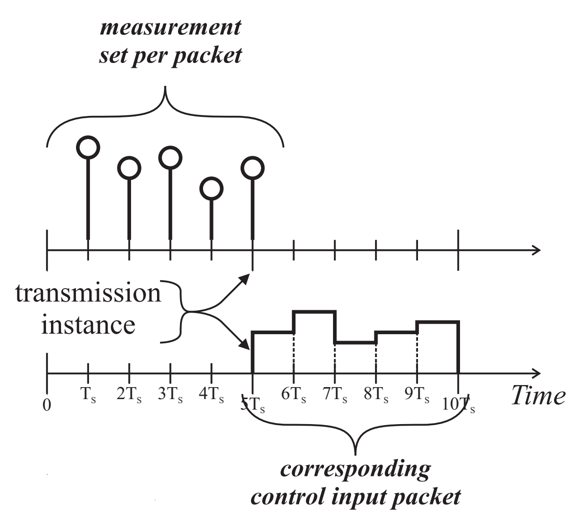

Fig. 1. Typical Ethernet packet

The downside of this protocol is pretty clear. Mainly, if the period between consecutive packet transmissions is

This work was supported in part by the University of Michigan ERC/RMS under grant NSF 95-29125 and NSF 98-76039.

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equal to ten sample periods, the first sample to be stored will be delayed 9 sample periods by the time it reaches the destination node. Less obvious are the advantages. First, transmitting a finite number of samples together in one packet decreases the network traffic by a factor approximately equivalent to the number of samples per packet. Second, depending on the network protocol, each packet carries a significant amount of overhead. Figure 2 contains the structure of a typical Ethernet packet. There

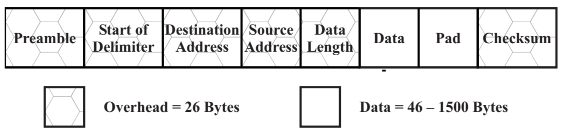

Fig. 2. Typical Ethernet packet

are several components only one of which can be associated with the data. Furthermore, each packet contains a minimum data size requirement of 46 bytes, while an average data sample may consume only 2 bytes. Therefore, a traditional packet, which would contain only one data point, carries a total of 70 bytes of overhead. Finally, the division of the control interval is closely related to multirate control, which has been shown to improve the system performance in many cases. In this paper, we evaluate the tradeoff between the pros and cons mentioned above. More specifically, the performance of a system using the above communication strategy is compared to two other, more traditional, systems; one operating solely at the sampling rate and the other operating solely at the packet transmission rate.

II. Problem Statement

To permit a fair comparison of control mechanisms operating at different sampling rates, we will approach the problem in a sampled-data setting. The H_2 optimal, sampled data problem has received many different treatments of varying complexity, for example see [4], [5], [6]. In [5] the problem is structured such that sampling of impulsive signals is avoided, yet H_2 problem remains well-posed. For this reason, our sampled-data system configuration uses many attributes from [5].

Consider the system whose interconnection is illustrated in Fig. 3. The solid and dotted lines represent continuous and discrete signals respectively. The signals w, z, and v are the continuous time inputs, outputs and discrete-time measurement noise respectively. S represents an ideal sampler and H represents a zero-order hold A/D converter – both share the same sampling period T_s. Notice that the measurement disturbance is placed after the sampler. This allows for measurement disturbances to be included without risking sampling of impulsive signals, which, as shown in [4], causes the sampling operation to be unbounded. The periodic delay operator, $\Delta : \mathbb{N} \times \mathbb{R}^{p_2} \to \mathbb{R}^{p_2}$ defined as $\Delta(i, y(k)) = y(k - N_p + i)$, represents the communication constraint imposed on the output measurements. In Fig 3, the arguments of Δ are the packet index and the measured output. If the multipoint protocol described above calls for N_p output measurements to be collected before the packet is transmitted, then the packet index represents the number of samples collected since the last transmission instance. Here, Δ will be absorbed into the controller and treated as an extra causality constraint. This causality constraint will be referred to as *packet-causality constraint*.

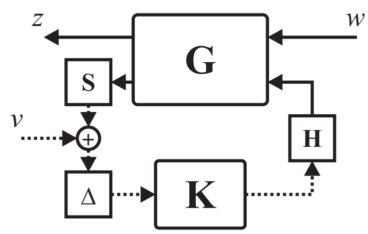

Fig. 3. Generalized plant with packet-causality delay

Let the plant in Fig. 3 represent a linear, time-invariant, finite-dimensional, continuous-time, generalized plant with the following state space representation.

$$G \sim \begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) & = & C_1 x(t) + D_{12} u(t) \\ y(t) & = & C_2 x(t) \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system, $w(t) \in \mathbb{R}^{m_1}$ is the continuous-time external input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output, and $y(t) \in \mathbb{R}^{p_2}$ is the measured output. The feedthrough matrices from $w(t)$ to $z(t), y(t)$ and from $u(t)$ to $y(t)$ were excluded to ensure the boundedness of the H_2 norm and well posedness of the feedback loop.

The controller in Fig. 3 is confined to the class of linear, N_p-periodically time-varying (commutes with the N_p-delay operator), finite-dimensional, discrete-time systems and has the following state space representation.

$$K \sim \begin{array}{rcl} \xi(k+1) & = & \Phi_k \xi(k) + \Gamma_k \eta(k) \\ \psi(k) & = & \Theta_k \xi(k) + \Upsilon_k \eta(k) \end{array} \tag{2}$$

where $\eta(k)$ is the sampled output combined with the discrete measurement disturbance $v_k \in \mathbb{R}^{m_3}$ and can be written as

$$\eta(k) = C_2 x(kT_s) + D_{21} v(k) \tag{3}$$

and $\psi(k)$ is the control input before passing through the zero-order hold and can be written as

$$u(t) = \psi(k), \; kT_s < t \leq (k+1)T_s$$

This explicit distinction between the discrete and continuous signals is needed for the derivation of the closed loop model which accepts the mixed continuous/discrete exogenous inputs. Notice, that the closed loop system, resulting from the linear fractional interconnection of the plant and the controller, is periodically time-varying. In fact, the problem has two underlying periods, the sampling period, T_s and the period of the controller, $T_p = N_p T_s$.

It is also assumed that the plant in (1) together with the discrete, measured output disturbance in (3) satisfy the standard conditions for existence and uniqueness of a single-rate, sampled-data, H_2-optimal controller, see [5]. These conditions are defined below. It will be shown later that they are also the necessary and sufficient conditions for the existence and uniqueness of the packet-based, H_2-optimal controller.

Definition 1 (H_2 solvability conditions). Consider the plant in (1) with its measured output affected according to (3). Then a unique, sampled-data, H_2-optimal controller exists if the following conditions are satisfied.

i) (A, B_2) is stabilizable and (C_2, A) is detectable;
ii) $P \triangleq D_{12}^T D_{12}$ and $Q \triangleq D_{21} D_{21}^T$ are nonsingular;
iii) The following matrices have, respectively, a full column and row rank for all λ on the unit circle.

$$\begin{bmatrix} A-\lambda & B_2 \\ C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} A-\lambda & [\;B_1 & 0\;] \\ C_2 & [\;0 & D_{21}\;] \end{bmatrix}$$

III. H_2-OPTIMAL PACKET-BASED CONTROL

In this section, the H_2-optimal packet-based controller will be derived. First, the generalized H_2 measure for the closed loop system is defined. Then, the periodically time-varying problem is transformed into a time-invariant one via the lifting method. Finally, the state space representation for the optimal packet-based controller is presented.

A. Sampled-Data Formulation

Our sampled-data state space representation builds off of the sampled-data, state space representation presented in [5]. The derivation of this representation and some of the formulas involved are not important in the context of this paper therefore, our formulation will only summarize the steps covered in [5] and present only the results that are either unique to our problem or will be used in the synthesis problem. The system is treated as follows. The sampler and the zero-order-hold are absorbed into the generalized plant model. As a result, the plant model itself becomes periodically time varying (where period = T_s). A discrete, time-invariant, operator-valued representation is then obtained via continuous-time lifting. Next, we define the generalized H_2 measure for the closed loop system T_{zw} that takes into account the intersample impulse response over the entire period T_p, as

$$J = \left[\frac{1}{T_p} \int_0^{T_p} \left(\sum_{i=1}^{m_1} ||T_{zw} \delta_\tau e_i||_2^2 \right) d\tau \right]^{1/2} \tag{4}$$
$$= \left[\frac{1}{N_p} \sum_{j=0}^{N_p-1} \frac{1}{T_s} \int_0^{T_s} \left(\sum_{i=1}^{m_1} ||T_{zw} \delta_{(\tau + jT_s)} e_i||_2^2 \right) d\tau \right]^{1/2}$$

where T_{zw} is the closed-loop system and $\delta_t e_i$ is an impulse applied at time t at the ith input. Finally, we can express the generalized measure of (4) in terms of the Hilbert-Schmidt norm and an H_2 norm of an equivalent discrete-time plant as

$$J = \left[||\underline{D}_{11}||_{\mathbb{HS}}^2 + \frac{1}{N_p} \sum_{j=0}^{N_p-1} \sum_{i=1}^{m_1} ||\hat{T}_{zw} \delta_{(\tau + jT_s)} e_i||_2^2 \right]^{1/2} \tag{5}$$

where

$$||\underline{D}_{11}||_{\mathbb{HS}}^2 = trace \left(B_1^T \int_0^{T_s} \int_0^t e^{\tau A^T} C_1^T C_1 e^{\tau A} d\tau dt B_1 \right)$$

and

$$\hat{T}_{zw} = \mathcal{F}(G_{eq}, K)$$

with $\mathcal{F}(G_{eq}, K)$ representing the lower, linear fractional transformation and G_{eq} defined as

$$G_{eq} \sim \left[\begin{array}{c|ccc} A_d & [\;B_{1d} & 0\;] & B_{2d} \\ \hline C_{1d} & [\;0 & 0\;] & D_{12d} \\ C_2 & [\;0 & D_{21}\;] & 0 \end{array} \right]. \tag{6}$$

Values of the matrices associated with the representation for G_{eq} are omitted here for the sake of brevity but are well documented in [4], [5], or [6].

B. Lifting and the packet-causality constraint

Discrete-time *lifting* is a powerful tool in the analysis of discrete, periodically time-varying systems. The idea is to eliminate the time-varying nature of the system by assembling or *lifting* the signal values over one period into a single vector which then becomes the value of the lifted signal. The lifting operator $L_{N_p} : l^p \to l^{pN_p}$, where l^p is the space of p-dimensional sequences, is defined as follows. Let $\{y_k\} \in l^p$ be expressed as

$$\{y_k\} = \{y(1), \cdots, y(N_p), y(N_p+1), \cdots\}$$

then

$$L_{N_p}(\{y_k\}) = \left\{ \begin{bmatrix} y(1) \\ \vdots \\ y(N_p) \end{bmatrix}, \begin{bmatrix} y(N_p+1) \\ \vdots \\ y(2N_p) \end{bmatrix}, \cdots \right\}.$$

Furthermore, a representation of the equivalent plant G_{eq} operating on the lifted spaces can be written as

$$\underline{G}_{eq} = L_{N_p} G_{eq} L_{N_p}^{-1} \sim \left[\begin{array}{c|cc} \underline{A} & \underline{B}_1 & \underline{B}_2 \\ \hline \underline{C}_1 & \underline{D}_{11} & \underline{D}_{12} \\ \underline{C}_2 & \underline{D}_{21} & \underline{D}_{22} \end{array} \right] \tag{7}$$

$$\underline{A} = A_d^{N_p}$$
$$\underline{B}_2 = \begin{bmatrix} A_d^{N_p-1} B_{2d} & \cdots & B_{2d} \end{bmatrix}$$
$$\underline{C}_1 = \begin{bmatrix} C_{1d}^T & \cdots & (C_{1d} A_d^{N_p-1})^T \end{bmatrix}^T$$
$$\underline{D}_{12L} = \begin{bmatrix} D_{12d} & 0 & \cdots & 0 \\ C_{1d} B_{2d} & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{12d} & 0 \\ C_{1d} A_d^{N_p-2} B_{2d} & \cdots & C_{1d} B_{2d} & D_{12d} \end{bmatrix}$$

Where the definitions of the omitted matrices $\underline{B}_1$, $\underline{C}_2$, $\underline{D}_{11}$, $\underline{D}_{21}$, and $\underline{D}_{22}$ follow immediately from the definitions of $\underline{B}_2$, $\underline{C}_1$, and $\underline{D}_{12}$.

The lifting transformation is an isomorphism and therefore, it can be shown (see [4]) that the H_2 norm of a system is preserved under lifting, more precisely $||G_{eq}||_2^2 = \frac{1}{N_p}||L_{N_p}G_{eq}L_{N_p}^{-1}||_2^2$. Stability is also preserved under lifting, in other words $L_{N_p}KL_{N_p}^{-1}$ stabilizes $L_{N_p}G_{eq}L_{N_p}^{-1}$ iff K stabilizes G_{eq}. Moreover, stabilizability, observability and the invariant-zero structure are all also preserved. The structure of the lifted matrices also shows that the injective property of D_{12} and the surjective property of D_{21} imply that $\underline{D}_{12}$ and $\underline{D}_{21}$ are themselves injective and surjective respectively. This leads to the following lemma which is a direct result of the above properties and implications of the conditions which were imposed on the plant in (1)

Lemma 1. *A unique H_2-optimal controller exists for the lifted system if and only if conditions in Definition 1 are satisfied.*

Lastly, lifting also allows for a more compact representation of the packet-causality constraint and the generalized H_2 measure in (4).

In the lifted signal domains, the packet-causality constraint on the controller translates into a constraint on the structure of the lifted controller's feedthrough matrix, $\underline{\Upsilon}$.

Theorem 1. *Consider the signal value $\eta(k) \in \mathbb{R}^{p_2}$ and the associated lifted value $\underline{\eta}(k) \in \mathbb{R}^{p_2 N_p}$ and let P_{p_2} be the projection operator onto $\mathbb{R}^{p_2}$ defined by simply setting the last $(N_p - 1)p_2$ elements of $\underline{\eta}(k)$ to zero. Then a system, $\underline{K}$, satisfies the packet-causality constraint iff the feedthrough matrix $\underline{\Upsilon}$ of its associated lifted state space representation meets the following criteria*

$$\underline{\Upsilon}\underline{\eta}(k) = \underline{\Upsilon}P_{p_2}\underline{\eta}(k), \ \forall \ \underline{\eta}(k) \ \in \ \mathbb{R}^{p_2 N_p} \tag{8}$$

Proof. Result follows immediately from the structure of the lifted signal and definition of matrix multiplication. □

Next, we can express the generalized measure in (4) in terms of the lifted closed loop system, $\underline{T}_{zw} = \mathcal{F}_l(\underline{G}_{eq}, \underline{K})$ as

$$J = \left[||\underline{D}_{11}||_{\mathbb{HS}}^2 + \frac{1}{N_p}||\underline{T}_{zw}||_2^2 \right]^{1/2} \tag{9}$$

This result follows directly from the definition of the multivariable H_2 norm.

C. H_2-optimal controller w/ generalized causality constraint

We are now ready to synthesize the packet-based controller. It follows directly from (9) that a controller minimizes the generalized measure in (4) of the sampled data system if and only if it minimizes the H_2 norm of the equivalent, discrete, and lifted plant, $\underline{G}_{eq}$. Therefore, the synthesis problem can be formally stated as the following minimization problem

$$\begin{aligned} &\underline{K}_{opt} = \underset{\underline{K}}{\operatorname{argmin}}\{\mathcal{F}(\underline{G}_{eq}, \underline{K})\} \\ &\text{such that (8) is satisfied.} \end{aligned} \tag{10}$$

Notice that this problem is also unique in that it is not necessary to inverse-lift the optimal controller obtained for the lifted system, since the time instances of signal updates (measurement/input) are compatible with the lifted representation. In other words, the *control packet* and the *output packet* equal $\underline{\psi}(k)$ and $\underline{\eta}(k)$ respectively.

Before solving the H_2-optimal packet-based control problem which is characterized by the imposed packet-causality constraint, we first solve the more general H_2-optimal control problem with generalized causality constraint. A controller is defined to be causal if the control input is dependent on past measured outputs including the current output. A controller is defined as being strictly causal if it is only dependent on past measured outputs but not on the current output. A controller for which the dependence of the control input on the current measured output is arbitrary will be referred to as being generally causal and will be said to satisfy a generalized causality constraint. To the best of the authors' knowledge, this problem has not been solved. Different versions of this problem have been addressed in literature, for example see [7], [8], [9]. The most common version arises in the design of multirate control systems, which, if solved via lifting, inherently posses a causality constraint. This constraint requires the feedthrough matrix to be lower-block triangular and is therefore not general. Moreover, in most cases the solution is obtained in the frequency domain. It is also interesting to note that the solution to the H_2-optimal control problem with generalized causality constraint has a larger set of possible applications besides the design of packet-based control systems. For example, any system possessing some type of one step delay pattern can incorporate this design and avoid the higher dimensional controllers or complexity of implementing reduced-order solution methods which result from augmenting the plant dynamics with the delayed measurement states.

Consider the system in (7). A generalized causality constraint on a controller is equivalent to a null constraint on individual elements of the controller feedthrough matrix $\underline{\Upsilon}$. It is immediately apparent that an expression for the generalized causality constraint in $\mathbb{R}^{m_2 N_p \times p_2 N_p}$ will be cumbersome. It would be more convenient if instead of working with the matrix representation of $\underline{\Upsilon}$ we could work with the vector representation. This can be done with the introduction of two operators, the stack operator and the matrix Kronecker product. The stack operator associates $\underline{\Upsilon} \in \mathbb{R}^{m_2 N_p \times p_2 N_p}$ with $\underline{\Upsilon}^{\mathbb{S}} \in \mathbb{R}^{m_2 p_2 N_p^2}$ in a natural way, $\underline{\Upsilon}_{ij}$ – the element in the i^{th} row and j^{th} column – translates to $\underline{\Upsilon}^{\mathbb{S}}_{i+(j-1)m_2 N_p}$. In other words the columns are simply stacked and for that reason the stack operator applied to a matrix is signified with the superscript $\mathcal{S}$. We will also use the notation $(\cdot)^{-\mathcal{S}}:\mathbb{R}^{m_2 p_2 N_p^2}\to\mathbb{R}^{m_2 N_p\times p_2 N_p}$ to denote the transformation back into the appropriate matrix space. Although, the definition for $(\cdot)^{-\mathcal{S}}$ is somewhat vague if used out of context, it will only be used here following a stack operator and therefore the row and column dimensionality of the image space will equal those of the original matrix space to which the stack operator was applied. The matrix Kronecker product $\otimes:(\mathbb{R}^{l\times k},\mathbb{R}^{r\times s})\to\mathbb{R}^{lr\times ks}$ is defined as

$$A\otimes B=\begin{bmatrix} a_{11}B & \cdots & a_{1k}B\\ \vdots & & \vdots\\ a_{l1}B & \cdots & alkB\end{bmatrix}$$

For details on the Kronecker product we refer the reader to [10]. Using the *stack* operator, a clean representation of the generalized causality constraint can be formed; $\underline{\psi}_i(k)$ is independent of $\underline{\eta}_j(k)$ if and only if $\underline{\Upsilon}^{\mathcal{S}}_{i+(j-1)m_2N_p}$ is equal to zero. This simple relation can be further used to restate the generalized causality constraint using an orthogonal projection operator. Let $\mathcal{D}\subset\mathbb{R}^{m_2p_2N_p^2}$ represent the allowable subspace for $\underline{\Upsilon}^{\mathcal{S}}$ and $P_{\mathcal{D}}$ be the associated orthogonal projection operator. Then $\underline{\Upsilon}^{\mathcal{S}}$ satisfying the causality constraint implies $\underline{\Upsilon}^{\mathcal{S}}=P_{\mathcal{D}}\underline{\Upsilon}^{\mathcal{S}}$.

Consider the following pair of Riccati Equations.

$$\begin{aligned} X &= \underline{A}^T X\underline{A}-\underline{C}_x^T\underline{D}_x(\underline{D}_x^T\underline{D}_x)^{-1}\underline{D}_x^T\underline{C}_x+\underline{C}_1^T\underline{C}_1\\ Y &= \underline{A}Y\underline{A}^T-\underline{B}_y\underline{D}_y^T(\underline{D}_y\underline{D}_y^T)^{-1}\underline{D}_y\underline{B}_y^T+\underline{B}_1\underline{B}_1^T\end{aligned}\tag{11}$$

where

$$\begin{aligned} \underline{D}_x^T\underline{D}_x &= \underline{D}_{12}^T\underline{D}_{12}+\underline{B}_2^T X\underline{B}_2\\ \underline{D}_x^T\underline{C}_x &= \underline{D}_{12}^T\underline{C}_1+\underline{B}_2^T X\underline{A}\\ \underline{D}_y\underline{D}_y^T &= \underline{D}_{21}\underline{D}_{21}^T+\underline{C}_2 Y\underline{C}_2^T\\ \underline{B}_y\underline{D}_y^T &= \underline{B}_1\underline{D}_{21}^T+\underline{A}Y\underline{C}_2^T\end{aligned}$$

By Lemma 1 the above Riccati equations satisfy the necessary conditions needed for the existence and uniqueness of positive-semidefinite stabilizing solutions. For more on the existence and uniqueness of stabilizing solutions to Discrete Algebraic Riccati Equations, see for example [11]. Taking X and Y to be the positive-semidefinite stabilizing solutions to (11), define

$$\begin{aligned} \underline{F} &= -(\underline{D}_x^T\underline{D}_x)^{-1}\underline{D}_x^T\underline{C}_x\\ \underline{L} &= -\underline{B}_y\underline{D}_y^T(\underline{D}_y\underline{D}_y^T)^{-1}\\ \underline{R} &= \underline{D}_x^T\underline{C}_x Y\underline{C}_2^T+\underline{B}_2^T X\underline{B}_1\underline{D}_{21}^T+\underline{D}_{12}^T\underline{D}_{11}\underline{D}_{21}^T\\ \underline{\Upsilon}_o &= -\left[(P_{\mathcal{D}}(\underline{D}_y\underline{D}_y^T\otimes\underline{D}_x^T\underline{D}_x)P_{\mathcal{D}})^{\dagger}\underline{R}^{\mathcal{S}}\right]^{-\mathcal{S}}\end{aligned}$$

where $P_{\mathcal{D}}$ is the generalized causality constraint orthogonal projection operator defined above and $(\cdot)^{\dagger}$ symbolizes the generalized inverse. Notice that thus far the approach has been completely general – no assumptions besides those in Definition 1 were made. In fact, the only assumption that will need to be made concerns the structure of $\underline{D}_{22}$. A nonzero feedthrough matrix from the control input to the measured output introduces an algebraic loop in the feedback system. Therefore, if the feedback system is well posed this algebraic loop can be eliminated. The same idea applies here. The only exception is that the feedthrough matrix also has to be well posed with respect to the causality constraint. In other words, if the i^{th} control input is to be independent of the j^{th} measured output and the k^{th} control input is dependent on the j^{th} measured output then all the output measurements that the i^{th} control input is dependent on have to be independent of the k^{th} control input as well. This condition can be easily checked explicitly by forming the algebraic equation of the feedback loop.

The expression for the H_2-optimal controller with a generalized causality constraint is now stated in the following theorem.

Theorem 2. *Let all terms be as defined above, and assume that the feedback loop is well posed with respect to the generalized causality constraint. Then the unique H_2-optimal controller that satisfies the generalized causality constraint – characterized by the orthogonal projection operator $P_{\mathcal{D}}$ – is*

$$\underline{K}_{opt}=\mathcal{F}\left(\begin{bmatrix}0 & I\\ I & -\underline{D}_{22}\end{bmatrix},\tilde{K}\right)\tag{12}$$

where

$$\tilde{K}\sim\left[\begin{array}{c|c}\underline{A}+\underline{B}_2\underline{F}+\underline{L}\underline{C}_2-\underline{B}_2\underline{\Upsilon}_o\underline{C}_2 & \underline{L}-\underline{B}_2\underline{\Upsilon}_o\\ \hline \underline{\Upsilon}_o\underline{C}_2-\underline{F} & \underline{\Upsilon}_o\end{array}\right]$$

Moreover,

$$\begin{aligned}\min_{\underline{K}}||\underline{T}_{zw}||_2^2 &= \mathrm{trace}\left(\underline{B}_1^T X\underline{B}_1\right)+\mathrm{trace}\left(\underline{D}_{11}^T\underline{D}_{11}\right)\\ &+\mathrm{trace}\left((\underline{A}^T X\underline{A}-X+\underline{C}_1^T\underline{C}_1)Y\right)\\ &-\mathrm{trace}\left((\underline{D}_x^T\underline{D}_x)^{-1}\underline{\Upsilon}_o(\underline{D}_y\underline{D}_y^T)^{-1}\underline{\Upsilon}_o^T\right)\end{aligned}\tag{13}$$

Proof. Making the assumption that the matrix $\underline{D}_{22}$ is consistent with the generalized causality constraint, we solve the problem as if $\underline{D}_{22}$ was identically zero and account for it in the end by solving the algebraic equation of the feedback loop explicitly. The first part of the proof is similar to that of [11], the main differences being the inclusion of the matrix $\underline{D}_{11}$ and the use of the stack operator. Applying ideas from geometric control theory, it can be shown that the infimum of the H_2 norm of the closed loop system taken over all proper controllers is a function of the plant state space representation and the controller feedthrough matrix only. This equation is

$$\begin{aligned}\min_{\underline{K}}||\underline{T}_{zw}||_2^2 &= \mathrm{trace}\left(\underline{B}_1^T X\underline{B}_1\right)+\mathrm{trace}\left(\underline{D}_{11}^T\underline{D}_{11}\right)\\ &+\mathrm{trace}\left((\underline{A}^T X\underline{A}-X+\underline{C}_1^T\underline{C}_1)Y\right)+\Omega_*\end{aligned}$$

where

$$\Omega_*\triangleq\min_{\underline{\Upsilon}}\Omega(\underline{\Upsilon})\ \text{and}$$

$$\Omega(\underline{\Upsilon})=2(\underline{R}^{\mathcal{S}})^T\underline{\Upsilon}+[(\underline{D}_y^T\otimes\underline{D}_x)\underline{\Upsilon}^{\mathcal{S}}]^T[(\underline{D}_y^T\otimes\underline{D}_x)\underline{\Upsilon}^{\mathcal{S}}]$$

Next we proceed by completing the square. Using the generalized inverse (see [12]) formulations of orthogonal projection operators and after some algebraic simplification of the resulting expressions, $\Omega(\underline{\Upsilon})$ becomes

$$\Omega(\underline{\Upsilon}) = \left[\underline{R}_*^{\mathcal{S}} + (\underline{D}_y^T \otimes \underline{D}_x)\underline{\Upsilon}^{\mathcal{S}}\right]^T \left[\underline{R}_*^{\mathcal{S}} + (\underline{D}_y^T \otimes \underline{D}_x)\underline{\Upsilon}^{\mathcal{S}}\right] - (\underline{R}_*^{\mathcal{S}})^T \underline{R}_*^{\mathcal{S}}$$

where

$$\underline{R}_*^{\mathcal{S}} = (\underline{D}_y^T \otimes \underline{D}_x)(P_{\mathcal{D}}(\underline{D}_y\underline{D}_y^T \otimes \underline{D}_x^T\underline{D}_x)P_{\mathcal{D}})^{\dagger}\underline{R}^{\mathcal{S}}$$

which implies

$$\Omega_* = -(\underline{R}_*^{\mathcal{S}})^T \underline{R}_*^{\mathcal{S}}$$

and the unique value of $\underline{\Upsilon}_o$ is

$$\underline{\Upsilon}_o = -\left[(P_{\mathcal{D}}(\underline{D}_y\underline{D}_y^T \otimes \underline{D}_x^T\underline{D}_x)P_{\mathcal{D}})^{\dagger}\underline{R}^{\mathcal{S}}\right]^{-\mathcal{S}}$$

The next few steps are fairly straightforward and therefore, for the sake of brevity, will be merely summarized. First, $\underline{\Upsilon}_o$ is absorbed into the plant to form a new plant $\underline{G}_{eq}^{\Upsilon}$. Since $\underline{\Upsilon}_o$ is unique, this reduces the problem to finding a strictly proper controller for the new plant $\underline{G}_{eq}^{\Upsilon}$. The solution to this problem is readily available, see for example [4] or [11]. The controller for the original system is then simply obtained by absorbing $\underline{\Upsilon}_o$ back into the controller equations. This expression is then further simplified by making the observation that the solutions to the Riccati equations in (11) are invariant under feedback. Finally, matrix $\underline{D}_{22}$ is accounted for by solving the algebraic equation of the feedback loop explicitly. □

Using the actual values of the matrices involved in (7) and the the expression for the packet-causality constraint presented in Theorem 1, a more compact expression for the packet-based controller can be obtained.

Corollary 1 (H_2-optimal packet-based controller). *The unique H_2 optimal controller that satisfies the packet-causality constraint is*

$$\underline{K}_p = \mathcal{F}\left(\begin{bmatrix} 0 & I \\ I & -\underline{D}_{22} \end{bmatrix}, \tilde{K}_p\right) \quad (14)$$

where

$$\tilde{K}_p \sim \left[\begin{array}{c|c} \underline{A} + \underline{B}_2\underline{F} + \underline{L}\underline{C}_2 - \underline{B}_2\underline{\Upsilon}_p\underline{C}_2 & \underline{L} - \underline{B}_2\underline{\Upsilon}_p \\ \hline \underline{\Upsilon}_p\underline{C}_2 - \underline{F} & \underline{\Upsilon}_p \end{array}\right]$$

with

$$\underline{\Upsilon}_p = \begin{bmatrix} \underline{\Upsilon}_{p1} & 0 \end{bmatrix}$$
$$\underline{\Upsilon}_{p1} = (\underline{D}_x^T\underline{D}_x)^{-1}\underline{D}_x^T\underline{C}_x Y \underline{C}_2^T(D_{21}D_{21}^T + C_2 Y C_2^T)^{-1}$$

The rest of the terms are defined in the same way as before.

Proof. The uniqueness and existence conditions are automatically satisfied for the lifted system as was stated in Lemma 1. The structure of $\underline{D}_{22}$ is well posed with respect to the packet-causality constraint since all inputs are dependent on the same current measured outputs. The remainder is a direct result of Theorem 2 and the structure of the lifted matrices described earlier. □

D. Performance analysis

A direct relationship between the minimum achievable generalized H_2 measure of the packet-based controller and that of a single rate controller would be convenient to evaluate suitability of the packet-based controller to a specific problem. Since the packet-based controller is periodic and does not have a nice representation in the non-lifted domain this comparison has to be done in the lifted domain. The structure of the packet-based controller is most closely related to the lifted representation of the single rate controller operating at the sampling period T_s. As was pointed out in [13] the solutions of the Riccati equations in (11) are invariant under the lifting operation. Moreover, the H_2 optimal periodic controller for a time-invariant system is also time-invariant. Therefore, the H_2-optimal generalized causality constraint controller solution can be used to derive a compact expression for the lifted single rate controller and, since the generalized measure of the resulting plant will not be affected, can be used to calculate the difference between the generalized measure of the single rate controller and the packet-based controller. The difference can be written as

$$J_p^2 - J_{T_s}^2 = -\frac{1}{N_p}\text{trace}\left((\underline{D}_x^T\underline{D}_x)^{-1}\underline{\Upsilon}_p(\underline{D}_y\underline{D}_y^T)^{-1}\underline{\Upsilon}_p^T\right) + \frac{1}{N_p}\text{trace}\left((\underline{D}_x^T\underline{D}_x)^{-1}\underline{\Upsilon}_{T_s}(\underline{D}_y\underline{D}_y^T)^{-1}\underline{\Upsilon}_{T_s}^T\right)$$

where J_p and J_{T_s} are the generalized measures for the packet-based system and the lifted single-rate system respectively and $\underline{\Upsilon}_{T_s}$ is the lifted feedthrough matrix of the single-rate system. Furthermore, in [14], it was shown that the generalized H_2 measure for the sampled-data system increases as the sampling period squared (for small sampling periods). In the case where the network imposes a communication constraint that forces a slower transmission rate to be used, the relationship between the result in [14] and the results presented here would be a useful measure in determining the advantage of using the packet-based controller over just simply down-sampling. Unfortunately, this is not clear from our expression. Perhaps a more detailed, element-wise analysis could provide some answers.

IV. Example

In this section we compare the H_2-optimal packet-based controller with two different H_2-optimal single-rate controllers via simulation. An interesting yet simple plant to consider is a double integrator $(1/s^2)$. The state vector $[x_1\ x_2]^T$ is defined as the plant output (position) and its derivative (velocity) respectively. The disturbance and controlled output matrices are defined as

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 10 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$D_{12} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0.1 \end{bmatrix}$$

The sampling period T_s and the packet size N_p are left as variables.

Fig. 4 and 5 contain the position and velocity impulse responses. Here the packet size, N_p, is equal to 10 and T_s=0.01 sec. In this figure, the packet-based controller is compared against a single rate controller that purely operates at sampling period equal to T_s and a single rate controller that purely operates at sampling period equal to T_p. If we qualitatively asses this figure by simply comparing the areas under the impulse responses then it appears that the packet-based controller recovers about 50% of the performance lost due to down-sampling by the slow rate controller. Another interesting feature is that the packet-based controller results in a more continuous velocity response.

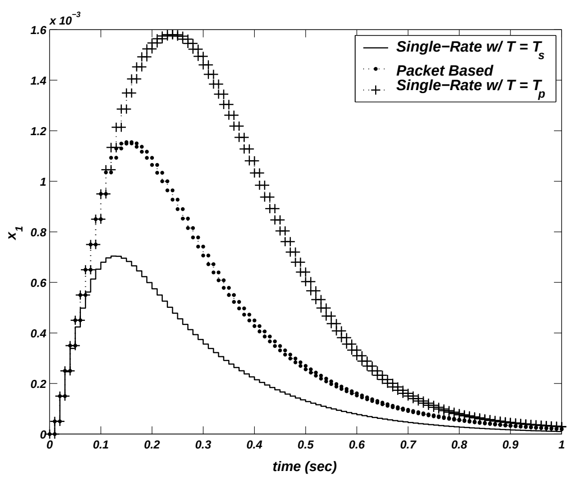

Fig. 4. Impulse response – position vs. time: T_s=0.01sec N_p=10

Fig. 6 contains the control input corresponding to the responses in Fig. 4 and 5. This figure illustrates how the control input is redistributed over the transmission period. The result is a larger control input in the first half and a smaller control input in the second half of the transmission period. Depending on the actuator, this apparently inherent quality of the packet-based controller, may not be desirable.

Fig. 7 compares the increase of the generalized H_2 measure with increase in transmission period for the doubled integrator plant. Again, the packet-based controller is compared against the two single rate controllers. In this figure T_s is held fixed and at each transmission period, N_p is adjusted to equal the number of samples collected during

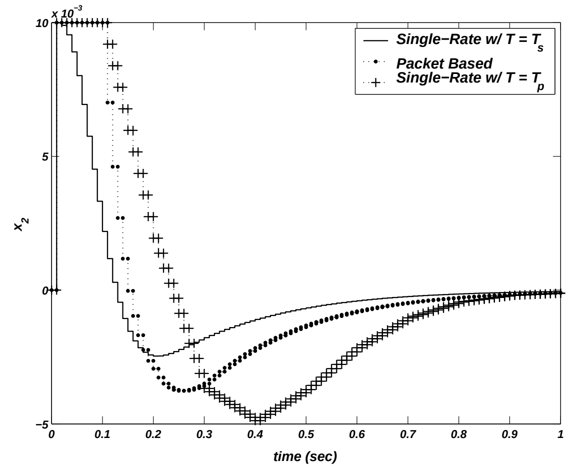

Fig. 5. Impulse response – velocity vs. time: T_s=0.01sec N_p=10

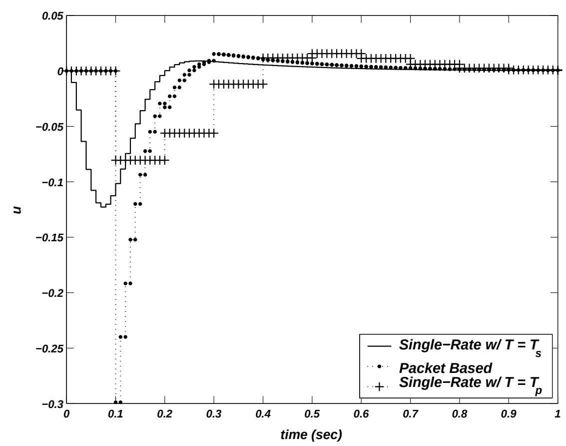

Fig. 6. Impulse response – control input vs. time: T_s=0.01sec, N_p=10

that transmission period. This figure provides a clearer indication of the amount of performance recovered that was lost by down-sampling.

Fig. 8 contains the plot of generalized H_2 measure vs. transmission period for different fixed packet sizes. All cases are compared with the continuous time case. Again, this figure indicates that the system performance can be significantly improved using packet-based control.

V. Conclusion

In this paper, we presented a new approach to networked control system design. The synthesis problem was motivated by traditional design criteria of most control networks together with an efficient use of the packet structure. A specific protocol, consisting of a multipoint packet structure, was proposed. It was pointed out that the extra samples in each packet come at no extra price, in that the size of the packet remains the same, and therefore no extra load on the network is introduced. Next, the H_2-optimal control problem with a generalized causality constraint was defined. The solution to this problem was presented and

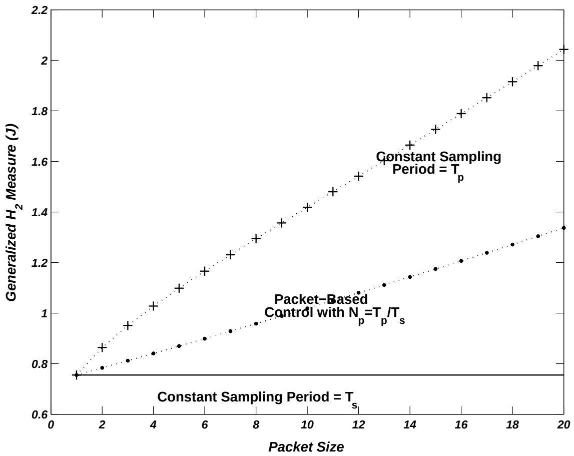

Fig. 7. Generalized measure vs. packet size for fixed T_s=0.01sec, packet-based controller is compared against a single rate controller w/ $T = T_s$ and a single rate controller w/ $T = T_p$

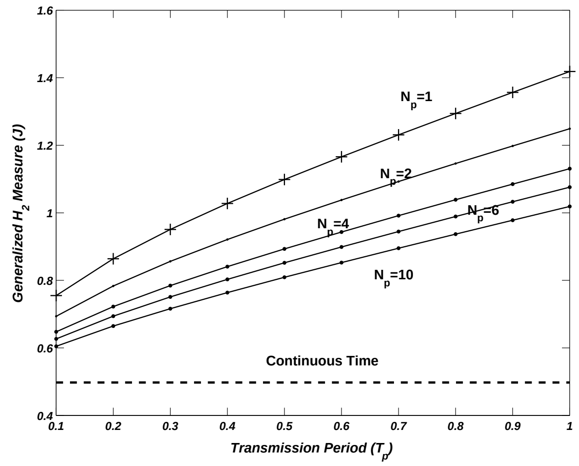

Fig. 8. Generalized measure vs. transmission period for different packet sizes, here T_s is adjusted to equal T_p/N_p

then used to solve the packet-based control problem. The methods developed were then used to design a packet-based controller for a double integrator. The performance gained using the packet-based approach appears to be significant enough to be considered for some applications. Lastly, there are many aspects left unanswered. For example: exactly how much performance can be gained in general and how does it depend on the plant dynamics, how do delay and lost packets affect the performance and can more flexible control structures be used to counter some of these effects.

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