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Piecewise-affine Lyapunov Functions for Continuous-time Linear Systems with Saturating Controls

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***Abstract*— This paper is concerned with piecewise-affine functions as Lyapunov function candidates for stability analysis of time-invariant continuous-time linear systems with saturating closed-loop control inputs. Using a piecewise-affine model of the closed-loop system, new necessary and sufficient conditions for a piecewise-affine function be a Lyapunov function are presented. Based on linear programming formulation of these conditions, an effective backward set simulation procedure is proposed for construction of such Lyapunov functions for estimation of the region of local asymptotic stability. Compared to quadratic functions, piecewise-affine functions showed to be more adequate to capture the dynamical effects of saturation nonlinearities, giving strictly less conservative results.**

I. INTRODUCTION

Saturation nonlinearities are ubiquitous in control engineering. They are usually caused by limits on components size, available power, being often associated with amplifiers and actuators, which are important components of control systems [1]. Regardless of how saturation arises, the analysis and design of a system containing saturation nonlinearities is an important problem. Not only is this problem theoretically challenging, but it is also practically imperative [2]. The estimation of the stability region of time-invariant continuous-time systems with saturating closed-loop control inputs has been a focus of study in recent years. In particular, simple and general methods have been derived applying absolute stability analysis tools such as the circle and Popov criteria [3], [4]. Since the circle criterion is applicable to general memoriless sector bounded nonlinearities, its application to saturation nonlinearities leads to conservative results. Based on special property of saturation nonlinearities, [2] proposed a less conservative constructive method for determination of an hellipsoidal estimate of the asymptotic stability region, formulated as a linear matrix inequalities (LMI) optimization problem. Using a piecewise-affine model of the saturating closed-loop system, [5] presented a linear programming (LP) characterization of piecewise-affine (PWA) Lyapunov functions for discrete-time systems. Based on this LP characterization, a computationally efficient constructive procedure was proposed for determination of a low conservative polyhedral estimate of the asymptotic stability region. The success of this approach is due to the ability showed by PWA functions to capture the dynamical effects of the imputs saturation combined with the computational efficiency of their LP characterization.

This paper extends the results in [5] to time-invariant continuous-time linear systems with saturating closed-loop control imputs. New necessary and sufficient conditions are derived for positive definite PWA functions be Lyapunov functions. Based on LP formulation of these necessary and sufficient conditions, a new backward set simulation procedure is proposed for construction of such PWA Lyapunov functions and estimation of the region of local asymptotic stability of origin. A numerical example compares the performances of the proposed PWA function based approach and the quadratic function LMI based approach [2] in the estimation of the region of asymptotic stability of a saturating closed-loop system.

Throughout this paper: for two $n \times m$ real matrices $A = (a_{ij})$ and $B = (b_{ij})$, $A \leq B$ is equivalent to $a_{ij} \leq b_{ij}$ for all i, j such that $1 \leq i \leq n$ and $1 \leq j \leq m$.

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II. PRELIMINARIES

Consider the continuous-time system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where function $f : \Re^n \rightarrow \Re^n$ is possibly nonlinear and satisfies sufficient conditions for existence and uniqueness of solutions and $f(0) = 0$.

Consider the set

$$\mathcal{B}(\Psi, \epsilon) = \{x \in \Re^n : \Psi(x) \leq \epsilon\} \tag{2}$$

and its boundary set

$$\partial\mathcal{B}(\Psi, \epsilon) = \{x \in \Re^n : \Psi(x) = \epsilon\}, \tag{3}$$

here called ϵ ball and ϵ level set of the function $\Psi : \Re^n \rightarrow \Re$, respectively.

Definition 2.1: A positive definite [6] locally Lipschitz function [7] $\Psi : \Re^n \rightarrow \Re$ is a Lyapunov function in strong sense, for time-invariant continuous-time system (1) in $\mathcal{B}(\Psi, 1)$ (2), if for all $x \in \mathcal{B}(\Psi, 1)$, the upper Dini derivative [8] satisfies

$$D^+\Psi(x) = \lim_{\tau \rightarrow 0^+} sup \frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \leq -\beta\Psi(x), \tag{4}$$

for some $\beta > 0$. The ball $\mathcal{B}(\Psi, 1)$ is a region of asymptotic stability of the origin with convergence rate β.

Proposition 2.1: A positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for system (1) in $\mathcal{B}(\Psi, 1)$, iff there is a $\beta > 0$ such that:

$$D^+\Psi(x) \leq -\beta\epsilon \quad \forall x \in \partial\mathcal{B}(\Psi, \epsilon)\ ,\ 0 \leq \epsilon \leq 1. \tag{5}$$

Proof: Noting that $\Psi(x) = \epsilon$ for $x \in \partial\mathcal{B}(\Psi, \epsilon)$ and that $\forall x \in \partial\mathcal{B}(\Psi, \epsilon)$, $0 \leq \epsilon \leq 1$ is equivalent to $\forall x \in \mathcal{B}(\Psi, 1)$, the proof is immediate from Definition 2.1. □

Proposition 2.2: A positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for system (1) in $\mathcal{B}(\Psi, 1)$, iff there are $\beta > 0$ and $\bar{\tau} > 0$ such that:

$$\frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \leq -\beta\Psi(x) \;\; \forall \;\; 0 < \tau \leq \bar{\tau}. \quad (6)$$

Proof: (Sufficiency:) If (6) holds for all $x \in \mathcal{B}(\Psi, 1)$, then (4) in Definition 2.1 is also satisfied. (Necessity:) From (4) in Definition 2.1, there must be an infinitesimal $\bar{\tau}$ such that (6) holds. □

Proposition 2.3: A convex positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for system (1) in $\mathcal{B}(\Psi, 1)$, iff there are $\beta > 0$ and $\bar{\tau} > 0$ such that

$$\Psi(x + \bar{\tau} f(x)) \leq (1 - \bar{\tau}\beta)\Psi(x) \quad (7)$$

holds for all $x \in \mathcal{B}(\Psi, 1)$.

Proof: Equation (7) can be rewritten as:

$$\frac{\Psi(x + \bar{\tau} f(x)) - \Psi(x)}{\bar{\tau}} \leq -\beta\Psi(x). \quad (8)$$

For convex functions $\Psi(x)$ [6], it can be verified that the following property holds for all $0 < \tau \leq \bar{\tau}$:

$$\frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \leq \frac{\Psi(x + \bar{\tau} f(x)) - \Psi(x)}{\bar{\tau}}. \quad (9)$$

From (8), (9), one has

$$\frac{\Psi(x + \tau f(x)) - \Psi(x)}{\tau} \leq -\beta\Psi(x) \;\; \forall \;\; 0 < \tau \leq \bar{\tau},$$

which makes the proof immediate from Proposition 2.2. □

Consider the discrete-time Euler Approximating Systems (EAS) of system (1):

$$x(t + 1) = x(t) + \tau f(x(t)) \;\;\; , \;\;\; \tau > 0. \quad (10)$$

Definition 2.2: The one-step admissible set to $\mathcal{B}(\Psi, \epsilon)$ (2) w.r.t. EAS (10) is given by:

$$\mathcal{B}(\Psi_f, \epsilon) = \{x : \Psi(x + \tau f(x)) \leq \epsilon\}. \quad (11)$$

In other words, it is the set of all EAS states $x(t)$ such that $x(t + 1) \in \mathcal{B}(\Psi, \epsilon)$.

Corollary 2.1: A convex positive definite locally Lipschitz function $\Psi(x)$ is a Lyapunov function in strong sense, for system (1) in $\mathcal{B}(\Psi, 1)$, iff there are $\beta > 0$ and $\tau > 0$ such that:

$$\mathcal{B}(\Psi, \epsilon) \subset \mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon) \;\; \forall \;\; 0 \leq \epsilon \leq 1. \quad (12)$$

Proof: It can be verified from (11) and (2) that (12) holds iff (7) in Proposition 2.3 holds. □

Corollary 2.2: If $\Psi(x)$ is a convex locally Lipschitz Lyapunov function in strong sense, for continuous-time system (1) in $\mathcal{B}(\Psi, 1)$ with convergence rate β, then there is $\tau > 0$ such that the set $\mathcal{B}(\Psi_f, 1 - \tau\beta)$ (11) is a region of asymptotic stability of origin with convergence rate β.

Proof: Immediate from Proposition 2.3, Definitions 2.1 and 2.2. □

Consider the continuous-time linear system represented by the following state equations and constraints:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad (13)$$

$$-\check{u} \leq u \leq \hat{u} \;\; , \;\; \check{u}, \hat{u} \geq 0, \quad (14)$$

where $x(t) \in \Re^n$ and $u(t) \in \Re^m$ are the state and control variables, respectively. A, B, $\check{u}$, $\hat{u}$ are real and of appropriate dimensions. Assume the saturating feedback control law

$$u(k) = sat(Fx(t)), \quad (15)$$

where $F \in \Re^{m\times n}$ is constant and the components of $sat(Fx)$ are given by:

$$sat(Fx)_i = \begin{cases} -\check{u}_i & if \quad f_i x < -\check{u}_i, \\ f_i x & if \quad -\check{u}_i \leq f_i x \leq \hat{u}_i, \\ \hat{u}_i & if \quad f_i x > \hat{u}_i, \end{cases} \quad (16)$$

where f_i denotes the ith row of matrix F.

From (13), (15), the closed-loop system is given by the nonlinear model:

$$\dot{x}(t) = Ax(t) + Bsat(Fx(t)). \quad (17)$$

Considering all $x \in \Re^n$, each one of the m components of the saturating law (16) has 3 possible states: saturated at lower bound, not saturated and saturated at upper bound. Consequently, $\Re^n$ can be decomposed into $j = 1 : 3^m$ regions $S(R_j, d_j) \subset \Re^n$, denoted as saturation regions [9], given by polyhedra of the form:

$$S(R_j, d_j) = \{x \in \Re^n; R_j x \leq d_j\}, \quad (18)$$

$$R_j = \begin{bmatrix} F_{ns} \\ -F_{ns} \\ -F_{su} \\ F_{sl} \end{bmatrix} \;\; , \;\; d_j = \begin{bmatrix} \hat{u}_{ns} \\ \check{u}_{ns} \\ -\hat{u}_{su} \\ -\check{u}_{sl} \end{bmatrix}, \quad (19)$$

where F_{ns}, $\hat{u}_{ns}$, $\check{u}_{ns}$, F_{su}, $\hat{u}_{su}$, F_{sl}, $\check{u}_{sl}$ denote matrices and vectors appropriately formed by the rows of F, $\hat{u}$, $\check{u}$, related, respectively, to the components not saturated, saturated at upper level and saturated at lower level, which characterize the region.

Within each saturation region $S(R_j, d_j)$, closed-loop system (17) can be represented by an affine model [9]:

$$\begin{array}{c} \dot{x}(t) = A_j x(t) + p_j, \\ A_j = [A + B_{ns}F_{ns}], \\ p_j = B_{su}\hat{u}_{su} - B_{sl}\check{u}_{sl}, \end{array} \quad (20)$$

where B_{ns}, B_{su} and B_{sl} denote matrices appropriately formed by the columns of B related to F_{ns}, $\hat{u}_{su}$, $\check{u}_{sl}$, respectively. Throughout the paper, it will be assigned $j = 1$ for the region of linear behavior of $sat(F(x))$, described by:

$$\begin{array}{c} R_1 = \begin{bmatrix} F \\ -F \end{bmatrix} \;\; , \;\; d_1 = \begin{bmatrix} \hat{u} \\ \check{u} \end{bmatrix}, \\ A_1 = A + BF \;\; , \;\; p_1 = 0. \end{array} \quad (21)$$

III. PWA Lyapunov Functions

Consider the PWA function:

$$\Psi(x) = \max_{1\le i\le r} w_i^{-1}\{g_i x + c_i\}, \tag{22}$$

where: $x \in \Re^n$ and w_i, g_i, c_i are ith rows of $w > 0 \in \Re^r$, $G \in \Re^{r\times n}$, $c \le 0 \in \Re^r$, respectively. It is easy to verify that function $\Psi(x)$ (22) can also be defined as:

$$\Psi(x) = \min_{\epsilon\in\Re} \epsilon \quad s.t. \quad Gx + c \le w\epsilon. \tag{23}$$

It can be verified that PWA function $\Psi(x)$ (22), (23) is locally Lipschitz [7] and convex [6].

Proposition 3.1 [5]: PWA function $\Psi(x)$ (22), (23) is positive definite iff there is a permutation matrix P such that:

$$P\begin{bmatrix} G & c & w \end{bmatrix} = \begin{bmatrix} \tilde{G}_1 & \tilde{c}_1 & \tilde{w}_1 \\ \tilde{G}_2 & \tilde{c}_2 & \tilde{w}_2 \end{bmatrix} \ , \ \tilde{c}_1 = 0 \ , \ \tilde{c}_2 < 0$$

and polyhedron $\tilde{G}_1 x \le \tilde{w}_1$ is bounded.

It can be verified that the ϵ ball (2) and the ϵ level set (3) of $\Psi(x)$ (22), (23) are, respectively, given by:

$$\mathcal{B}(\Psi,\epsilon) = \{x \in \Re^n : Gx \le w\epsilon - c\}, \tag{24}$$

$$\partial\mathcal{B}(\Psi,\epsilon) = \bigcup_{i=1}^{i=r} \partial_i \mathcal{B}_\Psi(\epsilon), \tag{25}$$

$$\begin{aligned} \partial_i\mathcal{B}(\Psi,\epsilon) = \{x \in \Re^n : g_i x + c_i = w_i\epsilon \ ; \\ g_l x + c_l \le w_l\epsilon \ , \ l = 1:r \ , \ l \ne i\}. \end{aligned}$$

Now, considering the $j = 1 : 3^m$ saturation regions $S(R_j, d_j)$ (18), it can be verified that ϵ level set is also given by:

$$\partial\mathcal{B}(\Psi,\epsilon) = \bigcup_{i=1}^{i=r}\bigcup_{j=1}^{j=3^m} \partial_i^j \mathcal{B}(\Psi,\epsilon), \tag{26}$$

$$\begin{aligned} \partial_i^j\mathcal{B}(\Psi,\epsilon) = \{x \in S(R_j,d_j) : g_i x + c_i = w_i\epsilon \ ; \\ g_l x + c_l \le w_l\epsilon \ , \ l = 1:r \ , \ l \ne i\}. \end{aligned} \tag{27}$$

Within each saturation region $S(R_j, d_j)$ (18), (19), the EAS (10) of affine model (20) of closed-loop saturating system (17) is given by:

$$x(t+1) = (I + \tau A_j)x(t) + \tau p_j \ , \ \tau > 0. \tag{28}$$

From Definition 2.2, the one-step admissible set to (24) w.r.t. EAS (28) is given by:

$$\mathcal{B}(\Psi_f,\epsilon) = \bigcup_{j=1}^{j=3^m} \mathcal{B}_j(\Psi_f,\epsilon), \tag{29}$$

$$\begin{aligned} \mathcal{B}_j(\Psi_f,\epsilon) = \{x \in S(R_j,d_j) : \\ G(I+\tau A_j)x \le w\epsilon - G\tau p_j - c\}. \end{aligned} \tag{30}$$

Proposition 3.2: A positive definite PWA function $\Psi(x)$ (22), (23) is a Lyapunov function in strong sense, for closed-loop system (17) in $\mathcal{B}(\Psi, 1)$ (24), iff for the $i = 1 : r$ components of $\Psi(x)$ and the $j = 1 : 3^m$ saturation regions (18), (19), (20) there is a positive β such that:

$$g_i A_j x + g_i p_j \le -\beta w_i \epsilon, \tag{31}$$

holds for any (ϵ, x) satisfying:

$$\begin{aligned} & g_i x + c_i = w_i\epsilon, \\ & g_l x + c_l \le w_l\epsilon \ , \ l = 1:r \ , \ l \ne i, \\ & R_j x \le d_j, \\ & 0 \le \epsilon \le 1. \end{aligned} \tag{32}$$

Proof: From Proposition 2.1,

$$D^+\Psi(x) \le -\beta\epsilon \tag{33}$$

must hold $\forall x \in \partial\mathcal{B}(\Psi,\epsilon)$, $0 \le \epsilon \le 1$. From (26), (27), it can be verified that $\partial\mathcal{B}(\Psi,\epsilon)$ is given by:

$$\partial\mathcal{B}(\Psi,\epsilon) = \bigcup_{i=1}^{i=r}\bigcup_{j=1}^{j=3^m} \partial_i^j \mathcal{B}(\Psi,\epsilon), \tag{34}$$

where $\partial_i^j\mathcal{B}(\Psi,\epsilon)$ is given by:

$$\begin{aligned} & g_i x + c_i = w_i\epsilon, \\ & g_l x + c_l \le w_l\epsilon \ , \ l = 1:r \ , \ l \ne i, \\ & R_j x \le d_j. \end{aligned} \tag{35}$$

From (4), (20), (22), (35), for $x \in \partial_i^j \mathcal{B}_\Psi(\epsilon)$, (33) is given by:

$$g_i A_j x + g_i p_j \le \beta w_i \epsilon. \tag{36}$$

From (34), (35), it can be verified that (33), (36) hold $\forall x \in \partial\mathcal{B}_\Psi(\epsilon)$, $0 \le \epsilon \le 1$ iff (36) hold for all (x, ϵ) satisfying (35) for $i = 1 : r$, $j = 1 : 3^m$, $0 \le \epsilon \le 1$, concluding the proof. □

The following corollary gives a primal linear programming formulation to Proposition 3.2.

Corollary 3.1: A positive definite PWA function $\Psi(x)$ (22), (23) is a Lyapunov function in strong sense, for closed-loop system (17) in $\mathcal{B}(\Psi, 1)$, iff for the $i = 1 : r$ components of $\Psi(x)$ and the $j = 1 : 3^m$ saturation regions (18), (19), (20) there is a $\beta > 0$ such that:

$$\max_{i,j}\{\sigma_j^i\} \le 0, \quad 1 \le i \le r, \quad 1 \le j \le 3^m, \tag{37}$$

where σ_j^i are obtained solving the following independent linear programs:

$$\sigma_j^i = \max_{x,\epsilon} g_i A_j x + g_i p_j + \beta w_i \epsilon, \tag{38}$$

$$\begin{aligned} & g_i x + c_i = w_i\epsilon, \\ & g_l x + c_l \le w_l\epsilon \ , \ l = 1:r \ , \ l \ne i, \\ & R_j x \le d_j, \\ & 0 \le \epsilon \le 1. \end{aligned} \tag{39}$$

Furthermore, let (x_j^i, ϵ_j^i) be an optimal solution related to a $\sigma_j^i > 0$. This indicates that $(x_j^i, \epsilon_j^i) \in \partial_i^j\mathcal{B}(\Psi,\epsilon)$, $0 \le \epsilon \le 1$ (27), (39) is outside the ith half-space defining $\mathcal{B}_j(\Psi_f, (1-\tau\beta)\epsilon)$ (30) at jth saturation region:

$$\begin{aligned} \mathcal{B}_j^i = \{(x,\epsilon) \in \Re^{n+1} \ : \\ g_i(I+\tau A_j)x - (1-\tau\beta)w_i\epsilon \le -\tau g_i p_j - c_i\}. \end{aligned} \tag{40}$$

Or, equivalently, x_j^i is outside the ϵ_j^i ball of elementary PWA function given by:

$$\Psi^{ij}(x) = (1-\tau\beta)^{-1}w_i^{-1}\{g_i(I+\tau A_j)x+\tau g_i p_j+c_i\}. \quad (41)$$

Proof: (37)-(39) are immediate from Proposition 3.1. It can be verified that (40) corresponds to the ith half-space defining (30). From (38), (39), it can be verified that:

$$g_i x + c_i - w_i\epsilon + \tau(g_i A_j x + g_i p_j + \beta\epsilon) = \tau\sigma_j^i. \quad (42)$$

For an optimal (x_j^i, ϵ_j^i) with $\sigma_j^i > 0$ and $\tau > 0$, (42) gives

$$g_i(I + \tau A_j)x_j^i - (1 - \tau\beta)w_i\epsilon_j^i > -\tau g_i p_j - c_i,$$

showing that (x_j^i, ϵ_j^i) is outside (40) which is equivalent to x_j^i be outside the $\mathcal{B}(\Psi^{ij}, \epsilon_j^i)$ (41), concluding the proof. □

The following corollary gives a dual linear programming formulation to Proposition 3.2.

Corollary 3.2: A positive definite PWA function $\Psi(x)$ (22), (23) is a Lyapunov function in strong sense, for closed-loop system (17) in $\mathcal{B}(\Psi, 1)$, iff for the $i = 1 : r$ components of $\Psi(x)$ and the $j = 1 : 3^m$ saturation regions (18), (19), (20) there is a $\beta > 0$ such that:

$$\min_{i,j}\{\beta_j^i\} \geq \beta \ , \ 1 \leq i \leq r \ \ , \ \ 1 \leq j \leq 3^m, \quad (43)$$

where β_j^i are solutions of the following independent linear programs:

$$\begin{array}{c} \max \beta_j^i, \\ hG + kR_j = g_i A_j, \\ hw - t \leq \beta_j^i w_i, \\ -hc + kd_j + t \leq -g_i p_j - c_i, \\ h(h_l) \ : \ h_l \geq 0 \ , \ l \neq i, \\ k, t, \beta_j^i \geq 0. \end{array} \quad (44)$$

Furthermore, $\beta_j^i < \beta$ indicates that at jth saturation region, $\partial_i^j \mathcal{B}(\Psi, \epsilon)$, $0 \leq \epsilon \leq 1$ (27) has elements outside the ith half-space defining $\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon)$ (30) at jth saturation region:

$$\begin{array}{c} \mathcal{B}_j^i = \{(x, \epsilon) \in \Re^{n+1} \ : \\ g_i(I + \tau A_j)x - (1 - \tau\beta)w_i\epsilon \leq -\tau g_i p_j - c_i\}. \end{array} \quad (45)$$

Or, equivalently, $\partial_i^j \mathcal{B}(\Psi, \epsilon)$ has a x outside an ϵ ball of the elementary PWA function given by:

$$\Psi^{ij}(x) = (1-\tau\beta)^{-1}w_i^{-1}\{g_i(I+\tau A_j)x+\tau g_i p_j+c_i\}. \quad (46)$$

Proof: Inspecting (31), (32), it can be verified that Proposition 3.2 is satisfied iff there is $\beta > 0$ such that for $j = 1 : 3^m$, $i = 1 : r$,

$$\begin{array}{c} g_i A_j x - \beta_j^i w_i \epsilon \leq -g_i p_j, \\ \beta_j^i \geq \beta \end{array} \quad (47)$$

hold for all (x, ϵ), satisfying:

$$\begin{array}{l} g_i x + c_i = w_i\epsilon, \\ g_l x + c_l \leq w_l\epsilon \ , \ l = 1 : r \ , \ l \neq i, \\ R_j x \leq d_j, \\ 0 \leq \epsilon \leq 1. \end{array} \quad (48)$$

Using the Extended Farkas Lemma [10] and few routine algebraic manipulations, it can be verified that polyhedron (48) is a subset of polyhedron (47) iff there are vectors h, k of appropriate dimensions and positive scalars t, β_j^i such that:

$$\begin{array}{c} \beta_j^i \geq \beta, \\ hG + kR_j = g_i A_j, \\ hw - t \leq \beta_j^i w_i, \\ -hc + kd_j + t \leq -g_i p_j - c_i, \\ h(h_l) \ : \ h_l \geq 0 \ , \ l \neq i, \\ k, t, \beta_j^i \geq 0. \end{array} \quad (49)$$

It can be verified that (49) are equivalent to linear programs (44). Similarly to Corollary 3.1, it can be verified that $\beta_j^i < \beta$ implies that (45) is not satisfied by some (x, ϵ) in $\partial_i^j \mathcal{B}(\Psi, \epsilon)$, $0 \leq \epsilon \leq 1$ (27), (48), which is equivalent to a x be outside $\mathcal{B}(\Psi^{ij}, \epsilon)$ (46), concluding the proof. □

The following remarks about Corollaries 3.1 and 3.2 are opportune:

- Corollary 3.1, for a given β, corresponds to a set of $3^m r$ linear programs (LPs) (38), (39). Using interior point algorithms, the complexity of each LP (38), (39) is polynomial in n [11]. Similarly, it can be verified that Corolary 3.2 corresponds to a set of $3^m r$ LPs, each one with complexity polynomial in r. From Proposition 3.1, it can be verified that positive definite PWA functions must have $r > n$. Consequently, Corollary 3.1 has lower computational complexity than Corollary 3.2.;
- Corollary 3.2 is indicated when the objective is to find the maximum convergence rate β of a Lyapunov function candidate. Corollary 3.1 is more convenient than Corollary 3.2 when the objective is to check if a Lyapunov function candidate satisfies a convergence rate β known a priori. In this case, besides its lower complexity, if the check result is negative, Corollary 3.1 gives the elements (x, ϵ) of $\mathcal{B}(\Psi, \epsilon)$ responsible for the failure.

IV. Construction of PWA Lyapunov functions

Consider a positive definite PWA function $\Psi(x)$ (22), its ϵ ball $\mathcal{B}(\Psi, \epsilon)$ (24) and its one-step admissible set $\mathcal{B}(\Psi_f, \epsilon)$ (29) w.r.t. EAS of closed-loop saturating system (28). According to Corollary 2.1, $\Psi(x)$ is a Lyapunov function with convergence rate $\beta > 0$ iff there is $\tau > 0$ such that

$$\mathcal{B}(\Psi, \epsilon) \subset \mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon) \ \ \forall \ \ 0 \leq \epsilon \leq 1, \quad (50)$$

which is equivalent to:

$$\mathcal{B}(\Psi_f, (1-\tau\beta)\epsilon) \cap \mathcal{B}(\Psi, \epsilon) = \mathcal{B}(\Psi, \epsilon) \ \ \forall \ \ 0 \leq \epsilon \leq 1 \quad (51)$$

If (51) is not satisfied, the next natural candidate should be a PWA function with ϵ ball given by $\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon) \cap \mathcal{B}(\Psi, \epsilon)$. However, $\mathcal{B}(\Psi_f, (1 - \tau\beta)\epsilon)$ is possibly non convex, being, hence, not possible to assure the convexity of the resulting intersection set and its related PWA function. A next satisfactory convex PWA Lyapunov function candidate can be often obtained taking the PWA function with ϵ ball given by the intersection $\mathcal{B}(\Psi,\epsilon)\cap\mathcal{B}(\Psi^{ij},\epsilon)$ where Ψ^{ij} is conveniently selected among elementary PWA functions (41) given by Corollary 3.1. It can be verified from (22), (41), that this procedure corresponds to the following recurrence:

$$\Psi_{k+1}(x) = max\{\Psi_k(x)\ ,\ \Psi_k^{ij}(x)\}\ ,\ \Psi_0(x) = \Psi(x). \tag{52}$$

An important motivation for construction of Lyapunov functions is the estimation of regions of asymptotic stability of nonlinear systems. Let $\Psi(x)$ and $\tilde{\Psi}(x)$ be convex PWA Lyapunov functions with convergence rate β such that the unit ball $\mathcal{B}(\tilde{\Psi},1)$ is a convex polyhedral approximation of the one-step admissible set $\mathcal{B}(\Psi_f,(1-\tau\beta))$. From Corollary 2.1, it can be verified that the following recurrence gives a sequence of convex PWA Lyapunov functions with convergence rate β and monotonically increasing unit balls:

$$\Psi_{k+1}(x) = \tilde{\Psi}_k(x)\ ,\ \Psi_0(x) = \Psi(x). \tag{53}$$

Lyapunov functions $\tilde{\Psi}_k(x)$ can be computed using recurrence (52) started from a PWA Lyapunov function candidate with unit ball given by a convex polyhedral approximation of $\mathcal{B}(\Psi_{kf},(1-\tau\beta))$. Recurrence (53) is expected to be convergent for asymptoticaly stable closed-loop saturating systems which are unstable in open-loop. According to Definition 2.1 and Corollary 2.2, the unit ball of the limit solution of recurrence (53) and its one-step admissible set are, respectively, convex and nonconvex polyhedral estimates of the region of asymptotic stability of origin with convergence rate β.

Based on Corollary 3.1, the following procedure gives an effective computational approach to recurrence (53).

In what follows, PWA function $\Psi(x)$ (22), (23) and ϵ ball $\mathcal{B}(\Psi,\epsilon)$ (24) will be denoted in compact form as $\Psi[G,c,w]$ and $\mathcal{B}_\Psi[G,c,\epsilon w]$, respectively.

Procedure 4.1: Construction of a PWA Lyapunov function for estimation of the region of local asymptotic stability of closed-loop saturating system (17).

1 - Initialization:

β - desired convergence rate;
$\delta_m > 1$ - expansion tolerance for convergence test;
$\Psi[G,c,w]$ - initial Lyapunov function with convergence rate β.

2 - Convex expansion:

2.1 Set: $G_a = G$, $w_a = w$, $c_a = c$.
2.2 Find $\rho > 0$ such that $\mathcal{B}_\Psi[G,c,\rho]$ is an outer polyhedral approximation of one-step admissible set $\mathcal{B}(\Psi_f,(1-\tau\beta))$ (backward set simulation).
Set $w = \rho$.
2.3 - Check if $\Psi[G,c,w]$ is a Lyapunov function using Corollary 3.1:

Yes : Set $w_a = w$, $c_a = c$.
Return to 2.2.
No: Identify Ψ^*, the elementary PWA function with the most violated unit ball at the innermost saturation region not satisfying Corollary 3.1:

$$\Psi^* = \Psi[g_{i^*}(I+\tau A_{j^*}),\tau g_{i^*}p_{j^*}+c_{i^*},(1-\tau\beta)w_{i^*}].$$

3 - Construction of a Lyapunov function $\Psi[G,c,w]$ such that:

$$\Psi[G_a,c_a,\delta_m w_a] \geq \Psi[G,c,w] \geq \Psi[G_a,c_a,\rho].$$

3.1 - Check if [10]:

$$\mathcal{B}_\Psi[G_a,c_a,\delta_m\epsilon w_a] \subset \mathcal{B}(\Psi^*,\epsilon)\ ,\ 0 \leq \epsilon \leq 1.$$

No: $\Psi[G,c,w] = \Psi[G_a,c_a,w_a]$ is the desired Lyapunov function. **Stop**
Yes: update PWA Lyapunov function candidate:

$$\Psi[G,c,w] = \max\{\Psi[G,c,w]\ ,\ \Psi^*\}$$

3.2 - Check if $\Psi[G,c,w]$ is a Lyapunov function:

Yes : Eliminate redundant inequalities in $\mathcal{B}_\Psi[G,c,\epsilon w]$, $0 \leq \epsilon \leq 1$ [12].
Return to 2.
No: Identify Ψ^*, the elementary PWA function with the most violated unit ball at the innermost saturation region not satisfying Corollary 3.1:

$$\Psi^* = \Psi[g_{i^*}(I+\tau A_{j^*}),\tau g_{i^*}p_{j^*}+c_{i^*},(1-\tau\beta)w_{i^*}].$$

Return to 3.1

Some remarks about Procedure 4.1:

- The selection of the elementary PWA function Ψ^* at the innermost saturation region in steps 2.3, 3.2 and the inclusion check in step 3.1, are an attempt to avoid ϵ balls of elementary PWA functions Ψ^+ intersecting $\mathcal{B}_\Psi[G_a,c_a,w_a]$ due to non convexity of $\mathcal{B}(\Psi_f,\epsilon)$. See Fig. 1.
- The elimination of redundant inequalities in step 3.2 is strongly recommended, not only to obtain a concise representation of $\Psi[G,c,w]$, but also for the overall computational effectiveness of the procedure.
- The iteration of steps 2 and 3 gives a monotonic decreasing convergent sequence of Lyapunov functions. The unit balls of these Lyapunov functions form a monotonic increasing sequence of convex polyhedral regions of local asymptotic stability. The overall convergence rate cannot be easily determined due to the nonconvex nonlinear nature of the problem formulation.
- If convenient, any Lyapunov function $\Psi[G,c,w]$ can be used as initial function in step 1.

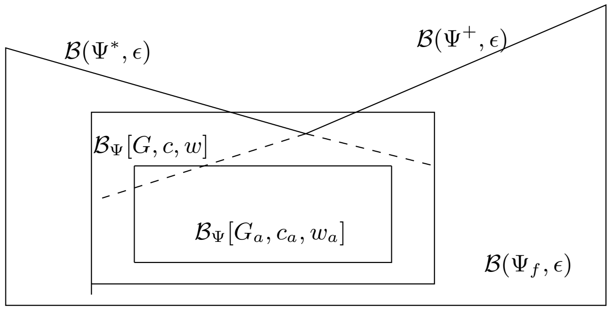

Figure 1: $\mathcal{B}(\Psi^*,\epsilon)$, $\mathcal{B}(\Psi^+,\epsilon)$ not satisfying Corollary 3.1.

V. NUMERICAL EXAMPLE

Consider the following system with saturating feedback control law [4], [2]:

$$\dot{x}(t) = Ax(t) + Bu(t),$$

$$u(t) = sat(Fx(t)) \quad , \quad -\check{u} \leq u \leq \hat{u},$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \ , \ B = \begin{bmatrix} 0 \\ 5 \end{bmatrix},$$

$$F = \begin{bmatrix} -2 & -1 \end{bmatrix} \quad , \quad \check{u} = \hat{u} = 1.$$

Fig. 2 shows the convex polyhedral estimate of the region of asymptotic stability obtained by Procedure 4.1. The inner hellipsoid is the estimate obtained by the quadratic function LMI based approach in [2]. The pay off for the strictly better result is the PWA function more complex representation: the polyhedral region is the unity ball of a PWA function with 30 symmetrical facets, while the ellipsoidal region is the unit ball of a positive definite two dimensional quadratic function.

VI. CONCLUSION

This paper dealt with PWA functions as Lyapunov function candidates for stability analysis of continuous-time linear systems with saturating closed-loop controls. Using a piecewise-affine model of the saturating closed-loop system, new necessary and sufficient conditions for PWA functions be Lyapunov functions were derived. Based on linear programming formulation of these conditions, an effective backward set simulation procedure is proposed for construction of such Lyapunov functions for estimation of the region of local asymptotic stability. A numerical example showed that PWA functions are more adequate than quadratic functions to capture the dynamical effects of saturation nonlinearities, giving strictly less conservative results.

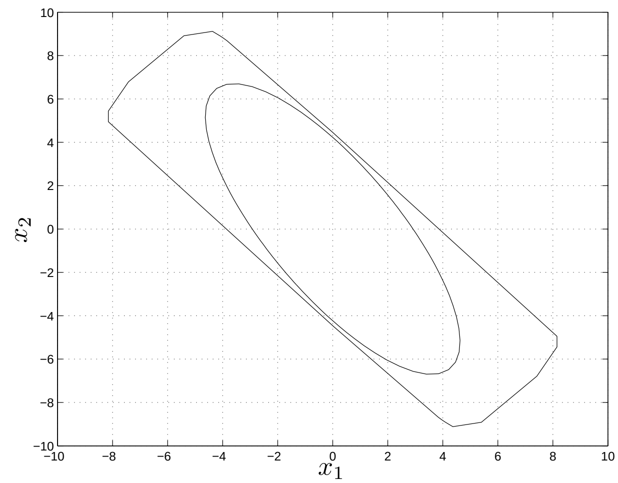

Figure 2: Asymptotic stability regions.

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