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A Hybrid Predictive Control Approach for Output Feedback Stabilization of Constrained Linear Systems

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Abstract— **This work proposes a hybrid control approach, uniting bounded control with model predictive control (MPC), for the stabilization of constrained linear systems under output feedback. The approach is predicated upon the idea of switching between a bounded controller, for which a region of closed–loop stability under constraints is explicitly characterized, and a predictive controller that minimizes a quadratic performance objective subject to constraints. The state–feedback controllers are combined with a Luenberger state observer that guarantees arbitrarily fast decay of the state estimation error. Switching laws, that monitor the evolution of the closed–loop state estimates, are derived to orchestrate the transition between the two controllers, in a way that guarantees asymptotic closed–loop stability for all initial conditions within arbitrarily large compact subsets of the bounded controller's state–feedback stability region, provided that the observer gain is sufficiently large. The hybrid control scheme is shown to provide a safety net for the practical implementation of MPC under output feedback, by providing a fall–back controller for which there exists a priori knowledge of a large set of initial conditions for which closed–loop stability is guaranteed.**

I. INTRODUCTION

Stabilization of dynamical systems using constrained control is an important problem that has been the subject of significant research work in control theory. Input constraints, typically a manifestation of the physical limitations on the capacity of control actuators, impose fundamental restrictions on our ability to steer the dynamics of the closed–loop system at will, and can cause severe performance deterioration, and even closed–loop instability, if not explicitly taken into account at the stage of controller design. Currently, model predictive control (MPC) is one of the few control methods available for handling constraints within an optimal control setting. Here the control action is obtained by solving repeatedly, on–line, a finite–horizon constrained open–loop optimal control problem. The popularity of this approach stems largely from its ability to handle, among other issues, multi–variable interactions, constraints on controls and states, and optimization requirements. Its success in many commercial applications is also well–documented in the literature (e.g., see [15]).

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Numerous research studies have investigated the stability properties of MPC and led to a plethora of MPC formulations that focus on closed–loop stability (e.g., see [13], [10] for extensive surveys of these developments). This progress notwithstanding, the issue of obtaining, a priori (i.e., before controller implementation), an explicit characterization of the region of constrained closed–loop stability for MPC remains to be adequately addressed. Part of the difficulty in this direction owes to the fact that the stability of finite–horizon model predictive controllers depends on a complex interplay between several factors such as the choice of the horizon length, the penalties in the performance index, and, for open–loop unstable plants, the fundamental feasibility of the optimization for a given initial condition. A priori knowledge of the stability region requires an explicit characterization of these interplays which is a difficult task in general. This difficulty can impact on the practical implementation of MPC by imposing the need for extensive closed–loop simulations over the whole set of possible initial conditions, to check for closed–loop stability, or by potentially limiting operation within an unnecessarily small neighborhood of the nominal equilibrium point.

The desire to implement control approaches that allow for an explicit characterization of their stability properties has motivated significant work on the design of stabilizing bounded control laws that provide large, explicitly defined regions of attraction for the constrained closed–loop system (e.g., see [8], [17], [3], [4]). Despite their well characterized stability and constraint–handling properties, the above controllers are not necessarily optimal with respect to an arbitrary performance criterion. In a previous work [5], we developed a hybrid control scheme, uniting bounded control with MPC, for the stabilization of linear systems with input constraints. The scheme was based on the idea of switching between a bounded controller, for which the region of constrained closed–loop stability is explicitly characterized, and a model predictive controller that minimizes a given performance objective subject to constraints. Switching laws were derived to orchestrate the transition between the two controllers in a way that reconciles the tradeoffs between their respective stability and optimality properties, and guarantees asymptotic closed–loop stability for all initial

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conditions within the stability region of the bounded controller. The hybrid scheme was shown to provide a safety net for the practical implementation of MPC under state–feedback.

In most practical applications, however, not all states are available for measurement and, therefore, this issue needs to be addressed in implementing the control strategy. Output feedback stabilization of constrained systems has been the subject of several research studies. Examples include scalar output feedback control of linear systems [16], stability analysis of a composite system comprising of a moving horizon regulator and a moving horizon observer for control of nonlinear systems [12] and moving horizon estimation as an extension of Kalman filtering, for constrained and nonlinear processes [14]. However, in these works the stability region of the constrained closed–loop system is not explicitly characterized.

Motivated by the above considerations, we propose in this paper a controller switching strategy that extends the hybrid control structure in [5] to the case of output feedback. The guiding principle in realizing this strategy is to embed the implementation of MPC within the output feedback stability region of the bounded controller and design both the state estimator and the supervisory switching logic in a way that ensures a stabilizing transition to the fall–back controller in the event that MPC is unable to achieve closed–loop stability. The reader may refer to [11] for further results and examples, as well as a discussion on the robustness properties of the proposed control strategy to model uncertainty and measurement noise.

II. PRELIMINARIES

In this work, we consider the problem of output feedback stabilization of continuous–time linear time–invariant (LTI) systems with input constraints, with the following state–space description:

$$\dot{x}(t) \ = \ Ax(t) + Bu(t) \tag{1}$$

$$y(t) \ = \ Cx(t) \tag{2}$$

$$u(t) \ \in \ \mathcal{U} \subset \mathrm{IR}^m \tag{3}$$

where $x = [x_1, \cdots, x_n]^T \in \mathrm{IR}^n$ denotes the vector of state variables, $y = [y_1, \cdots, y_k]^T \in \mathrm{IR}^k$ denotes the vector of output variables and $u = [u_1, \cdots, u_m]^T$ is the vector of manipulated inputs, taking values in a compact and convex subset, $\mathcal{U} := \{u \in \mathrm{IR}^m : \|u\| \leq u_{max}\}$, where $\|\cdot\|$ denotes the standard Euclidean norm of a vector and $u_{max} > 0$ is the magnitude of input constraints. The matrices A, B and C are constant $n \times n$, $n \times m$ and $k \times n$ matrices, respectively. The pair (A, B) is assumed to be controllable and the pair (C, A) is assumed to be observable. Throughout the paper, the notation $\| \cdot \|_Q$ refers to the weighted norm, defined by $\| x \|_Q^2 = x'Qx$ for all $x \in \mathrm{IR}^n$, where Q is a positive definite symmetric matrix and x' denotes the transpose of x.

We now review the design of the state observer. We will focus on the output feedback problem where measurements of $y(t)$ are assumed to be available for all t.

A. State observer design

For the system of Eqs.1-2, we consider a standard Luenberger observer of the form

$$\dot{\hat{x}} = A\hat{x} + Bu(t) + L(y - C\hat{x}) \tag{4}$$

where $\hat{x} = [\hat{x}_1, \cdots, \hat{x}_n]^T \in \mathrm{IR}^n$ denotes the vector of state estimates, L is a constant $n \times k$ matrix that multiplies the discrepancy between the actual and estimated outputs. Under the state observer of Eq.4, the estimation error in the closed–loop system, defined as $e = x - \hat{x}$, evolves, independently of the controller, according to the following equation:

$$\dot{e} = (A - LC)e \tag{5}$$

The pair $(C,\ A)$ is assumed to be observable in the sense that the observer gain matrix L can be chosen such that the norm of the estimation error in Eq.5 evolves according to $\|e(t)\| \leq \kappa(\beta)\|e(0)\|\exp(-\beta t)$, where $-\beta < 0$ is the largest eigenvalue of $A - LC$ and $\kappa(\beta)$ is a polynomial function of β. In this manner, the dynamics of the error equation can be manipulated at will by appropriate choice of the observer gain (i.e. pole placement of the matrix $A - LC$).

Remark 1: Referring to the state observer of Eq.4, it should be noted that the results presented in this work are not restricted to this particular class of observers. Any other observer that allows us to control the rate of decay of the estimation error at will, can be used. Our choice of using this particular observer design is motivated by the fact that it provides a transparent relationship between the temporal evolution of the estimation error bound and the observer parameters. For example, this design guarantees convergence of the state estimates in a way such that for larger values of β, the error decreases faster. As we discuss later (see section V), the ability to ensure a sufficiently fast decay of the estimation error is necessary to guarantee closed–loop stability under output feedback control. This requirement or constraint on the error dynamics is present even when other estimation schemes, such as moving horizon observers, are used (e.g., see [12]) to ensure closed–loop stability. For such observers, however, it is difficult, in general, to obtain a transparent relationship between the tunable observer parameters and the error decay rate.

III. MODEL PREDICTIVE CONTROL

In MPC, the control action at state x and time t is conventionally obtained by solving, on–line, a finite horizon optimal control problem of the form

$$P(x, t) \ : \ \min\{J(x, t, u(\cdot)) | u(\cdot) \in S\} \tag{6}$$

where $S = S(t, T)$ is the family of piecewise continuous functions (functions continuous from the right), with period Δ, mapping $[t, t+T]$ into $\mathcal{U}$ and T is the specified horizon.

A control $u(\cdot)$ in S is characterized by the sequence $\{u[k]\}$ where $u[k] := u(k\Delta)$. A control $u(\cdot)$ in S satisfies $u(t) = u[k]$ for all $t \in [k\Delta, (k+1)\Delta)$. The performance index is given by $J(x, t, u(\cdot)) =$

$$\int_t^{t+T} \left[\|x^u(s; x, t)\|_Q^2 + \|u(s)\|_R^2 \right] ds + F(x(t+T)) \tag{7}$$

where R and Q are strictly positive–definite, symmetric matrices, $x^u(s; x, t)$ denotes the solution of Eq.1, due to control u, with initial state x at time t, and $F(\cdot)$ denotes the terminal penalty. In addition to penalties on the state and control action, the objective function may also include penalties on the rate of change of the input, reflecting limitations on actuator speed (e.g., a large valve requiring few seconds to change position). The minimizing control $u^0(\cdot) \in S$ is then applied to the plant over the interval $[k\Delta, (k+1)\Delta)$ and the procedure is repeated indefinitely. This defines an implicit model predictive control law

$$M_s(x) \ := \ u^0(t; x, t) \tag{8}$$

It is well known that, even when complete measurements of the state are available, the control law defined by Eqs.6-8 is not necessarily stabilizing. To achieve closed–loop stability, early versions of MPC focused on tuning the horizon length, T, and/or increasing the terminal penalty (see [1] for a survey of these approaches), while, in more recent formulations, closed–loop stability is typically addressed by introducing penalties and constraints on the state at the end of the finite optimization horizon (see [10] for surveys of different constraints proposed in the literature and the concomitant theoretical issues). The implicit nature of the predictive control law obtained through repeated on–line optimization, however, limits our ability to obtain, a priori (i.e. before controller implementation), an explicit characterization of the admissible initial conditions starting from where a given predictive controller (with a fixed performance index and horizon length) is guaranteed to enforce asymptotic closed–loop stability. The difficulties encountered in characterizing the stability region under state–feedback carry over to the case of output feedback, where the lack of state measurements requires that the control action be computed using the state estimates. Feasibility of the MPC optimization problem based on the state estimates, however, does not guarantee closed–loop stability or even the continued feasibility of the optimization problem based on state estimates.

IV. Bounded Lyapunov-based control

We first present the design of the state–feedback bounded controller and then characterize the stability properties of the composite system comprising of the state–feedback bounded controller and the state estimator presented in section II.

A. State feedback controller design

Consider the Lyapunov function candidate $V = x'Px$, where P is a positive–definite symmetric matrix that satisfies the Riccati equation

$$A'P + PA - PBB'P = -\bar{Q} \tag{9}$$

for some positive–definite matrix $\bar{Q}$. Using this Lyapunov function, we can construct, using a modification of Sontag's formula for bounded controls proposed in [8] (see also [3]), the following bounded nonlinear controller

$$u(x) \ = \ -2k(x)B'Px \ := \ b(x) \tag{10}$$

where $k(x) =$

$$\left(\frac{L_f^* V + \sqrt{\left(L_f^* V\right)^2 + (u_{max}\|(L_g V)'\|)^4}}{\|(L_g V)'\|^2 \left[1 + \sqrt{1 + (u_{max}\|(L_g V)'\|)^2}\right]} \right) \tag{11}$$

with $L_f^* V = x'(A'P + PA)x + \rho x'Px$, $(L_g V)' = 2B'Px$, $\rho > 0$, and $u_{max} > 0$ is the size of input constraints. This controller is continuous everywhere in the state space and smooth away from the origin. Using a Lyapunov argument, one can show that whenever the closed–loop state trajectory evolves within the state–space region described by the set:

$$\Phi(u_{max}) \ = \ \{x \in \mathbb{R}^n : L_f^* V < u_{max}\|(L_g V)'\|\} \tag{12}$$

the resulting control action respects the constraints (i.e., $\|u\| \leq u_{max}$) and enforces, simultaneously, the negative–definiteness of the time–derivative of the Lyapunov function, $\dot{V} < 0$, along the trajectories of the closed–loop system.

An estimate of the stability region is generated by constructing an invariant subset of $\Phi(u_{max})$, using the level sets of V (see chapter 4 in [7] for details), i.e.

$$\Omega(u_{max}) \ = \ \{x \in \mathbb{R}^n : x'Px \leq c_{max}\} \tag{13}$$

where $c_{max} > 0$ is the largest number for which all nonzero elements of $\Omega(u_{max})$ are contained within $\Phi(u_{max})$. To simplify notation, we will suppress the dependence of the sets Ω and Φ on u_{max} in the remainder of the paper.

B. Stability properties under output feedback

The lack of state measurements motivates the use of a state estimator that provides estimates of the state variables. When a state estimator of the form of Eq.4 is used, the resulting closed–loop system is composed of a cascade, between the error system and the plant, of the form

$$\begin{aligned} \dot{x} \ &= \ Ax + Bu(x - e) \\ \dot{e} \ &= \ (A - LC)e \end{aligned} \tag{14}$$

Note that the values of the states used in the controller contain errors. The state–feedback stability region, therefore, is not exactly preserved under output feedback. However, by exploiting the error dynamics of Eq.5, it is possible to recover arbitrarily large compact subsets of the state–feedback stability region, provided that the poles of the observer are placed sufficiently far in the left half of the complex plane (which can be accomplished by choosing the observer gain parameter β sufficiently large). This idea is consistent with earlier results on semi–global output feedback stabilization of unconstrained systems using high–gain observers (e.g., see [18], [9], [2]) and is formalized in Propositions 1 and 2 below.

Proposition 1: *Consider the constrained LTI system of Eqs.1-3 under the bounded control law of Eqs.10-11. Then, there exists a positive real number, e_m, such that if $x(0) \in \Omega$ and $\|e(t)\| \leq e_m \; \forall \; t \geq 0$, then $x(t) \in \Omega \; \forall \; t \geq 0$.*

Proof of Proposition 1:
Part 1: Substituting the state–feedback control law of Eq.10-11 into the system of Eq.1 and evaluating the time–derivative of the Lyapunov function along the closed–loop trajectories, it can be shown that

$$\begin{aligned}\dot{V}(x) &= L_f V(x) + L_g V(x) u(x) \\ &< \frac{-\rho x' P x}{\left[1 + \sqrt{1 + (2u_{max}\|B'Px\|)^2}\right]}\end{aligned} \tag{15}$$

for all $x \in \Phi(u_{max})$, and hence for all $x \in \Omega$, where Φ and Ω were defined in Eq.12 and Eq.13, respectively. Note that the denominator term in Eq.15 is bounded on Ω. Therefore, there exists a positive real number, ρ^*, such that

$$\dot{V} < -\rho^* x' P x \tag{16}$$

for all $x \in \Omega$, which implies that the origin of the closed–loop system, under the control law of Eqs.10-11, is asymptotically stable, with Ω as an estimate of the domain of attraction.

Part 2: In this part, we analyze the behavior of $\dot{V}$ on the boundary of Ω (i.e. the level set described by $V(x) = c_{max}$) under bounded measurement errors, $\|e\| \leq e_m$. To this end, consider $\dot{V}(x)$

$$\begin{aligned} &= L_f V(x) + L_g V(x) u(x-e) \\ &= L_f V(x) + L_g V(x) u(x) + L_g V(x)\left[u(x-e) - u(x)\right] \\ &\leq -\rho^* c_{max} + \|L_g V\| \|u(x-e) - u(x)\| \\ &\leq -\rho^* c_{max} + M\|u(x-e) - u(x)\| \end{aligned} \tag{17}$$

for all $x = V^{-1}(c_{max})$, where $M = \max_{V(x)=c_{max}} (\|L_g V(x)\|)$ (note that M exists since $\|L_g V(\cdot)\|$ is continuous and the maximization is considered over a closed set). Since $u(\cdot)$ is continuous, then given any positive real number r such that $\mu = \dfrac{\rho^* c_{max} - r}{M} > 0$, there exists $e_m > 0$ such that if $\|(x-e)-x\| = \|e\| \leq e_m$, then $\|u(x-e) - u(x)\| \leq \mu$ and, consequently,

$$\dot{V}(x) \;\leq\; -\rho^* c_{max} + M\mu \;=\; -r \;<\; 0 \tag{18}$$

for all $x = V^{-1}(c_{max})$. This implies that for all measurement errors such that $\|e\| \leq e_m$, we have $\dot{V} < 0$ on the boundary of Ω. Therefore, under the bounded controller, any closed–loop state trajectory, starting within Ω, cannot escape this region, i.e. $x(t) \in \Omega \; \forall t \geq 0$. This completes the proof of the proposition.

Proposition 2: *Consider the constrained LTI system of Eqs.1-3, the state observer of Eq.4 and the bounded control law of Eqs.10-11. Then, given any positive real number, δ_b, such that $\Omega_b = \{x \in \mathbb{R}^n : \|x\|_P^2 \leq \delta_b\} \subset \Omega$, there exists a positive real number β^* such that if $\|x(0)\|_P^2 \leq \delta_b$, $\|\hat{x}(0)\|_P^2 \leq \delta_b$, $\beta \geq \beta^*$, the origin of the constrained closed–loop system is asymptotically stable.*

Proof of Proposition 2:
Part 1: From the discussion in Section II-A, we have that the error dynamics are given by $\dot{e} = (A - LC)e$, where $A - LC$ is Hurwitz, and all the eigenvalues of $A - LC$ satisfy $\lambda \leq -\beta$. It then follows that an estimate of the form $\|e(t)\| \leq \kappa(\beta)\|e(0)\|\exp(-\beta t)$ holds for some $\kappa > 0$, for all $t \geq 0$. Given any positive real number, δ_b, such that $\Omega_b = \{x \in \mathbb{R}^n : \|x\|_P^2 \leq \delta_b\} \subset \Omega$, let $T_{min} = \min\{t \geq 0 : V(x(0)) = \delta_b, V(x(t)) = c_{max}, u(t) \in \mathcal{U}\}$ (i.e. T_{min} is the shortest time in which the closed–loop state trajectory can reach the boundary of Ω starting from the boundary of Ω_b using *any* admissible control action). Further, let $e_{max}(0) = \max_{x,y \in \Omega_b} \|x - y\|$ (i.e., $e_{max}(0)$ is the largest possible initial error given that both the states and state estimates are initialized within Ω_b). Choose T_d such that $0 < T_d < T_{min}$ and let β^* be such that

$$e_m \leq \kappa(\beta^*) e_{max}(0) \exp(-\beta^* T_d) \tag{19}$$

(we can find such a β^* since $\kappa(\beta)$ is polynomial in β). For any choice of $\beta \geq \beta^*$, therefore, it follows that $\|e(T_{min})\| \leq e_m$, since $\|e(T_d)\| \leq e_m$, $T_{min} \geq T_d$ and the bound on the norm of the estimation error decreases monotonically with time for all $t \geq 0$. This implies that the norm of the estimation error decays to a value less than e_m before the closed–loop state trajectory, starting within Ω_b, could reach the boundary of Ω. It then follows from Proposition 1 that the closed–loop state trajectory cannot escape Ω for all $t \geq 0$, i.e. the trajectories are bounded, $\|x(t)\|_p^2 \leq c_{max} \; \forall \; t \geq 0$.

Part 2: To prove asymptotic stability, we note, from Eq.17, that for all $x \in \Omega$

$$\begin{aligned}\dot{V}(x) &= L_f V(x) + L_g V(x) u(x-e) \\ &\leq -\rho^* \|x\|_p^2 + M\|u(x-e) - u(x)\|\end{aligned} \tag{20}$$

The term $\|u(x-e) - u(x)\|$ is continuous and vanishes when $e = 0$. Therefore, since both x and e are bounded, there exists a positive real number $\varphi(e_m)$ such that $\|u(x-e) - u(x)\| \leq \varphi\|e\|$ for all $\|x\|_p^2 \leq c_{max}$, $\|e\| \leq e_m$. Substituting this estimate into Eq.20, we obtain

$$\begin{aligned}\dot{V}(x) &\leq -\rho^* \|x\|_p^2 + M\varphi\|e\| \\ &\leq -\frac{\rho^*}{2}\|x\|_p^2 \quad \forall \;\; \|x\|_p \geq \sqrt{\frac{2M\varphi\|e\|}{\rho^*}} := \gamma_1(\|e\|)\end{aligned} \tag{21}$$

where $\gamma_1(\cdot)$ is a class $\mathcal{K}$ function of its argument. The above inequality implies that $\dot{V}$ is negative outside some residual set whose size depends on $\|e\|$. Using the result of Theorem 5.1-Corollary 5.2 in [7], this implies that, for any $x(0) \in \Omega_b$, there exists a class $\mathcal{KL}$ function $\bar{\beta}(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma_2(\cdot)$, such that

$$\|x(t)\| \;\leq\; \bar{\beta}(\|x(0)\|, t) + \gamma_2(\sup_{\tau\geq 0}\|e(\tau)\|) \;\forall\; t \geq 0 \tag{22}$$

implying that the x subsystem of Eq.14, with e as input, is input-to-state stable (recall, from Proposition 1, that $x(t) \in \Omega_b \;\forall t \geq 0$). Noting also that the e subsystem of Eq.14 is asymptotically stable ($\lim\limits_{t\to\infty}\|e(t)\| = 0$), and using Lemma 5.6 in [7], we get that the interconnected system of Eq.14 is asymptotically stable. This completes the proof of Proposition 2.

Remark 2: The only assumption on C is that the pair (C, A) is observable. Understandably, the estimates can become very large before converging to the true values. This, however, does not pose a problem in our design because: (a) the physical constraints on the manipulated input eliminates occurrence of instability due to peaking of the state estimates (they prevent transmission of peaking to the plant), and (b) by "stepping back" from the state–feedback stability region and choosing an appropriate value for β, the design ensures that the system states cannot leave the region of stability for the bounded controller before the estimation errors have gone below the permissible value.

Remark 3: In principle, the stability region under output feedback, Ω_b, can be chosen as close as desired to Ω by increasing the observer gain parameter β. However, it is well known that large observer gains can amplify measurement noise and induce poor performance (see Section VI for how this issue is addressed in observer implementation). This points to a fundamental tradeoff that cannot be resolved by simply changing the estimation scheme. For example, while one could replace the high–gain observer design with other observer designs (e.g., a moving horizon estimator) to get a better handle on measurement noise, it is difficult in such schemes to obtain an explicit relationship between the observer tuning parameters and the output feedback stability region.

Proposition 3: *Consider the constrained LTI system of Eqs.1-3, the state observer of Eq.4 and the bounded control law of Eqs.10-11. Let* $T_d^* := \dfrac{1}{\beta}\ln\left(\dfrac{\kappa(\beta)e_{max}(0)}{\epsilon\sqrt{c_{max}/\lambda_{max}(P)}}\right)$ *for some* $0 < \epsilon < 1$. *Then there exists a positive real number* $\delta_s^* < c_{max}$ *such that for all* $\delta_s \leq \delta_s^*$, *and for all* $t \geq T_d^*$, $\hat{x}'(t)P\hat{x}(t) \leq \delta_s \Longrightarrow x'(t)Px(t) \leq c_{max}$.

Proof of Proposition 3: Since the error dynamics obey a bound of the form $\|e(t)\| \leq \kappa(\beta)\|e_{max}(0)\|\exp(-\beta t)$, substituting T_d^* into this expression yields $\|e(T_d^*)\| := e^* \leq \epsilon\sqrt{c_{max}/\lambda_{max}(P)}$. Then, for all $t \geq T_d^*$, if $\hat{x}'(t)P\hat{x}(t) \leq \delta_s$ for some $\delta_s > 0$, we can write

$$\begin{aligned} x'(t)Px(t) &= (\hat{x}(t)+e(t))'P(\hat{x}(t)+e(t)) \\ &= \hat{x}'(t)P\hat{x}(t) + 2\hat{x}'(t)Pe(t) + e'(t)Pe(t) \\ &\leq \delta_s + 2\|P\hat{x}(t)\|\|e(t)\| + \|e(t)\|_p^2 \\ &\leq \delta_s + 2\sqrt{\frac{\lambda_{max}(P^2)\delta_s}{\lambda_{min}(P)}}\; e^* + \lambda_{max}(P)e^{*2} \\ &:= f(\delta_s) \end{aligned} \tag{23}$$

Note that the right hand side of the last inequality is a continuous, monotonically increasing function of $\delta_s \geq 0$, with $f(0) = \epsilon^2 c_{max} < c_{max}$ and $f(c_{max}) > c_{max}$. This implies that there exists $0 < \delta_s^* < c_{max}$ such that, for all $\delta_s \leq \delta_s^*$, $f(\delta_s) \leq c_{max}$, i.e. $x'(t)Px(t) \leq c_{max}$. This completes the proof of the proposition.

V. Hybrid predictive Output feedback control

Theorem 1: *Consider the constrained LTI system of 1-3, the bounded controller of Eqs.10-11, the state estimator of Eq.4 and the MPC law of Eqs.6-8. Let* $x(0) \in \Omega_b$, $\hat{x}(0) \in \Omega_b$, $\beta \geq \beta^*$, $\delta_s \leq \delta_s^*$, *and* $\Omega_s(T_d^*) = \{x \in \mathbb{R}^n : x'Px \leq \delta_s(T_d^*)\}$, *where* Ω_b, β^* *were defined in Proposition 2, and* Ω_s, δ_s^*, T_d^* *were defined in Proposition 3. Let* $T_m \geq \max\{T_d, T_d^*\}$ *(where* T_d *was defined in Eq.19) be the earliest time for which* $\hat{x}(T_m) \in \Omega_s$, *and let* $T_f > T_m$ *be the earliest time for which* $\dot{V}(\hat{x}(T_f)) \geq 0$. *Then, the following switching rule*

$$i(t) \;=\; \left\{\begin{array}{ll} 1, & 0 \leq t < T_m \\ 2, & T^* \leq t < T_f \\ 1, & t \geq T_f \end{array}\right\} \tag{24}$$

where $i(t) = 1 \Leftrightarrow u_i(\hat{x}(t)) = b(\hat{x}(t))$ *and* $i(t) = 2 \Leftrightarrow u_i(\hat{x}(t)) = M_s(\hat{x}(t))$, *asymptotically stabilizes the origin of the closed–loop system.*

Proof of Theorem 1: Let $x(0) \in \Omega_b$, $\hat{x}(0) \in \Omega_b$, $\beta \geq \beta^*$ (see Proposition 2), and $\delta_s \leq \delta_s^*$ (see Proposition 3). Since $T_m \geq T_d$ and $\|e(T_d)\| \leq e_m$ (see part 1 of the proof of Proposition 2), we have that $\|e(T_m)\| \leq e_m$. Since only the bounded controller is implemented, i.e. $i(t) = 1$, for $0 \leq t < T_m$, it follows from Proposition 1 that $x(t) \in \Omega$ for all $0 \leq t < T_m$ (or that $x(T_m^-) \in \Omega$). This fact, together with the continuity of the solution of the switched closed–loop system – which follows from the fact that the right hand side of Eqs.1-3 is continuous in x and piecewise continuous in time since only a finite number of switches is allowed over any finite time interval – implies that, upon switching (instantaneously) to MPC at $t = T_m$, we have $x(T_m) \in \Omega$. For $t \geq T_m$, one of the following two scenarios is possible:
Case 1: Consider first the case when $T_f = \infty$. From the definition of T_f in Theorem 1, it follows that $u(\hat{x}(t)) = M_s(\hat{x}(t))$ and $\dot{V}(\hat{x}(t)) < 0$ for all $t \geq T_m$. This implies that $\hat{x}(t) \in \Omega_s$ for all $t \geq T_m$ and, consequently, from the definition of T_m (note that $T_m \geq T_d^*$ where T_d^* was defined in Proposition 3), that $x(t) \in \Omega$ for all $t \geq T_m$ (i.e. the closed–loop trajectories under MPC are bounded).

Furthermore, we have that $\lim_{t\to\infty} \hat{x}'(t)P\hat{x}(t) = 0$; and, therefore, $\lim_{t\to\infty} x'(t)Px(t) = 0$ since $\hat{x}(t) = x(t) - e(t)$ and $\lim_{t\to\infty} \|e(t)\| = 0$. Therefore, the origin of the switched closed–loop system is asymptotically stable.

Case 2: Consider now the case when $T_f < \infty$. From the analysis in case 1 above, we have that $x(t) \in \Omega \quad \forall\ 0 \leq t < T_f$ (or that $x(T_f^-) \in \Omega$). This fact, together with the continuity of the solution of the switched closed–loop system, implies that, upon switching (instantaneously) from MPC to the bounded controller at $t = T_f$, we have $x(T_f) \in \Omega$ and $u(t) = b(\hat{x}(t))$ for all $t \geq T_f$. We also have $\|e(t)\| \leq e_m$ for all $t \geq T_f$ since $T_f > T_d$. Therefore, from Proposition 1 it is guaranteed that $x(t) \in \Omega$ for all $t \geq T_f$. Finally, following the same arguments presented in Part 2 of the proof of Proposition 2, it can be shown that $\|x(t)\| \to 0$ as $t \to \infty$, which together with boundedness of the state, establishes asymptotic stability of the origin of the closed–loop system. This completes the proof of Theorem 1.

Remark 4: The proposed approach does not provide any information as to whether the predictive controller itself is asymptotically stabilizing or not, starting from any initial condition within Ω_b. The approach does not turn Ω_b into a stability region for MPC; instead, it turns Ω_b into a stability region for the switched closed–loop system. The value of this can be understood in light of the difficulty in obtaining, a priori, an analytical characterization of the set of admissible initial conditions that MPC can steer to the origin in the presence of input constraints. Given this difficulty, by using the bounded controller as a fall–back controller, the switching scheme of Theorem 1 allows us to safely initialize the closed–loop system anywhere within Ω_b with the guarantee that the bounded controller can intervene, at any moment, to preserve closed–loop stability in the event that MPC is unable to achieve it (due, for example, to improper tuning of MPC).

Remark 5: Note that, once MPC is switched in, if $V(\hat{x})$ continues to decrease monotonically, then the predictive controller will be implemented for all $t \geq T_m$. In this case, the optimal performance of the predictive controller is practically recovered. Note also, that in this approach, the state–feedback predictive controller design is not required to be robust with respect to state measurement errors (see [6] for examples when MPC is not robust) because even if it is not robust, closed–loop stability can always be guaranteed by switching back to the bounded controller (within its associated stability region) which provides the desired robustness with respect to the measurement errors.

Remark 6: The hybrid control structure proposed in this work is not restricted to the conventional MPC formulation considered in Theorem 1, and can be used to provide a safety net for the implementation of advanced MPC formulations that employ stability constraints (e.g., terminal inequality or equality constraints). For these formulations, however, the implementation of MPC depends on whether the optimization problem, subject to the stability constraints, is feasible or not (note that a priori knowledge of the set of initial conditions and/or horizon lengths that guarantee feasibility is difficult to ascertain, even under state–feedback). This observation suggests that, when these formulations are to be used within the hybrid control structure, the switching logic of Theorem 1 should be slightly modified to accommodate the issue of feasibility. In particular, after $\hat{x}$ enters Ω_s, the supervisor needs to check feasibility of MPC (based on $\hat{x}$) and switch to MPC only if it is feasible. If no feasible solution is obtained, then the bounded controller is kept active until such time that MPC becomes feasible. Once MPC is switched in, the supervisor continues to check feasibility as well as $\dot{V}(\hat{x})$ under MPC. At the earliest time that either MPC becomes infeasible again or $\dot{V}(\hat{x})$ becomes zero, the supervisor switches back to the bounded controller.

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