Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Infinite Horizon Robustly Stable Seismic Protection of Cable-Stayed Bridges Using Cost Cumulants

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***Abstract*— The basic idea behind infinite horizon state feedback k-Cost-Cumulant (kCC) control is to seek a constant state feedback law which minimizes the value of a finite linear combination of the first k cost cumulants of a traditional integral quadratic cost associated with a linear stochastic system, when there is no specification of a large terminal time. The paper begins with the development of matrix algebraic equations for the cost cumulants. These equations permit the incorporation of classes of linear feedback controllers for linear dynamical systems defined on infinite horizon. The infinite horizon control problem with the optimization goal of a finite linear combination of the first k cost cumulants is then stated. Because the control optimization problem at hand involves matrix equality constraints, the optimal feedback solution is investigated by utilizing a Lagrange multipliers technique. The efficacy and applicability of this control paradigm, based upon the first three cost cumulants, are demonstrated on the Second Earthquake Generation Benchmark for Response Control of Cable-Stayed Bridges [1]. The simulation results indicate that the cost cumulants in the infinite horizon case offer significant improvements in robust stability margin while keeping comparable levels of structural performance when comparing to those of the baseline LQG in the benchmark.**

I. PRELIMINARIES

Throughout the paper, the set of symmetric $n \times n$ matrices with real elements is denoted by $\mathbb{S}^n$. We consider controlling a linear dynamical system modeled by the stochastic differential equation

$$dx(t) = (Ax(t) + Bu(t))dt + Gdw(t), \;\; x(t_0) = x_0, \quad (1)$$

where the system noise $w(t) \in \mathbb{R}^p$ is a standard stationary Wiener process on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with correlation of increments

$$E\left\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\right\} = W|\tau - \xi|, \quad W > 0.$$

Assume $x_0 \in \mathbb{R}^n$ is known and matrix coefficients $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times p}$. The admissible control input $u(t)$ belongs to the Hilbert space $L^2_{\mathcal{F}}(\Omega; \mathcal{C}([t_0, \infty); \mathbb{R}^m))$ so that the system state $x(t)$ is in $L^2_{\mathcal{F}}(\Omega; \mathcal{C}([t_0, \infty); \mathbb{R}^n))$ i.e., $E\left\{\int_{t_0}^{\infty} x^T(\tau)x(\tau)d\tau\right\} < \infty$. For the given $(t_0, x_0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, associate with the system (1) a traditional integral quadratic form (IQF) random cost $J : L^2_{\mathcal{F}}(\Omega; \mathcal{C}([t_0, \infty); \mathbb{R}^m)) \mapsto \mathbb{R}^+$ such that

$$J(u) = \int_{t_0}^{\infty} \left[x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau)\right] d\tau\,, \quad (2)$$

where the weighting matrices $Q \in \mathbb{S}^n$ and $R \in \mathbb{S}^m$ are positive semidefinite with R invertible.

This work was supported in part by the Center of Applied Mathematics and the Frank M. Freimann Chair in Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 U.S.A.

Definition 1. *A feedback control $u(t) = k(t, x(t))$ is called stabilizing if the system (1) defined on $[t_0, \infty)$ by*

$$dx(t) = (Ax(t) + Bk(t, x(t)))dt + Gdw(t)\,, \quad x(t_0) = x_0\,,$$

is bounded input-bounded state stable. In addition, if $w = 0$, then the origin $(x = 0)$ is asymptotically stable.

Consider the class of linear state-feedback constant control laws given as follows.

Definition 2. *A feedback gain $K \in \mathbb{R}^{m \times n}$ is stabilizing if the state-feedback control law*

$$k(t, x(t)) = K\,x(t)\,, \qquad t \in [t_0, \infty)\,, \quad (3)$$

is stabilizing.

The induced system thus becomes

$$dx(t) = (A + BK)x(t)dt + Gdw(t)\,, \qquad x(t_0) = x_0\,,$$

in which the state $x(t) \in L^2_{\mathcal{F}}(\Omega; \mathcal{C}([t_0, \infty); \mathbb{R}^n))$.

Theorem 1. Cost Cumulants.
Fix $k \in \mathbb{Z}^+$. Suppose that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times p}$, $Q \in \mathbb{S}^n$ is positive semidefinite and that $R \in \mathbb{S}^m$ is positive definite. If (A, B) is stabilizable, there exists a stabilizing feedback gain $K \in \mathbb{R}^{m \times n}$ such that all eigenvalues of the closed-loop matrix $A+BK$ have negative real parts. The kth cost cumulant is given by

$$\kappa_{\infty,k} = \mathrm{Tr}\left\{\mathcal{H}_k GWG^T\right\}\,, \quad (4)$$

where the cumulant-forming matrices $\{\mathcal{H}_i \geq 0\}_{i=1}^{k}$ satisfy the algebraic equations

$$0 = (A + BK)^T\mathcal{H}_1 + \mathcal{H}_1(A + BK) + K^TRK + Q\,, \quad (5)$$

$$0 = (A + BK)^T\mathcal{H}_i + \mathcal{H}_i(A + BK) + \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!}\mathcal{H}_j GWG^T\mathcal{H}_{i-j}\,, \quad 2 \leq i \leq k\,. \quad (6)$$

Proof. See [5].

II. kCC CONTROL STATEMENTS

Let the right members of the algebraic equations (5)-(6) be denoted by the mappings, $1 \le i \le k$,

$$\mathcal{F}_i : (\mathbb{S}^n)^k \times \mathbb{R}^{m\times n} \mapsto \mathbb{S}^n ,$$

with the actions

$$\begin{aligned}\mathcal{F}_1(\mathcal{H}, K) &\triangleq (A+BK)^T\mathcal{H}_1 + \mathcal{H}_1(A+BK) \\ &\quad + K^TRK + Q , \\ \mathcal{F}_i(\mathcal{H}, K) &\triangleq (A+BK)^T\mathcal{H}_i + \mathcal{H}_i(A+BK) \\ &\quad + \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!}\mathcal{H}_j GWG^T \mathcal{H}_{i-j}, \quad 2 \le i \le k,\end{aligned}$$

where the k-tuple matrix variable $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_k)$. Then the product system of (5)-(6) under the obvious definition $\mathcal{F} \triangleq \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ is written as follows

$$\mathcal{F}(\mathcal{H}, K) = 0 .$$

$\mathcal{F}$ defines the object which is to be controlled and K is the control.

Definition 3. Well Posed Control Law.
A feedback gain K is well posed if the equation

$$\mathcal{F}(\mathcal{H}, K) = 0 \tag{7}$$

has a unique solution for $\mathcal{H}$.

On the infinite horizon, however, we require a stronger property.

Definition 4. Admissible Feedback Control Law.
A well posed feedback control law $K(\mathcal{H})$ is admissible if the unique solution for $\mathcal{H}(K)$ satisfies $\{\mathcal{H}_i(K) \ge 0\}_{i=1}^k$ where ≥ 0 is to be understood as positive semidefinite.

We denote the admissible set of feedback control laws by $\mathcal{Q}$. On $\mathcal{Q}$, we now introduce a performance index.

Definition 5. Performance Index.
Fix $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$. The performance index in the infinite horizon state feedback kCC control problem is defined by the mapping

$$\phi_\infty : \mathcal{Q} \mapsto \mathbb{R}^+$$

with the action

$$\phi_\infty(\mathcal{H}(K)) \triangleq \sum_{i=1}^k \mu_i \kappa_{\infty,i} = \sum_{i=1}^k \mu_i \mathrm{Tr}\left\{\mathcal{H}_i(K)GWG^T\right\} .$$

The infinite horizon state feedback kCC control problem is formulated as a constrained optimization problem.

Definition 6. Infinite Horizon kCC Optimization.
For the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$, the infinite horizon kCC control optimization problem is defined as

$$\min_{K\in\mathcal{Q}} \left\{ \phi_\infty(\mathcal{H}(K)) = \sum_{i=1}^k \mu_i \mathrm{Tr}\left\{\mathcal{H}_i(K)GWG^T\right\} \right\} \tag{8}$$

subject to the equality constraints

$$\mathcal{F}(\mathcal{H}, K) = 0 .$$

It is necessary to abstract the infinite horizon kCC control optimization problem so that the general Lagrange multiplier theory can be applied. The natural approach is to treat the product constraint $\mathcal{F}(\mathcal{H}, K)$ as a constraint connecting $\mathcal{H}$ and K. In what follows, the necessary conditions for optimality according to Lagrange multiplier theory in [3] are restated in the context of kCC control, which will be used in deriving an optimal feedback solution.

Theorem 2. Regularity Condition.
Let $k \in \mathbb{Z}^+$ and $\mathcal{F}(\mathcal{H}, K) \in (\mathbb{S}^n)^k$ where $K \in \mathcal{Q}$. Let $(\mathcal{H}^0, K^0) \in (\mathbb{S}^n)^k \times \mathcal{Q}$ be such that $\mathcal{F}(\mathcal{H}^0, K^0) = 0$. Then $(\mathcal{H}^0, K^0)$ is a regular point of the constraints

$$\mathcal{F}(\mathcal{H}, K) = 0$$

if, for $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_k) \in (\mathbb{S}^n)^k$,

$$\text{grad}\, \mathrm{Tr}\left\{\mathcal{F}(\mathcal{H}^0, K^0)\mathcal{P}^T\right\} = 0 \tag{9}$$

has the unique solution $\mathcal{P} = 0$.

Theorem 3. Necessary Condition for Optimality.
If $(\mathcal{H}^0, K^0) \in (\mathbb{S}^n)^k \times \mathcal{Q}$ is such that the functional $\phi_\infty(\mathcal{H})$ attains its minimum under the constraints $\mathcal{F}(\mathcal{H}, K) = 0$, and is a regular point of the hyper-surface $\mathcal{F}(\mathcal{H}, K) = 0$, then there is an element $\mathcal{P} \in (\mathbb{S}^n)^k$ such that the Lagrange functional

$$\mathcal{L}(\mathcal{H}, K, \mathcal{P}) = \phi_\infty(\mathcal{H}) + \mathrm{Tr}\left\{\mathcal{F}(\mathcal{H}, K)\mathcal{P}^T\right\}$$

is stationary at $(\mathcal{H}^0, K^0)$. Or, equivalently,

$$\text{grad}\ \phi_\infty(\mathcal{H}^0) + \text{grad}\ \mathrm{Tr}\left\{\mathcal{F}(\mathcal{H}^0, K^0)\mathcal{P}^T\right\} = 0 . \tag{10}$$

III. INFINITE HORIZON kCC CONTROL SOLUTION

For the given $k \in \mathbb{Z}^+$, the set of scalar coefficients $\mu = \{\mu_i \ge 0\}_{i=1}^k$ with $\mu_1 > 0$ and the well posed feedback gain $K \in \mathcal{Q}$, the constraints of state feedback kCC constrained optimization problem are restated, for convenience:

$$(A+BK)^T\mathcal{H}_1 + \mathcal{H}_1(A+BK) + K^TRK + Q = 0 ,$$

and, for $2 \le i \le k$,

$$\begin{aligned}&(A+BK)^T\mathcal{H}_i + \mathcal{H}_i(A+BK) \\ &\qquad + \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!}\mathcal{H}_j GWG^T \mathcal{H}_{i-j} = 0 .\end{aligned}$$

Consider the change of variables

$$\begin{aligned}\mathcal{Y}_1 &= \mu_1\mathcal{H}_1 + \cdots + \mu_k\mathcal{H}_k , \\ \mathcal{Y}_i &= \mathcal{H}_i , \qquad 2 \le i \le k .\end{aligned}$$

Then, for $\mathcal{Y} = (\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_k) \in (\mathbb{S}^n)^k$, the well posed feedback gain $K \in \mathcal{Q}$, and the Lagrange multiplier $\mathcal{P} =$

$(\mathcal{P}_1, \ldots, \mathcal{P}_k) \in (\mathbb{S}^n)^k$, the Lagrange functional may be defined as

$$\mathcal{L}(\mathcal{Y}, K, \mathcal{P}) \triangleq \mathrm{Tr}\left\{\mathcal{Y}_1 GWG^T\right\} + \mathrm{Tr}\left\{\mathcal{F}(\mathcal{Y}, K)\mathcal{P}^T\right\}$$

in which $\mathcal{F} \triangleq \mathcal{F}_1 \times \cdots \times \mathcal{F}_k$ and

$$\mathcal{F}_1(\mathcal{Y}, K) \triangleq (A+BK)^T\mathcal{Y}_1 + \mathcal{Y}_1(A+BK) + \mu_1 K^T RK + \mu_1 Q + \sum_{r=2}^{k} \mu_r \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!}\mathcal{Y}_s GWG^T \mathcal{Y}_{r-s} = 0\,,$$

and, for $2 \le i \le k$,

$$\mathcal{F}_i(\mathcal{Y}, K) \triangleq (A+BK)^T\mathcal{Y}_i + \mathcal{Y}_i(A+BK) + \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!}\mathcal{Y}_j GWG^T \mathcal{Y}_{i-j} = 0\,.$$

We first verify the regularity condition

$$\mathrm{grad}\ \mathrm{Tr}\left\{\mathcal{F}(\mathcal{Y}, K)\mathcal{P}^T\right\} = 0\,,$$

where components of the left member can be written as

$$\frac{\partial}{\partial \mathcal{Y}_1}\sum_{r=1}^{k} \mathrm{Tr}\left\{\mathcal{F}_r(\mathcal{Y}, K)\mathcal{P}_r^T\right\} = (A+BK)\mathcal{P}_1 + \mathcal{P}_1(A+BK)^T + \sum_{s=2}^{k} 2s\left(GWG^T\mathcal{Y}_{s-1}\mathcal{P}_s + \mathcal{P}_s\mathcal{Y}_{s-1}GWG^T\right)\,, \quad (11)$$

$$\frac{\partial}{\partial \mathcal{Y}_i}\sum_{r=1}^{k} \mathrm{Tr}\left\{\mathcal{F}_r(\mathcal{Y}, K)\mathcal{P}_r^T\right\} = (A+BK)\mathcal{P}_i + \mathcal{P}_i(A+BK)^T + \sum_{s=i+1}^{k} \frac{2s!}{i!(s-i)!}\Big(GWG^T\mathcal{Y}_{s-i}\mathcal{P}_s + \mathcal{P}_s\mathcal{Y}_{s-i}GWG^T\Big)\,, \quad 2 \le i \le k-1\,, \quad (12)$$

$$\frac{\partial}{\partial \mathcal{Y}_k}\sum_{r=1}^{k} \mathrm{Tr}\left\{\mathcal{F}_r(\mathcal{Y}, K)\mathcal{P}_r^T\right\} = (A+BK)\mathcal{P}_k + \mathcal{P}_k(A+BK)^T\,, \quad i = k\,, \quad (13)$$

$$\frac{\partial}{\partial K}\sum_{r=1}^{k} \mathrm{Tr}\left\{\mathcal{F}_r(\mathcal{Y}, K)\mathcal{P}_r^T\right\} = 2B^T\sum_{s=1}^{k}\mathcal{Y}_s\mathcal{P}_s + 2\mu_1 RK\mathcal{P}_1\,. \quad (14)$$

It is required to have the feedback gain K well posed at the minimum point. Thus, the regularity condition (13)

$$(A+BK)\mathcal{P}_k + \mathcal{P}_k(A+BK)^T = 0$$

is satisfied only when $\mathcal{P}_k = 0$. It is important to note that the effect of the solution $\mathcal{P}_k = 0$ and the fact that K is well posed ripple through the regularity conditions (12) which then yield $\mathcal{P}_i = 0$ for $2 \le i \le k-1$. Moreover, these solutions $\{\mathcal{P}_i = 0\}_{i=2}^{k}$ quickly simplify the regularity condition (11) into

$$(A+BK)\mathcal{P}_1 + \mathcal{P}_1(A+BK)^T = 0\,,$$

which is satisfied only if $\mathcal{P}_1 = 0$ when K is well posed. In view of $\mathcal{P} = 0$, the last regularity condition (14) holds true. In other words, the regularity conditions (11)-(14) are indeed satisfied by the unique solution $\mathcal{P} = 0$.

Second, the necessary condition for optimality employed to find an extremum $(\mathcal{Y}^*, K^*)$ of the Lagrange functional $\mathcal{L}(\mathcal{Y}, K, \mathcal{P})$ requires that

$$\mathrm{grad}\ \mathcal{L}(\mathcal{Y}^*, K^*, \mathcal{P}^*) = 0\,,$$

which is equivalent to having the following conditions held

$$\frac{\partial}{\partial \mathcal{Y}}\mathcal{L}(\mathcal{Y}^*, K^*, \mathcal{P}^*) = 0\,,$$
$$\frac{\partial}{\partial K}\mathcal{L}(\mathcal{Y}^*, K^*, \mathcal{P}^*) = 0\,,$$
$$\frac{\partial}{\partial \mathcal{P}}\mathcal{L}(\mathcal{Y}^*, K^*, \mathcal{P}^*) = 0\,,$$

where these conditions are evaluated at the point of local extremum $\mathcal{Y} = \mathcal{Y}^*$, $K = K^*$, and $\mathcal{P} = \mathcal{P}^*$. The stationary conditions on the dependent state variable $\mathcal{Y}$ are found as

$$(A+BK^*)\mathcal{P}_1^* + \mathcal{P}_1^*(A+BK^*)^T + GWG^T + \sum_{s=2}^{k} 2s\left(GWG^T\mathcal{Y}_{s-1}^*\mathcal{P}_s^* + \mathcal{P}_s^*\mathcal{Y}_{s-1}^*GWG^T\right) = 0\,, \quad (15)$$

for, $2 \le i \le k-1$,

$$(A+BK^*)\mathcal{P}_i^* + \mathcal{P}_i^*(A+BK^*)^T + \sum_{s=i+1}^{k} \frac{2s!}{i!(s-i)!}\left(GWG^T\mathcal{Y}_{s-i}^*\mathcal{P}_s^* + \mathcal{P}_s^*\mathcal{Y}_{s-i}^*GWG^T\right) = 0, \quad (16)$$

and when $i = k$

$$(A+BK^*)\mathcal{P}_k^* + \mathcal{P}_k^*(A+BK^*)^T = 0\,. \quad (17)$$

The stationary condition on the independent feedback gain K is given by

$$2B^T\sum_{s=1}^{k}\mathcal{Y}_s^*\mathcal{P}_s^* + 2\mu_1 RK^*\mathcal{P}_1^* = 0\,. \quad (18)$$

The stationary condition on the multiplier $\mathcal{P}$ yields

$$(A+BK^*)^T\mathcal{Y}_1^* + \mathcal{Y}_1^*(A+BK^*) + \mu_1 K^{*T}RK^* + \mu_1 Q + \sum_{r=2}^{k}\mu_r\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{Y}_s^*GWG^T\mathcal{Y}_{r-s}^* = 0\,,$$

and, for $2 \le i \le k$,

$$(A+BK^*)^T\mathcal{Y}_i^* + \mathcal{Y}_i^*(A+BK^*) + \sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{Y}_j^*GWG^T\mathcal{Y}_{i-j}^* = 0\,.$$

which are of course the constraint equations as desired. It should be pointed out that if K^* is well posed then the equation (17) yields a unique symmetric solution $\mathcal{P}_k^* = 0$. Similarly, unique symmetric solutions $\mathcal{P}_i^* = 0$ of the equation (16) can be solved sequentially by using the collection of previously determined solutions $\mathcal{P}_{i+1}^*, \dots, \mathcal{P}_k^*$, down to $\mathcal{P}_2^*$. $\mathcal{P}_1^*$ is found from (15) after the other $\mathcal{P}_i^*$'s are set equal to zero.

Further, the equation (18) gives a candidate feedback gain which locally extremizes the Lagrange functional:

$$K^* = -(\mu_1 R)^{-1} B^T \mathcal{Y}_1^* .$$

Finally, the second order necessary condition that ensures the Lagrange functional locally achieves its minimum requires that the Hessian matrix

$$\frac{\partial^2}{\partial K^2} \mathcal{L}(\mathcal{Y}^*, K^*, \mathcal{P}^*) = 2\mu_1 R \otimes \mathcal{P}_1^* , \tag{19}$$

is positive definite, where $\otimes$ stands for the Kronecker matrix product. The condition (19) will be satisfied if $\mathcal{P}_1^*$ is positive definite. In that case, the local extremizer with $\widehat{\mu}_r = \mu_i / \mu_1$

$$K^* = -R^{-1} B^T \left[\widehat{\mu}_1 \mathcal{H}_1^* + \widehat{\mu}_2 \mathcal{H}_2^* + \cdots + \widehat{\mu}_k \mathcal{H}_k^* \right] \tag{20}$$

becomes a local minimizer.

Theorem 4. Infinite Horizon kCC Control Solution. *Consider a stochastic linear dynamical system described by*

$$dx(t) = (Ax(t) + Bu(t))dt + Gdw(t) , \qquad x(t_0) = x_0 ,$$
$$J(u) = \int_{t_0}^{\infty} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt ,$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, G \in \mathbb{R}^{n \times p}$, positive semidefinite $Q \in \mathbb{S}^n$, positive definite $R \in \mathbb{S}^m$, with the Wiener process $w(t) \in \mathbb{R}^p$ having correlation of increments $E\left\{ [w(\tau) - w(\xi)] [w(\tau) - w(\xi)]^T \right\} = W|\tau - \xi|$ and $W > 0$. Suppose that (A, B) is stabilizable and (Q, A) is detectable. Then, for the given $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$, the locally extremalizing control law $u(t) = K^* x(t)$ in the infinite horizon kCC control problem is achieved by the gain

$$K^* = -R^{-1} B^T \sum_{r=1}^{k} \widehat{\mu}_r \mathcal{H}_r^* , \qquad \widehat{\mu}_r = \frac{\mu_i}{\mu_1} , \tag{21}$$

whenever the equations (21)-(23) have a solution $\{\mathcal{H}_r^* \geq 0\}_{r=1}^k$

$$(A + BK^*)^T \mathcal{H}_1^* + \mathcal{H}_1^* (A + BK^*) + K^{*T} R K^* + Q = 0 , \tag{22}$$

and, for $2 \leq r \leq k$,

$$(A + BK^*)^T \mathcal{H}_r^* + \mathcal{H}_r^* (A + BK^*) + \sum_{s=1}^{r-1} \frac{2r!}{s!(r-s)!} \mathcal{H}_s^* G W G^T \mathcal{H}_{r-s}^* = 0 . \tag{23}$$

Furthermore, in such a case the feedback control law (21) is well posed and stabilizing. Finally, if (A, G) is a controllable pair, then K^* is locally minimizing.

Proof. If $\mathcal{H}_1^* \geq 0$, it follows from (22) that $A + BK^*$ has all its eigenvalues in the open left-half plane. Then from (15), with $\mathcal{P}_2^* = \mathcal{P}_3^* = \cdots = \mathcal{P}_k^* = 0$, we find that $\mathcal{P}_1^*$ is positive definite, and thus from (19) that the solution is a local minimum.

IV. CABLE-STAYED BRIDGE BENCHMARK

The Second Generation benchmark problem for response control of seismically excited cable-stayed bridges consists of the Bill Emerson Memorial Bridge spanning the Mississippi River (on Missouri 74–Illinois 146) that currently undergoes construction near Cape Girardeau, Missouri, USA. See Figs. 1 and 2. Three historical El Centro, Mexico City and Gebze earthquake records with different characteristics are used to excite the bridge in any arbitrary direction using the two horizon components of the historical earthquake with a specified incidence angle. Multi-support excitation is now taken into consideration in this benchmark study. The detail development of the linear evaluation model along with its set of evaluation criteria for seismic response control of the cable-stayed bridge is available from [1]. Focusing on the evaluation model of the 3956 ft long bridge consisting of 2 towers, 128 cables and 12 additional piers together with 24 hydraulic actuators installed between the deck and abutment, and the deck and the towers, and oriented to apply forces longitudinally, the implementation of a time invariant state feedback 3CC controller is done with the set of values $\mu_1 = 1.00, \mu_2 = 4.00 \times 10^{-8}, \mu_3 = 8.30 \times 10^{-19}$. For a simple comparison, the weighting matrices of the system performance measure together with models of the control devices and sensors, as well as the location and direction of the control forces applied to the bridge structure in the design of 3CC controller are further selected as the same as those in the baseline LQG.

The goal for a controller design is to have all performance criteria $J1 - J18$ as small as possible while keeping control actions within the control limits. Two groups of measures, $J1 - J6$ and $J7 - J11$, respectively, correspond to normalized peak and normed structural responses. The smaller numbers here mean smaller shear, moment, and cable tension of base towers, decks and cable deviations. From Table I, it is observed that the use of additional second and third cost cumulants over and above the mean cost offers better bridge vibration attenuation, say respectively 6%, 3%, and 11% further reduction in peak base shear, base moment, and cable deviation when comparing to those of the baseline LQG design. The results in Table I indicate that the 3CC controller continues to decrease the normed deck moment and cable deviation of the baseline LQG by 4% and 5%. In regard to minimizing peak deck moment, displacement, and normed base shear, the baseline LQG outperforms the cumulants-based design by factors of 4%,

5% and 8%, respectively. The other groups $J12 - J15$ are used to measure peak control force, device stroke and total power quantities. It is shown from Table I that the 3CC controller has larger peak responses in control force by 6%, device stroke by 5%, power by 9%, and total power by 9%.

The effects of frequent rain and snow loads on the bridge deck are anticipated to change the nominal mass of the original model by a factor of 3.5%. Thus, it is necessary to evaluate the robust performance of a candidate controller using the perturbed model. Due to page limitation, only the case of 45^o earthquake excitation angle and with snow loads is considered herein. It may be seen from the results presented in Table II that the 3CC control design performs adequately in the case of 3.5% mass uncertainty structure without violating hard constraints. The robust stability of the 3CC controller is also assessed by constructing multivariable Nyquist plots of the determinants of the return difference transfer functions, $I + L(j\omega)$, where the loop gains $L(j\omega)$ are formed by breaking the loop at the inputs of both nominal and perturbed design models. Referring to Figs. 4 and 5, the solid curve is that of the baseline LQG in the benchmark, whereas the dotted curve is associated with the 3CC control design. The distance from the origin to the Nyquist plot is a measure of the stability margin. As indicated in Figs. 4 and 5, there are significant improvements of 32% and 44% in the stability margins offered by the second and third cost cumulants in the 3CC controller design as compared to those of the baseline LQG. Note that in 3CC control, there are the mean, variance and skewness of the cost process that can be employed to achieve better structural performance. To see how sensitive the cost distribution function to the chosen values of μ_i for $i = 1, 2, 3$, readers should refer to Fig. 3 for an illustration.

V. CONCLUSIONS

As a direct outgrowth of the finite horizon kCC control problem described in the recent publication [4] and the references therein, the cost-cumulant control counterpart with infinite horizon and state feedback is developed in this paper. The use of cost cumulants in robust structural control has again continued its success in the civil engineering applications such as protecting buildings and cable-stayed bridges against severe earthquakes and strong wind. From the view of control theory, the cost-cumulant control is the generalization of LQG-type control designs. The properties of cost-cumulant controllers can be built directly on the existing knowledge of LQG control. Aiming at making better use of actuator capability and employing the same control setting as the baseline LQG, the cost-cumulant controller design performs respectably when compared to its first-order cousin, the baseline LQG, while it also offers rather a nontrivial improvement in robust stability margin.

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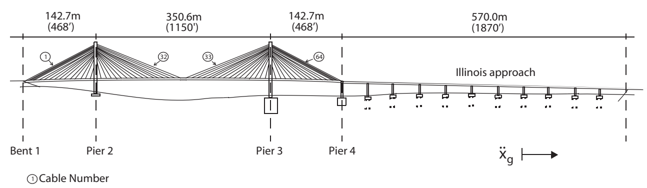

Fig. 1. Bill Emerson Memorial Bridge

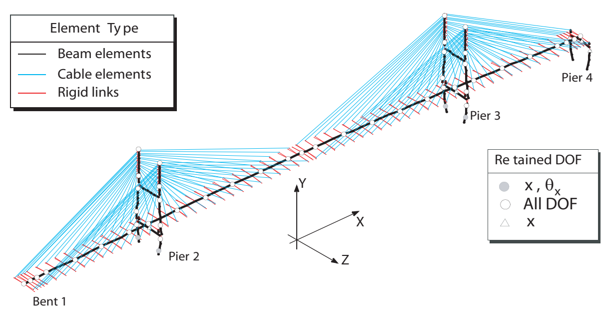

Fig. 2. Finite Element Model. Courtesy by Earthquake Engineering and Structural Control Laboratory at Washington Univeristy in St. Louis

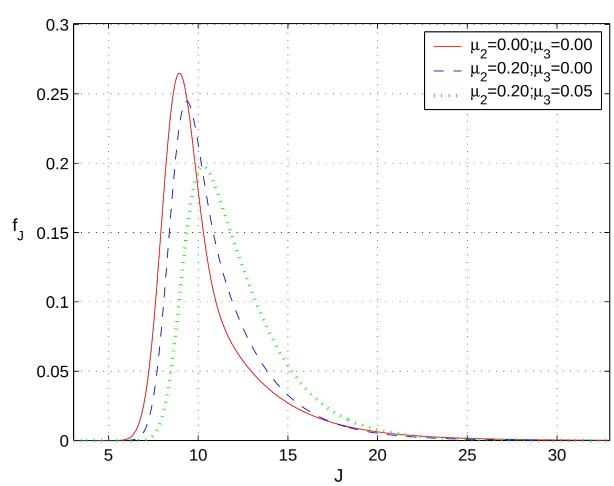

Fig. 3. Cost Probability Density Function, 3CC Control

TABLE I

BENCHMARK EVALUATION FOR 3CC CONTROLLER: NO SNOW LOADS AND 15^o ANGLE SEISMIC EXCITATION

Criteria	Direction	El Centro	Mexico	Gebze
J1	X	0.29634	0.35775	0.47310
	Z	1.02115	1.11662	1.03643
J2	X	0.79821	0.88259	0.94841
	Z	0.96636	0.99755	0.99607
J3	X	0.31830	0.37408	0.44215
	Z	1.09752	1.07941	1.04665
J4	X	0.59973	0.78982	0.92986
	Z	1.00977	0.99319	1.00118
J5	-	0.25202	0.11989	0.17562
J6	X	1.09860	1.87696	2.32410
J7	X	0.25422	0.32037	0.34150
	Z	1.01355	1.05532	1.05283
J8	X	0.82167	0.94315	0.94763
	Z	0.97802	0.99603	0.99121
J9	X	0.23719	0.31953	0.39634
	Z	1.00521	1.05195	1.03939
J10	X	0.59849	0.79675	0.80970
	Z	1.00185	1.00436	1.00330
J11	-	0.02343	0.01370	0.01628
J12	X	0.00294	0.00174	0.00285
	Z	0.00000	0.00000	0.00000
J13	X	0.67290	1.02213	1.01413
	Z	0.00000	0.00000	0.00000
J14	X	0.00384	0.00256	0.00693
	Z	0.00000	0.00000	0.00000
J15	X	0.000565	0.000372	0.000684
	Z	0.00000	0.00000	0.00000
J16	-	24	24	24
J17	-	14	14	14
J18	-	60	60	60
Force(kN)	Z	1500.00	887.626	1457.72
Stroke(m)	Z	0.11081	0.09766	0.16152
Vel(m/s)	Z	0.84429	0.42211	0.54059

TABLE II

BENCHMARK EVALUATION FOR 3CC CONTROLLER: SNOW LOADS AND 45^o ANGLE SEISMIC EXCITATION

Criteria	Direction	El Centro	Mexico	Gebze
J1	X	0.42903	0.44971	0.43175
	Z	0.98663	1.03476	0.99746
J2	X	0.69133	1.05991	0.99646
	Z	0.98302	0.97280	0.99020
J3	X	0.39892	0.43341	0.49577
	Z	0.97976	1.02930	0.99221
J4	X	0.63877	0.68889	0.80216
	Z	0.99710	0.99999	1.00219
J5	-	0.28164	0.11807	0.18569
J6	X	1.29793	2.28835	2.77199
J7	X	0.31459	0.41606	0.37658
	Z	0.98251	1.00251	1.02527
J8	X	0.83240	0.99357	0.99895
	Z	0.99206	0.97159	1.00153
J9	X	0.27323	0.38234	0.48375
	Z	0.98554	0.99723	1.02213
J10	X	0.66067	0.86314	0.97586
	Z	1.00328	1.00332	1.00487
J11	-	0.02549	0.01402	0.01777
J12	X	0.00294	0.00182	0.00250
	Z	0.00000	0.00000	0.00000
J13	X	0.67897	1.06582	1.23091
	Z	0.00000	0.00000	0.00000
J14	X	0.00440	0.00349	0.00544
	Z	0.00000	0.00000	0.00000
J15	X	0.000550	0.000445	0.00060
	Z	0.00000	0.00000	0.00000
J16	-	24	24	24
J17	-	14	14	14
J18	-	60	60	60
Force(kN)	Z	1500.00	928.919	1276.88
Stroke(m)	Z	0.12427	0.10215	0.15573
Vel(m/s)	Z	0.66430	0.47369	0.46895

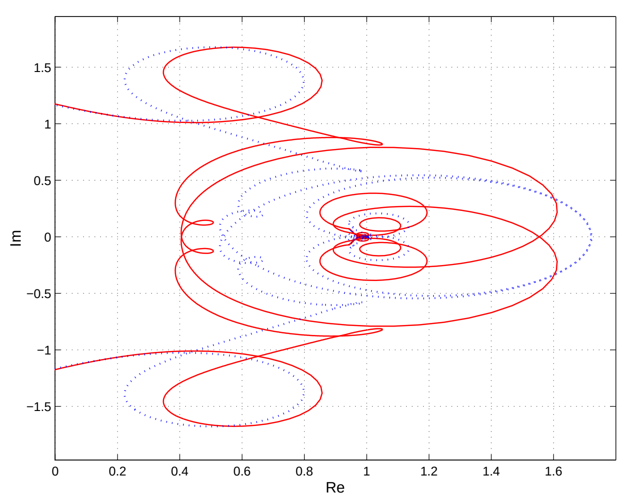

Fig. 4. Close-Up Nyquist Plot of Return Difference: No Snow Load and 15^o Angle Seismic Excitation

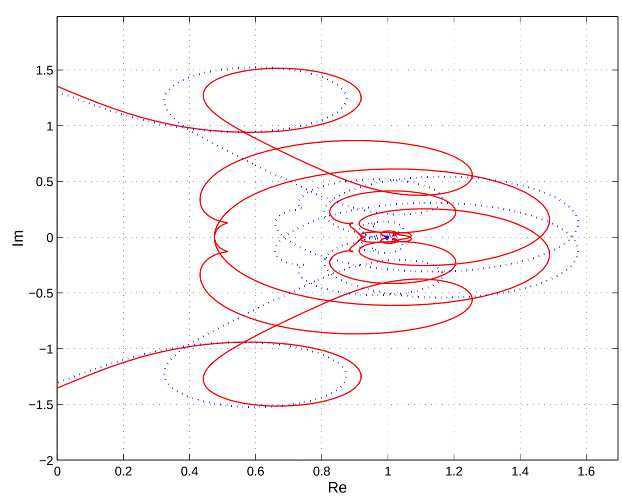

Fig. 5. Close-Up Nyquist Plot of Return Difference: Snow Loads and 45^o Angle Seismic Excitation