Stability Check Of Matrix Families: How And Why Vertex Solution For Multiple Vertex Case Is Different From Two Vertex Case

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***Abstract*— This paper presents a new insight into the problem of checking the stability of matrix families formed by the convex combination of Hurwitz stable matrices. The paper reviews the necessary and sufficient vertex solution offered by the author for testing the robust stability of this family and highlights the difference between the multiple vertex (i.e. number of vertex matrices being ≥ 3) case and the two vertex case. New insight is provided not only into this difference between these two cases, but also on the importance of computational awareness in the problem formulation and the resulting differences in the vertex algorithm, thereby clarifying many subtleties surrounding the proposed algorithm and helping to explain why it has been such a difficult task to understand and solve all this time. Examples illustrating the application of the vertex algorithm are given. Finally, some conclusions are drawn along with future research directions.**

Key Words: Robust Stability;Convex Combination; Real Parameter Variation; Structured Uncertainty;Polytopes of Matrices; Extreme Point Results

I. INTRODUCTION

The problem of analyzing the stability of matrix families arises in many applications of systems and control theory [1]. The most common matrix family of interest is the family generated by a convex combination of arbitrary Hurwitz stable matrices. Consider the matrix family given by

$$\mathcal{A} = \left\{ A = \sum_{i=1}^{h} \alpha_i A^i, \alpha_i > 0, \sum \alpha_i = 1 \right\}, \quad (1)$$

where h is an integer and all the vertex matrices A^i are Hurwitz stable (i.e. have eigenvalues with negative real parts). Then the issue of research is to ascertain if all the matrices belonging to the above convex combination are also Hurwitz stable or not. The above problem formulation of checking the Hurwitz stability of a convex combination of *arbitrary* Hurwitz stable matrices has not been researched even though it turns out to be a problem of interest in many applications such as linear switched systems [2].

The author recently presented a necessary and sufficient vertex algorithm as a solution to this problem in the journal publication [3]. From this solution, it is clear that there is an interesting and important difference between the multiple vertex case ($h \geq 3$) and the two vertex case ($h = 2$). In this paper, we delve deeper into the proof of the theorem provided in [3] and clearly explain the reasons for the difference in the solution for these two cases. In addition, this new insight and emphasis on this difference between these two cases is worth in its own right (as a separate paper like this) because it not only alerts researchers about the pitfalls in specializing multiple vertex result to two vertex result as well as generalizing two vertex result to the multiple vertex case, but also helps to apply these fundamental concepts that differentiate these two cases to any future, new problem formulation that involves checking the stability of matrix families. With this backdrop, the paper is organized as follows. In the next section, we briefly introduce the problem formulation of [3]. Then in section III, we revisit the strategy of converting the stability problem to that of checking nonsingularity via the 'Kronecker Lyapunov' matrix space, along with the preliminaries needed to state all the upcoming theorems.In section IV, we review the 'vertex solution' that was presented in [3], clearly restating the theorems that distinguish between the two vertex case and multiple vertex case. In section V, a new problem formulation with computational awareness is posed and the corresponding modification in the vertex algorithm is discussed. Then in section VI, few examples are presented illustrating the difference between these two cases. Finally in section VII, some concluding remarks are presented along with future directions of research.

II. Problem Formulation A:Conceptual Case:

A. Two Vertex Case:

First, let us consider the two vertex case, i.e.

$$\mathcal{A} = \left\{A = \alpha_1 A^1 + \alpha_2 A^2, \alpha_i > 0, \sum \alpha_i = 1\right\} \quad (2)$$

In this case, it may be noted that $\alpha_2 = 1 - \alpha_1$ (or $\alpha_1 = 1 - \alpha_2$) and thus there is only one freely varying coefficient.

B. Multiple Vertex Case:

Then, let us consider the multiple vertex case, i.e.

$$\mathcal{A} = \left\{A = \sum_{i=1}^{h} \alpha_i A^i, \alpha_i > 0, \sum \alpha_i = 1\right\}, \quad (3)$$

For better exposition and ease in notation, from now on let us take $h = 3$. Thus, consider

$$\mathcal{A} = \left\{A = \alpha_1 A^1 + \alpha_2 A^2 + \alpha_3 A^3\right\}, \quad (4)$$

where $0 < \alpha_i$ ¡ 1, and $\sum \alpha_i = 1$.
Note that in this case, *two* α*s* are free or arbitrary with the third coefficient being fixed. In addition, *any two* αs are free with the third one being fixed. It is this feature that becomes very crucial in explaining why two vertex case algorithm is different from multiple case algorithm. This will become clear in the development of proof of the final vertex algorithm for these two cases. Also another important point to note is that these αs are in the open interval $(0, 1)$. For this reason, this problem formulation is labelled as ‘conceptual’.
Note that in the two vertex case, there is a rigid constraint on the two coefficients (of adding upto one),which can be geometrically viewed as an ‘edge’ in the matrix space. However, in the multiple vertex case, the ‘connection’ between any *two* vertex matrices is not through rigid constraint (of those two coefficients adding up to one) but instead it is as though all these are ‘virtual edges’, ‘virtual faces’ etc and are all ‘collapsible’, and only the h vertices together keep the ‘polytope’ shape, because only the summation of all these h coefficients together is constrained but there is no constraint between any group of matrices less than h. In other words, between any group of matrices with less than h vertices, the family is a ‘positive real linear combination’. This subtle point of the existence of a positive real linear combination family within the convex combination of multiple vertices (and the absence of it in the two vertex case) can make a difference in the final solution. It is this aspect we want to elaborate in the next few sections.

III. Stability Problem as a Nonsingularity Problem via the ‘Kronecker Lyapunov’ Matrix:

It is known from references [4], [5], [6], [7], [8] that the stability assessment problems posed in the introduction for both formulations can be converted to a nonsingularity problem involving Kronecker based matrix operations.The above cited literature presents these conditions in terms of three matrices (each of which employs different Kronecker based operations) namely: (i) Kronecker Sum matrix (denoted as K matrix) (ii) Lyapunov matrix (later in this paper labeled as ‘Kronecker Lyapunov’ matrix to distinguish it from the standard and familiar Lyapunov matrix equation solution) denoted by L and iii) ‘Bialternate Sum’ matrix, denoted by B matrix. In this research, we consider only the second of these matrices i.e. the ‘Kronecker Lyapunov’ matrix denoted by L i.e.

$$L = A^{\dagger} = A \times I_n + I_n \times A \quad (5)$$

where ‘$\times$’ denotes an operation similar to the Kronecker Sum (see Jury [7]) for details). In order to conserve notation, henceforth, we will label the matrix given in (5) as the ‘Kronecker Lyapunov’ matrix and denote the matrix operation as ‘dagger’ operation. Note that L is a square matrix of dimension $m = \frac{1}{2}n(n+1)$,whose eigenvalues are the pairwise summation of the eigenvalues of A. In Tesi and Vicino [9] and in Jury [7] , simple computer amenable methodologies are given to form L matrix from the given matrix A.
Example 1: For n=2,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with μ_1 and μ_2 as eigenvalues, the Kronecker Lyapunov matrix L is given by

$$L = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $\mu_1 + \mu_2$, $2\mu_1$ and $2\mu_2$.
Note that there is an alternative form for the above L matrix where the element a_{ij} could be interchanged with the element a_{ji}.

Mathematical Preliminaries Related to 'Kronecker Lyapunov' Operation:

- **Property 1:**
  For two square matrices A_1 and A_2,
  $$(k_1 A_1 + k_2 A_2)^{\dagger} = k_1 {A_1}^{\dagger} + k_2 {A_2}^{\dagger} \quad (6)$$
  where k_1 and k_2 are scalars.

We **define** the vertex matrices L^i as follows:

$$L^i = (A^i)^{\dagger}. \quad (7)$$

Machinery and Concepts Needed to State the Main Theorems:

In this section we present the necessary concepts needed to set the stage to state the 'Theorems' of this paper. The following holds for any $h \geq 2$.

'Virtual Center' matrix:: Let $L_{vc,h}$ denote the 'virtual center' matrix formed with all the h vertex matrices L^1 ,L^2..L^h taken together at a time given by

$$L_{vc,h} = (L^1 + L^2 + L^3 + ...L^h) \quad (8)$$

The corresponding 'virtual center'(or 'summation') matrix $A_{vc,h}$ can also be easily defined for the original matrix space.

A Necessary Condition For Stability: All the matrices belonging to the given matrix family are Hurwitz stable **only if**

the 'virtual center' matrix $A_{vc,h}$ is Hurwitz stable.

So in the rest of the statements of the main theorems, we assume that this necessary condition is satisfied.

In order to state the theorems in later sections, we need a set of '**point**' matrices labeled 'Kronecker Nonsingularity Matrices' as follows:

'Kronecker Nonsingularity' Matrices: These special matrices are 'point' matrices given by

$$L(ns; h; j) = -[(L_(vc; h))^{-1} L^j] \quad (9)$$

$$(j = 1, 2, ..h) \quad (10)$$

For example, in a 3 vertex case, $L(ns; 3; 1)$ denotes the matrix $-[(L^1 + L^2 + L^3)^{-1} L^1]$ and $L(ns; 3; 2)$ denotes the matrix $-[(L^1 + L^2 + L^3)^{-1} L^2]$. Note that there are h KN matrices. For the two vertex case, the two KN matrices are given by $-[(L^1 + L^2)^{-1} L^1]$ and $-[(L^1 + L^2)^{-1} L^2]$.

- 'Real Axis Stability': We say a matrix is Real Axis Stable if its real eigenvalues are all negative.
- **Another Necessary Condition** [3]: The matrix family is stable only if all the KN matrices are real axis stable.

IV. Necessary and Sufficient Vertex Solutions for Checking the Robust Stability of a Convex Combination of Hurwitz Stable Matrices':[3]

A. Theorem for Two Vertex Case:

Theorem 1: All the matrices belonging to the convex combination matrix family (with vertex matrices A^1 and A^2 and the 'virtual center' matrix $A^1 + A^2$ being Hurwitz stable) are Hurwitz stable

if and only if

the two 'Kronecker Nonsingularity Matrices' (KN matrices), namely

$$-[(L^1 + L^2)^{-1} L^1] \; and \quad (11)$$
$$-[(L^1 + L^2)^{-1} L^2] \quad (12)$$

are Real Axis Stable.

Remark 1: The above theorem does not shed any light on the behavior of the complex eigenvalues of the two KN matrices involved. It leaves us with the question of whether one can have a Hurwitz stable matrix family with only negative real eigenvalues but with positive real part eigenvalues in *each* of the two KN matrices. But the next theorem answers that question (and rules out that possibility).

Thus we now state an alternative theorem for the two vertex case.

B. Alternative Theorem for Two Vertex Case:

Theorem 2: All the matrices belonging to the convex combination matrix family (with vertex matrices A^1 and A^2 and the 'virtual center' matrix $A^1 + A^2$ being Hurwitz stable) are Hurwitz stable

if and only if

out of the two 'Kronecker Nonsingularity Matrices' (KN matrices), namely

$$-[(L^1 + L^2)^{-1} L^1] \; and \quad (13)$$
$$-[(L^1 + L^2)^{-1} L^2] \quad (14)$$

one of them is Hurwitz stable and the other is real axis stable.

Next, we switch our attention to the multiple vertex case.

1) Theorem for the Multiple Vertex Case::

- **A Necessary Condition**[3]: The matrix family is stable only if $(h - 1)$ KN matrices are Hurwitz stable and the other one is real axis stable.

Now we state the main result taken from [3] specializing it for the present case.

Theorem 3: All the matrices belonging to the convex combination matrix family (with vertex matrices A^i and the 'virtual center' matrix $A^1+A^2+A^3+..A^h$ being Hurwitz stable) are Hurwitz stable
if and only if
the h 'Kronecker Nonsingularity Matrices' (KN matrices), namely

$$L(ns;h;j) = -[(L_(vc;h))^{-1}L^j] \quad (15)$$

$$(j = 1,2,..h) \quad (16)$$

are all Hurwitz stable.
For example, for $h = 3$ case, the above theorem reads as follows:

2) Illustrative Theorem for Three Vertex Case::

Theorem 4: All the matrices belonging to the convex combination matrix family (with vertex matrices A^1, A^2 and A^3 and the 'virtual center' matrix $A^1 + A^2 + A^3$ being Hurwitz stable) are Hurwitz stable
if and only if
the three 'Kronecker Nonsingularity Matrices' (KN matrices), namely

$$-[(L^1 + L^2 + L^3)^{-1}L^1] \; and \quad (17)$$
$$-[(L^1 + L^2 + L^3)^{-1}L^2] \; and \quad (18)$$
$$-[(L^1 + L^2 + L^3)^{-1}L^3] \quad (19)$$

are all Hurwitz stable.
Proof: It is available in [3]. However, here we discuss some salient points of that proof. In [3], it is shown that, because of the special nature of the dagger space matrices, in the linear domain of dagger space (i.e. addition of matrices), nonsingularity and stability are equivalent. In other words, the real parts of complex eigenvalues and the real eigenvalues are coupled and for nonsingularity (i.e. stability) the real parts are required to behave the same way as the real eigenvalues. Then the necessity of stability of the product domain KN matrices is established based on the generalized eigenvalue problem of the 'virtual ray' matrix $L_{vc} + \rho L^i$, where the positive scalar variable ρ varies within the open interval $(0, \infty)$.

V. Computational Awareness in the Problem Formulation and Solution:

It is important to realize that the vertex solution presented for problem formulation A can only be proved through analytical and conceptual arguments as is done in [3] and may not yield accurate results all the time for all problems because in a computational environment, there is no way to implement the open interval nature of αs (and ρ) described above. Interestingly, the open interval nature of αs (and ρ) is what imparts the Hurwitz stability of KN matrices as a necessity. So in a computational environment, while the $(h-1)$ KN matrices being Hurwitz stable as a necessity can be achieved, there is a possibility that this necessity may be lost for the lone remaining KN matrix. This is because in the computational implementation, αs can only belong to a semiclosed interval with $[\epsilon, 1)$ where ϵ is a very small positive scalar and the moment this happens, the necessity of stability of the lone KN matrix may be lost because the theoretically present strong coupling between real and real part eigenvalues in the KN matrix may not be manifested. Whether this happens or not depends on the conditioning of the matrices L^i and L_{vc}.This possibility exists especially when these matrices are near singularity because the KN matrix has an inverse operation,and a product operation and finally the operation of finding the eigenvalues. In the rare case this happens, the dilemma would be to decide whether the presence of positive real parts of any complex pair is due to instability in the family or is due to weak coupling between the real and real parts of that specific KN matrix, necessarily present in a computational environment. Fortunately, there is a simple way to decide. If the sum of the positive real parts of the complex conjugate pair in the KN matrix is less than a 'coupling bound', κ, then it is a case of weak coupling and then for that case, that complex conjugate pair with positive real part does not contribute to the stability condition. Obviously, this bound κ is very much dependent on the nature and conditioning of the vertex and virtual center matrices for the problem under consideration. For the 3 vertex problem, when the vertex matrices are arbitrary Hurwitz stable matrices, this coupling bound κ is shown to be equal to $m/3$. Because of space considerations, the derivation of this bound is not presented here. Similar condition can be obtained for the other multiple vertex case as well. An eigenvalue distribution in which all the eigenvalues except for one complex conjugate pair have negative real and real parts and the coupling bound κ on the positive real part is known (to render it inconsequential), can be labelled as 'imminently stable'. Suppose we label the concepts of 'imminent stability' and 'Hurwitz stability' together as 'virtual stability'. That is, 'virtual stability' includes both 'Hurwitz stability' as well as 'imminent stability'. Then, to summarize, we have a new computationally aware problem formulation and the corresponding vertex solution as follows:

A. Problem Formulation B: With Computational Awareness:

Now, consider the three vertex problem with

$$\mathcal{A} \;=\; \left\{A = \alpha_1 A^1 + \alpha_2 A^2 + \alpha_3 A^3\right\} \tag{20}$$

where $\epsilon_i \leq \alpha_i < 1$ and, $\sum \alpha_i = 1$.
For this case the vertex solution gets modified as follows:

Theorem 5: All the matrices belonging to the convex combination matrix family (with vertex matrices A^1, A^2 and A^3 and the 'virtual center' matrix $A^1 + A^2 + A^3$ being Hurwitz stable) are Hurwitz stable
if and only if
out of the three 'Kronecker Nonsingularity Matrices' (KN matrices), two are Hurwitz stable and the other one is 'virtually stable'.
Note that the computation of the 'coupling bound',κ is depenedent on the nature of vertex matrices. A method to compute this bound is presented in a future conference paper(currently under review). Note that this type of situation cannot occur for the two vertex problem.

VI. Illustrative Examples:

Example 1: Let us consider the convex combination of two 'vertex' matrices given by

$$A^1 = \begin{bmatrix} -1.1 & -1 \\ 5 & -4.1 \end{bmatrix} \; A^2 = \begin{bmatrix} -1.1 & 5 \\ -1 & -4.1 \end{bmatrix} \tag{21}$$

The two vertex matrices are Hurwitz stable and the 'Virtual Center' matrix is also verified to be Hurwitz stable. The two 'Kronecker Nonsingularity Matrices' $-(L^1+L^2)^{-1}L^1$ and $-(L^1+L^2)^{-1}L^2$ are seen to be real axis stable. Also it can be seen that one of the KN matrices is Hurwitz stable. In fact in this problem both KN matrices happen to be Hurwitz stable. From the necessary and sufficient conditions proposed in the Main theorem, we thus conclude that the above convex combination of matrices is stable.
Example 2: The author would like to thank one anonymous researcher who provided this as well as the next example: Let us consider a convex combination of two arbitrary Hurwitz stable matrices given by

$$A^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -0.5 \\ 0 & 0.5 & -1 \end{bmatrix} \; A^2 = \begin{bmatrix} 0 & 1 & 0.1 \\ -1 & 0 & 0.3 \\ -0.1 & -0.3 & -1 \end{bmatrix} \tag{22}$$

The two vertex matrices are Hurwitz stable and the 'Virtual Center' matrix is also verified to be Hurwitz stable. The two 'Kronecker Nonsingularity Matrices' $-(L^1+L^2)^{-1}L^1$ and $-(L^1+L^2)^{-1}L^2$ are seen to be real axis stable. In addition the KN matrix $-(L^1+L^2)^{-1}L^1$ is seen to be Hurwitz stable. It is interesting to observe that the KN matrix $-(L^1+L^2)^{-1}L^2$ is real axis stable but unstable with positive real part eigenvalues. However that is inconsequential because from the necessary and sufficient conditions proposed in the Main theorem, we can conclude that the above convex combination of matrices is Hurwitz stable. This fact can of course be independently verified by applying the result of [4].
Example 3: Now let us consider a convex combination of three Hurwitz stable matrices given by

$$\begin{aligned} A^1 &= \begin{bmatrix} 0 & 1 & -0.25 \\ -1 & 0 & 0 \\ 0.25 & 0 & -1 \end{bmatrix} \\ A^2 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -0.5 \\ 0 & 0.5 & -1 \end{bmatrix} \\ A^3 &= \begin{bmatrix} 0 & 1 & 0.1 \\ -1 & 0 & 0.3 \\ -0.1 & -0.3 & -1 \end{bmatrix} \end{aligned}$$

The three vertex matrices are Hurwitz stable. Note that the vertex matrices A^2 and A^3 are same as the two vertex matrices considered in the previous example. Considering two vertex matrices at a time, it can be verified that the individual 'edge' matrix families are Hurwitz stable by applying the theorem of two vertex case of the paper. But now considering all three vertex matrices at a time, we apply the theorem corresponding to the multiple vertex case. So we form the 'Virtual Center' matrix $(L^1 + L^2 + L^3)$ which is verified to be Hurwitz stable. Then we form the three 'Kronecker Nonsingularity (KN) Matrices', namely $-(L^1+L^2+L^3)^{-1}L^1$, $-(L^1+L^2+L^3)^{-1}L^2$ and $-(L^1+L^2+L^3)^{-1}L^3$. It turns out that one of these KN matrices is unstable with one positive real part complex conjugate pair. The summation of these positive real parts is 2.4 which is greater than 2 ($m = 6$ for this example). Hence we conclude that the above convex combination matrix family is unstable. Indeed, it can be seen that there are some unstable interior matrices such as $0.2A^1 + 0.3A^2 + 0.5A^3$.
Examples 2 and 3 clearly illustrate the difference in the proof between the multiple vertex case and the two

vertex case, which is the main emphasis of this paper.

VII. CONCLUSIONS:

This paper presents a new insight into the problem of checking the stability of matrix families formed by the convex combination of Hurwitz stable matrices. The paper reviews the necessary and sufficient vertex solution offered by the author for testing the robust stability of this family and highlights the difference between the multiple vertex (i.e. number of vertex matrices being ≥ 3) case and the two vertex case. Future research points to looking for methods to assess the bound on the coupling to guarantee virtual stability, which needs to consider the specific nature (conditioning) of the vertex matrices and the interrelationship (if any present) between these matrices. In other words, the result for the case of arbitrary Hurwitz stable convex combination problem solution may turn out to be different from the convex combinations generated by interval parameters,a case in which the vertex matrices are interrelated.

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