An Intrinsic Observer for a Class of Simple Mechanical Systems on a Lie Group

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***Abstract*— This paper presents an intrinsic formulation of an observer for an important class of simple mechanical systems on a Lie group. Recently, Aghannan and Rouchon have formulated an observer for a simple mechanical system on a general Riemannian manifold. The current paper specializes their result to the case where the manifold is a Lie group, the kinetic energy is left invariant, and the velocity variables are to be estimated based on measurement of the configuration variables. These restrictions allow a greatly simplified result, of interest in its own right. Most significantly, no coordinates need be introduced on the Lie group, hence a single formulation is valid for all coordinate patches. To illustrate the method, observers are computed for two simple mechanical systems, on the rotation group $SO(3)$ and on the Euclidian motion group $SE(3)$. Simulations of an example on $SO(3)$ show excellent performance.**

I. Introduction

A *simple mechanical system* on a Lie group consists of i) a Lie group, which corresponds to the configuration space of the system, ii) a Lagrangian defined by kinetic energy minus potential energy, and iii) a set of forces (or one-forms) [7], [15]. When some of these forces may be used for control, we refer to a simple mechanical *control* system [7]. Such systems provide a rich source of control problems. Some examples include underwater vehicles, satellites, surface vessels, airships, hovercrafts, and robots [8], [20], [21], [6], [2], [3]. Simple mechanical systems on Lie groups are also interesting as *subsets* of more complex interconnected systems. As an example, in [12] the authors model the electrostatically actuated micromirror component of a micro optoelectromechanical system (MOEMS) in this framework.

Many researchers have considered the stabilization and motion planning problems for simple mechanical control systems on Lie groups. For instance, open loop motion planning algorithms are developed using small amplitude forcing in [8] and by minimizing an appropriate cost functional in [21]. For examples of the many feedback stabilization methods we refer to [5], [6], [13] and the references therein. These stabilization techniques primarily depend on notions of energy shaping and passivity. The resulting control strategies typically involve both velocity and configuration feedback. However, depending on the application, one or the other of these may not be directly available, and therefore must be estimated. In some cases the velocity measurements may be easily obtained [3], [14], [17] while in others it is the configuration measurements [2], [19], [18], [12]. The latter case is the focus of this paper.

A central idea in differential geometry is that meaningful definitions should be *intrinsic*. A definition is intrinsic if its meaning is not dependent on the particular choice of coordinate representation, that is, if it can be considered to be coordinate independent. In particular, for a formulation to be general enough to apply to a wide class of Lie groups, it must be intrinsic, since the coordinate-dependent details are not known in advance. In the case of rigid-body rotation, many successful angular velocity estimation schemes have been published [18], [2], [19]. However, they require a particular parameterization of the rotation matrix, and the observers depend explicitly on this choice. Hence, they are not intrinsic, and so their methods cannot be extended in general. For instance in [18] the rotation matrix is parameterized by unit quaternions while in [3] the modified Rodrigues parameters are used.

A significant body of work exists on the design of observers for a class of nonlinear systems. A recent survey of the state of the field may be found in [11]. The general approach is contingent upon the system taking one of a number of special forms. For instance, a system is said to be *linear up to output injection* if, in a particular coordinate patch, the nonlinearities appear only in terms of the control and measured output. Then an observer may be designed that is guaranteed to converge exponentially as long as the system remains in the patch. This is an extremely valuable and powerful result, invaluable in situations where the system is to be confined to a relatively small set, such as output feedback stabilization of an equilibrium point or a periodic orbit. However, in cases where state trajectories are not so confined, it has some limitations. The special form is not intrinsic, and hence convergence is not guaranteed in any other coordinate patch. Furthermore, the structural requirement is extremely stringent, and is rarely naturally satisfied. To address this, a series of nonlinear transformations of sequentially higher order is applied in order to *approximately* transform a non-compliant system into the necessary form. However, these transformations rely on a power series expansion, and therefore are valid only locally. When the system is converted to the special form through such a series of transformations, convergence

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is guaranteed only locally. Thus this approach to nonlinear observer design does not seem extendable to an intrinsic formulation suitable for more general Lie groups.

In a recent paper [1], Aghannan and Rouchon present an intrinsic observer that provides estimates of the states of a simple mechanical system on a Riemannian manifold, given measurements of the configuration variables. They use a reformulation of the Luenberger observer, where the observation error is intrinsically defined by the geodesic distance between the actual and estimated configuration variables. This quantity is well-defined, provided the estimate and the true value are sufficiently close. Aghannan and Rouchon also show that an additional curvature term should be added to ensure local convergence. The key steps in their derivation are the computations of the Levi-Civita connection, the Riemannian curvature, the associated distance function, and the approximation to parallel transport on the manifold. While their observer is intrinsic, the formulation is presented in terms of coordinates. On a Lie group, the observer as formulated in [1] may be directly implemented once a suitable choice of coordinates are chosen. However, as on $SO(3)$, it is not always possible to pick a convenient, globally nonsingular choice of coordinates. In applications such as optimal control and long term trajectory tracking, these singularities may cause difficulties. A major contribution of the current paper is that the formulation of the observer is expressed independently of the choice of coordinates on the Lie group.

For any simple mechanical system, the kinetic energy may be used to define a metric on the configuration space, thus providing the structure of a Riemannian manifold. This in turn gives the unique Levi-Civita *connection* associated with this metric, allowing intrinsic differentiation of the velocity components. This is the minimum amount of structure required for an intrinsic observer to be defined. In this sense the result of [1] is the most general possible for simple mechanical systems. However, while the observer itself is intrinsic, it must be specified in each coordinate patch. That is, to explicitly write the equations of the observer, the configuration and velocity variables, as well as the connection coefficients, must be expressed in coordinates. While there is no way around this in general, the formulation may be greatly simplified in the important special case where the manifold is a Lie group, and the kinetic energy is left invariant. Locally, the tangent bundle of a Riemannian manifold looks like the product of the manifold and a Euclidean space. However, this product representation may not be valid globally. In contrast, Lie groups are *parallelizable*, meaning that they have a globally defined, smoothly varying set of vector fields that define a basis for the tangent space at any point. This implies that the product representation applies globally. The significance of this is that a single choice of coordinates for the tangent space at the identity may be left translated to provide coordinates for all tangent spaces. This means that the state equation for the configuration variable may be written without specifying coordinates. This holds for any Lie group. However, the connection coefficients required to write the state equations for the velocity components will be still functions of the configuration variables. If we further assume that the kinetic energy is invariant under left translation, as is typically the case for rigid body rotation and translation, then the connection coefficients become constant, and now the velocity component state equations are also simplified considerably. This is the case considered here. We note that though coordinates are not needed for the Lie group, it is still necessary to choose a basis (not necessarily orthonormal or orthogonal) for the Lie algebra.

In section II we show how the quantities we will need subsequently specialize from Riemannian geometry to Lie groups. In section III we use these to explicitly formulate the intrinsic observer of [1] for Lie groups with left invariant kinetic energy metric. The observer is calculated in section IV for two examples, one on $SO(3)$ and the other on $SE(3)$. Finally, section IV shows a simulation example on $SO(3)$ in which the angular velocities of a axisymmetric top are estimated. We also note that an application of this observer to $SE(3)$ for the closed-loop control of an electrostatically actuated MOEMS device has been given in [12], without details of the observer derivation.

II. The Riemannian Structure on Lie Groups

Let G be a Lie group and let $\mathcal{G} \simeq T_eG$ be its Lie algebra. The left translation of $\zeta \in \mathcal{G}$ to T_gG will be denoted by $g \cdot \zeta = DL_g \, \zeta$. The Jacobi Lie bracket of any two vector fields X, Y on G will be denoted by $[X, Y]$ while the Lie bracket on $\mathcal{G}$ for any two $\zeta, \eta \in \mathcal{G}$ will be denoted by $[\zeta, \eta]_{\mathcal{G}} = ad_\zeta \, \eta$ and the dual of the ad operator will be denoted by ad^*. Any smooth vector field $X(g)$ on G has the form $g \cdot \zeta(g)$ for some smooth $\zeta(g) \in \mathcal{G}$.

Let $I : \mathcal{G} \mapsto \mathcal{G}^*$ be an isomorphism such that the relation $<< \zeta \, , \, \eta >>_{\mathcal{G}} = < I\zeta \, , \, \eta >$ for $\zeta, \eta \in \mathcal{G}$ defines an inner product on $\mathcal{G}$. Here $< \, \cdot, \cdot \, >$ denotes the usual pairing between a vector and a co-vector. Identifying $\mathcal{G}^*$ and $\mathcal{G}$ with $\mathcal{R}^n$, let I_{ij} and I^{ij} be the matrix representation of I and I^{-1} respectively. I is symmetric and positive definite. Such an I induces a unique left invariant metric on G by the relation $<< g \cdot \zeta \, , \, g \cdot \eta >> = < I\zeta \, , \, \eta >$ and further it also follows that every left invariant metric has such an associated isomorphism. Thus by a choice of I the Lie group G can be endowed with the structure of a Riemannian manifold $(G \, , << \cdot \, , \cdot >>)$.

In what follows we will specialize the notions of a Levi-Civita connection, Riemannian curvature and the notion of a distance function to Lie groups equipped with a left invariant metric. Once this is accomplished, specialization of the result of [1] is reasonably straightforward.

A. Levi-Civita Connection

On any Riemannian manifold it is known that there exists a unique connection that is metric, and torsion free [4], [9], [10], [16]. This connection is referred to as the Riemannian or Levi-Civita connection. Let $\{e_i\}$ be any basis for the Lie algebra $\mathcal{G}$ and let $\{E_i(g) = g \cdot e_i\}$ be the associated left invariant basis vector field on G. The left invariant 1-form field dual to $\{E_i\}$ will be denoted by $\{\sigma^i\}$ (that is $\sigma^i(E_j) = \delta^i_j$). Now $[e_i, e_j]_{\mathcal{G}} = C^k_{ij} e_k$, where C^k_{ij} are the structure constants of the Lie algebra $\mathcal{G}$ (note that $C^k_{ij} = -C^k_{ji}$).

For any vector V at a point $g \in G$ and a smooth vector field Y, defined locally about $g \in G$, the Levi-Civita or Riemannian connection on G is defined as follows,

$$\nabla_V Y = (dY^k(V) + \omega^k_{ij} V^i Y^j) E_k \,, \tag{1}$$

where ω^k_{ij} are the Levi-Civita connection coefficient. Using the structure constants the Levi-Civita connection coefficients ω^k_{ij} are uniquely given by

$$\omega^k_{ij} = \frac{1}{2}\left(C^k_{ij} - I^{ks}(I_{ir}C^r_{js} + I_{jr}C^r_{is})\right). \tag{2}$$

Observe that the connection coefficients are constant. Thus the connection can be expressed as,

$$g \cdot \nabla_\zeta \eta = d\eta^k(\zeta)E_k + \frac{1}{2} g \cdot \left([\zeta,\eta]_{\mathcal{G}} - I^{-1}\left(ad^*_{(\zeta)} I(\eta) + ad^*_{(\eta)} I(\zeta)\right)\right) \tag{3}$$

where $V = g \cdot \zeta$, $Y = g \cdot \eta(g)$ and $\eta(g) \in \mathcal{G}$ is a smooth function from $G \mapsto \mathcal{G}$ and $g \cdot \nabla_\zeta \eta = \nabla_{g\cdot\zeta} g \cdot \eta$.

From the torsion free property of the connection the Jacobi Lie-bracket of two vector fields $X = g \cdot \zeta(g)$, $Y = g \cdot \eta(g)$ on G can be expressed as,

$$[X, Y] \;=\; \left\{d\eta^k(\zeta) - d\zeta^k(\eta)\right\} E_k + g \cdot [\zeta, \eta]_{\mathcal{G}}. \tag{4}$$

B. Curvature

Let u, v, w be tangent vectors to G at g and let the vector fields X, Y, Z be their arbitrary smooth extensions in a neighborhood of g. Then the curvature tensor on G is defined by [9], [10], [16],

$$R(u, v)w = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{5}$$

$R(u, v)w$ at g depends only on the values of u, v, w and not on their smooth extensions. Therefore at $g \in G$ if $u = g \cdot \zeta$, $v = g \cdot \eta$, $w = g \cdot \Omega$ for fixed $\zeta, \eta, \Omega \in \mathcal{G}$, then X, Y, Z can be chosen to be the corresponding left invariant vector fields. Thus if $g \cdot R(\zeta, \eta)\Omega = g \cdot R(g \cdot \zeta\,,\, g \cdot \eta) g \cdot \Omega$ from (5) we have

$$g \cdot R(\zeta, \eta)\Omega = E_k \otimes \{2\theta^k_j(g \cdot \zeta\,,\, g \cdot \eta)\Omega^j - \omega^k_j(g \cdot [\zeta, \eta]_{\mathcal{G}})\Omega^j\}, \tag{6}$$

where $\omega^k_j = \omega^k_{ij}\sigma^i$ are the connection 1-forms and

$$\theta^k_j = \frac{1}{2} R^k_{jab}\, \sigma^a \wedge \sigma^b \;= (-\frac{1}{2}\omega^k_{rj} C^r_{ab} + \omega^k_{ar}\omega^r_{bj})\, \sigma^a \wedge \sigma^b$$

are the curvature 2-forms with the curvature coefficients R^k_{jab} computed to be,

$$R^k_{jab} = (-\omega^k_{rj} C^r_{ab} + 2\omega^k_{ar}\omega^r_{bj}). \tag{7}$$

Hence the pull back of the curvature to $\mathcal{G}$ by left translation can be expressed by,

$$R(\zeta, \eta)\Omega = e_k \{R^k_{jab}\Omega^j(\zeta^a\eta^b - \zeta^b\,\eta^a) - \omega^k_{ij} C^i_{ab}\zeta^a\eta^b\Omega^j\} \tag{8}$$

C. The Local Distance Function

Let $||\zeta||_{\mathcal{G}} = \sqrt{I\zeta \cdot \zeta}$ define the norm on $\mathcal{G}$ induced by the metric $<< \cdot\,, \cdot >>$. For two sufficiently close points $\tilde{g}$ and g there exists a unique $\zeta_e \in \mathcal{G}$ such that $\exp \zeta_e = g^{-1}\tilde{g}$. Then the distance between two sufficiently close points $\tilde{g}$ and g can be defined as,

$$d(\tilde{g}, g) = ||\zeta_e||_{\mathcal{G}}. \tag{9}$$

For a fixed g define the function

$$F(\tilde{g}) = \frac{1}{2} d(\tilde{g}, g)^2 = \frac{1}{2} I\zeta_e \cdot \zeta_e. \tag{10}$$

For $\tilde{g}$ sufficiently close to g, the gradient of $F(\tilde{g})$, denoted by $grad\, F$, is uniquely given by the relationship

$$<< grad\, F\ ,\ \tilde{g} \cdot \eta >> = < dF\ ,\ \tilde{g} \cdot \eta > = I\zeta_e \cdot \eta, \tag{11}$$

for any $\eta \in \mathcal{G}$. Thus explicitly $grad\, F = \tilde{g} \cdot \zeta_e$.

D. Simple Mechanical Control Systems on Lie Groups

A simple mechanical control system evolving on a Lie group G equipped with a left invariant metric $<< \cdot\,, \cdot >>$ is defined as a system with kinetic energy, $E(g, \dot{g})$, given by,

$$E(g, \dot{g}) = \frac{1}{2} << \dot{g}\ ,\ \dot{g} >>, \tag{12}$$

and a Lagrangian $L(g, \dot{g}) = E - U(g)$ for some smooth function $U(g)$ on G, [7], [15]. Let $I : \mathcal{G} \mapsto \mathcal{G}^*$ be the associated isomorphism where now $<< g \cdot \zeta\ ,\ g \cdot \eta >> = I\zeta \cdot \eta$. Then the Euler-Lagrange equations of motion of the system are given by,

$$\begin{aligned} \dot{g} &= g \cdot \zeta, && (13)\\ \nabla_{\dot{g}}\, \dot{g} &= g \cdot I^{-1}\left(f^c(g) + \sum_i^m u_i f^i(g)\right), &&\\ &= g \cdot S(g), && (14) \end{aligned}$$

where $f^c(g)$, $f^i(g) \in \mathcal{G}^*$ and $u_i \in \mathcal{R}$. The conservative force term $f^c(g)$ satisfies the condition $< dU\ ,\ g \cdot \Omega > = -< f^c(g)\ ,\ \Omega >$ for any $\Omega \in \mathcal{G}$ and the f^i's denote the control directions and are assumed to be linearly independent. The u_i are the magnitude of the forces and are the controls of the system. Note that (13) are the kinematic conditions and (14) are the Euler-Lagrange equations of the system. Combined they define a dynamical system on TG. Using (3) these equations can also be expressed by,

$$\begin{aligned} \dot{g} &= g \cdot \zeta\,, && (15)\\ \dot{\zeta} &= I^{-1}\left(ad^*_\zeta\, I\zeta + f^c(g) + \sum_i^m u_i f^i(g)\right), && (16) \end{aligned}$$

where now (15) – (16) defines a dynamical system on $G \times \mathcal{G}$, the left trivialization of TG.

III. An Intrinsic Observer for Velocity Estimation

In this section we apply the computations of the previous section to the result derived in Theorem 1 of [1], and obtain our main result. Namely, we specialize the intrinsic observer of [1] to simple mechanical systems on Lie groups equipped with a left invariant metric. Consider (13) – (14). Let $\tilde{g}$ be the estimated value of g. If the two points are sufficiently close then the error between $\tilde{g}$ and g is defined by $||\zeta_e||_{\mathcal{G}}$ for $\zeta_e \in \mathcal{G}$ satisfying $\exp(\zeta_e) = g^{-1}\tilde{g}$.

For a fixed $\tilde{g}$ define the function $F(g)$ as in (10). Then the gradient of F is given by (11). Thus specializing Theorem 1 of [1] to a Lie group we have that the following observer converges locally for any $\alpha, \beta > 0$.

$$\begin{aligned} \dot{\tilde{g}} &= \tilde{g} \cdot (\tilde{\zeta} - 2\alpha\zeta_e), & (17)\\ \nabla_{\dot{\tilde{g}}}\, \tilde{g} \cdot \tilde{\zeta} &= \tilde{g} \cdot \Gamma(S, \zeta_e) + \tilde{g} \cdot R(\tilde{\zeta}, \zeta_e)\tilde{\zeta} - \beta\, \tilde{g} \cdot \zeta_e & (18) \end{aligned}$$

where,

$$\Gamma(S, \zeta_e) = (S^k - \omega^k_{ij} S^i \zeta^j_e)\, e^k. \quad (19)$$

Using (3) and expanding (18) we thus have the explicit observer,

$$\begin{aligned} \dot{\tilde{g}} &= \tilde{g} \cdot (\tilde{\zeta} - 2\alpha\zeta_e), & (20)\\ \dot{\tilde{\zeta}} &= I^{-1}\left(ad^*_{\tilde{\zeta}} I\tilde{\zeta} - \alpha(ad^*_{\zeta_e} I\tilde{\zeta} + ad^*_{\tilde{\zeta}} I\tilde{\zeta}_e)\right) \\ &\quad + [\,\zeta_e\,,\ \tilde{\zeta}\,]_{\mathcal{G}} + \Gamma(S, \zeta_e) + R(\tilde{\zeta}, \zeta_e)\tilde{\zeta} - \beta\zeta_e. & (21) \end{aligned}$$

This derivation of the intrinsic observer of [1] for Lie groups does not require coordinates to be introduced on the Lie group and is uniquely specified once the structure constants of the Lie group are specified.

IV. Examples

In this section we demonstrate the constructions presented in here for the two cases where the configuration space of the simple mechanical systems are the Lie groups $SO(3)$ and $SE(3)$. In the process we also compute the Riemannian connection, Riemannian curvature and the topological metric of the Lie groups $SO(3)$ and $SE(3)$.

A. The Rotation Group $SO(3)$

The rotation group $SO(3)$ is the group of matrices $R \in GL(3, \mathcal{R})$ that satisfy the conditions $R\,R^T = R^T\,R = I$ and $det(R) = 1$. The Lie algebra $so(3)$ of $SO(3)$ is the set of traceless skew symmetric three by three matrices. Note that $so(3) \simeq \mathcal{R}^3$ where the isomorphism is defined by,

$$\Omega \in \mathcal{R} \mapsto \hat{\Omega} = \begin{bmatrix} 0 & -\Omega^3 & \Omega^2 \\ \Omega^3 & 0 & -\Omega^1 \\ -\Omega^2 & \Omega^1 & 0 \end{bmatrix} \in so(3), \quad (22)$$

where $\Omega = [\Omega^1\ \Omega^2\ \Omega^3]^T$. From here on we will use both Ω and $\hat{\Omega}$ to mean the same element of $so(3)$.

Now let us investigate the Riemannian structure of $SO(3)$. Define the isomorphism $I : so(3) \simeq \mathcal{R}^3 \mapsto so(3)^* \simeq \mathcal{R}^3$ by the positive definite matrix I. This induces a left invariant metric on $SO(3)$ by the relation,

$$<< R \cdot \Omega\,,\, R \cdot \psi >> = << \Omega\,,\, \psi >>_{so(3)} = I\Omega \cdot \psi, \quad (23)$$

for any two elements $R \cdot \Omega, R \cdot \psi\ \in T_R SO(3)$. The Lie bracket on $so(3)$ is given by,

$$[\Omega\,,\,\psi]_{so(3)} = ad_\Omega\, \psi = \Omega \times \psi. \quad (24)$$

and the dual of the ad operator is given by,

$$ad^*_\Omega\, \Pi = \Pi \times \Omega, \quad (25)$$

where $\Pi \in so(3)^* \simeq \mathcal{R}^3$. Using equations (2) and (7) the connection coefficients ω^k_{ij} and the coefficients of the curvature tensor R^k_{jab} can be calculated once the structure constants C^k_{ij} of the Lie algebra are specified.

From (15) – (16) a simple control system on $SO(3)$ takes the form,

$$\begin{aligned} \dot{R} &= R \cdot \zeta\,, & (26)\\ \dot{\zeta} &= I^{-1}\left(I\zeta\, \times\, \zeta + f^c(R) + \sum_i^m u_i f^i(R)\right). & (27) \end{aligned}$$

The intrinsic observer (20) – (21) takes the form,

$$\begin{aligned} \dot{\tilde{R}} &= \tilde{R} \cdot (\tilde{\zeta} - 2\alpha\zeta_e), & (28)\\ \dot{\tilde{\zeta}} &= I^{-1}\left(I\tilde{\zeta} \times \tilde{\zeta} - \alpha(I\tilde{\zeta} \times \zeta_e + I\zeta_e \times \tilde{\zeta})\right) \\ &\quad + \alpha\zeta_e \times \tilde{\zeta} + \Gamma(S, \zeta_e) + R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta} - \beta\zeta_e, & (29) \end{aligned}$$

where ζ_e satisfies $exp(\zeta_e) = R^T\tilde{R}$ and is given by,

$$\hat{\zeta}_e = \frac{\psi}{2\sin\psi}(R^T\tilde{R} - \tilde{R}^T R), \quad (30)$$

where, $\cos\psi = (tr(R^T\tilde{R}) - 1)/2$, for $|\psi| < \pi$, and $\sin\psi = \sqrt{1 - \cos^2\psi}$, [13]. The parallel transport term $\Gamma(R, \zeta)$ is calculated from (19) where $S(R) = f^c(R) + \sum_i^m u_i f^i(R)$ and the curvature term $R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta}$ is calculated from (8).

1) Simulation Results: In this section we demonstrate the effectiveness of the observer (28)–(29) by means of simulation. Consider the classical problem of a axisymmetric top in a gravitational field. Let $P = \{P_1, P_2, P_3\}$ be an inertial frame fixed at the fixed point of the top and let $e = \{e_1, e_2, e_3\}$ be a body fixed orthonormal frame with the origin coinciding with that of P. At $t = 0$ the two frames coincide. Then let the coordinates of a point p in the inertial frame P be given by x and in the body frame e be given by X. They are related by $x(t) = R(t)X$ where $R(t) \in SO(3)$. Let $-P_3$ be the direction of gravity and let I be the inertia matrix. The kinetic energy of the top is $K = I\zeta \cdot \zeta/2$, where ζ is the body angular velocity and the potential energy is $U(R) = mglRe_3 \cdot P_3$. Here m is the mass of the top, g is the gravitational constant, l is the distance along the e_3 axis to the center of mass. For simplicity we assume the top to be symmetric about the e_3 axis. The generalized potential forces $f^c(R)$ in the body frame will be given by the relation $< f^c(R), \zeta > = - < dU, R \cdot \zeta > = -mglR\hat{\zeta}e_3 \cdot P_3$ for any $\zeta \in so(3)$, which yields $f(R) = mglR^T P_3 \times e_3$. The metric induced on $SO(3)$ by the kinetic energy is left invariant and the system is a simple mechanical system on $SO(3)$. Thus the equations of motion on $SO(3) \times so(3)$ are given by,

$$\begin{aligned} \dot{R} &= R \cdot \zeta\,, && (31)\\ \dot{\zeta} &= I^{-1}\left(I\zeta \times \zeta + mgl\, R^T P_3 \times e_3\right). && (32) \end{aligned}$$

Since in this example it is assumed that the top is symmetric about the e_3 axis, the inertia matrix is diagonal with $I_1 = I_2$, that is, $I = diag(I_1, I_1, I_3)$. In this case if e is the canonical basis, using (24), the nonzero structure constants C_{ij}^k on $so(3) \simeq \mathcal{R}^3$ are calculated to be,

$$C_{12}^3 = 1,\, C_{13}^2 = -1,\, C_{23}^1 = 1,$$

.

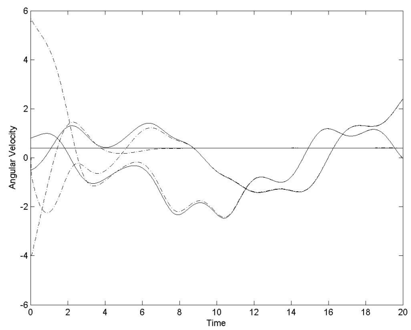

Fig. 1. Angular velocity estimates versus true values in axisymmetric top simulation. The solid curves correspond to the angular velocities of the axisymmetric top while the dashed dotted curves correspond to the estimated velocities.

With $\alpha = \beta = 10$ Implementing the observer (28) – (30) we estimate the angular velocities of the axi-symmetric top. The simulation results are shown in Fig. 1.

B. The Special Euclidian Motion Group $SE(3)$

The special Euclidian motion group $SE(3)$ is the semi-direct product $SO(3) \times_s \mathcal{R}^3$. As a matrix group, an element $A \in SE(3)$ can be represented by,

$$A = \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix}, \qquad (33)$$

where $R \in SO(3)$ and $b \in \mathcal{R}^3$. Then

$$A^{-1} = \begin{bmatrix} R^T & -R^T b \\ 0 & 1 \end{bmatrix}. \qquad (34)$$

The Lie algebra of $SE(3)$ denoted by $se(3)$ is the set of matrices,

$$\zeta = \begin{bmatrix} \hat{\Omega} & v \\ 0 & o \end{bmatrix}, \qquad (35)$$

where $\hat{\Omega} \in so(3)$ and $v \in \mathcal{R}^3$. Then $se(3) \simeq \mathcal{R}^3 \times \mathcal{R}^3$ by identifying $\zeta \in se(3)$ with $(\Omega\,,\, v) \in \mathcal{R}^3 \times \mathcal{R}^3$.

In order to investigate the Riemannian structure define the inner product on $se(3)$, $<<\cdot\,,\cdot>>_{se(3)}$ between the two elements $(\Omega, v), (\psi, u) \in se(3)$ as follows,

$$<<(\Omega, v), (\psi, u)>>_{se(3)} = I_b \Omega \cdot \psi + Mv \cdot u, \qquad (36)$$

where I_b is a positive definite matrix. Thus $I = diag(I_b,\, MI_{3\times3})$. This inner product on $se(3)$ defines a left invariant metric on $SE(3)$ in the usual way. The Lie bracket on $se(3)$ is given by,

$$\begin{aligned} ad_{(\Omega\,,\,v)}(\psi\,,\,u) &= [(\Omega\,,\,v), (\psi\,,\,u)]_{se(3)} \\ &= (\Omega \times \psi\,,\, \Omega \times u - \psi \times v). \quad (37) \end{aligned}$$

and the dual of the ad operator is given by,

$$ad^*_{(\Omega\,,\,v)} \begin{bmatrix} \Pi \\ \mu \end{bmatrix} = \begin{bmatrix} \Pi \times \Omega + \mu \times v \\ \mu \times \Omega \end{bmatrix}, \qquad (38)$$

where $(\Pi, \mu) \in se(3)^* \simeq \mathcal{R}^3 \times \mathcal{R}^3$.

Using equations (2) and (7) the connection coefficients ω_{ij}^k and the coefficients of the curvature tensor R_{jab}^k can be calculated once the structure constants of the Lie algebra $se(3) \simeq \mathcal{R}^6$ are specified.

From (15) – (16) a simple control system on $SE(3)$ takes the form,

$$\begin{aligned} \begin{bmatrix} \dot{R} & \dot{b} \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} R & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\Omega} & v \\ 0 & 0 \end{bmatrix} && (39)\\ \begin{bmatrix} \dot{\Omega} \\ \dot{v} \end{bmatrix} &= I^{-1}\Bigg(\begin{bmatrix} I_b\Omega \times \Omega \\ Mv \times \Omega \end{bmatrix} + f^c(R, b) \\ &\qquad + \sum_i^m u_i f^i(R, b) \Bigg). && (40) \end{aligned}$$

The intrinsic observer (20) – (21) takes the form,

$$\begin{aligned} \begin{bmatrix} \dot{\tilde{R}} & \dot{\tilde{b}} \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} \tilde{R} & \tilde{b} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\tilde{\Omega}} - 2\alpha\, \hat{\Omega}_e & \tilde{v} - 2\alpha\, v_e \\ 0 & 0 \end{bmatrix} && (41)\\ \dot{\Omega} &= I_b^{-1}\Big(I_b\tilde{\Omega} \times \tilde{\Omega} - \alpha(I_b\tilde{\Omega} \times \Omega_e + I_b\Omega_e \times \tilde{\Omega})\Big) \\ &\quad + \alpha\Omega_e \times \tilde{\Omega} + \Gamma(S, \zeta_e)^1 \\ &\quad + R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta}^1 - \beta\Omega_e && (42)\\ \dot{v} &= \tilde{v} \times \tilde{\Omega} - 2\alpha\tilde{v} \times \Omega_e + \Gamma(S, \zeta_e)^2 \\ &\quad + R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta}^2 - \beta v_e, && (43) \end{aligned}$$

where $\zeta_e = (\Omega_e,\, v_e)$ satisfies $\exp(\zeta_e) = A^{-1}\tilde{A}$ and is explicitly given by,

$$\begin{aligned} \Omega_e &= \frac{\psi}{2\sin\psi}(\tilde{R}R^T - R\tilde{R}^T), && (44)\\ v_e &= W^{-1}(R^T\tilde{b} - \tilde{R}^T b), && (45) \end{aligned}$$

where, $\cos\psi = (tr(R^T\tilde{R}) - 1)/2$, for $|\psi| < \pi$ [13] and

$$W = I_{3\times3} + \frac{(1 - \cos\psi)}{\psi^2}\Omega_e + \frac{(\psi - \sin\psi)}{\psi^3}\Omega_e^2.$$

The parallel transport term $\Gamma(S)$ is calculated from (19) where $S(R, b) = f^c(R, b) + \sum_i^m u_i f^i(R, b)$ and the curvature term $R_c(\tilde{\zeta}, \zeta_e)\tilde{\zeta}$ is calculated from (8). In [12] this observer is successfully employed in a observer based control strategy for the stabilization of an electrostatically actuated MOEMS device.

V. Conclusion

In this paper we present an intrinsic observer for simple mechanical systems with left invariant kinetic energy on a Lie group. The result is a specialization of a general result on arbitrary Riemannian manifolds [1], however the greatly simplified formulation, due to the the added structure, makes the result of significant interest for its own sake. In particular, the observer may be written explicitly without the need for coordinates on the Lie group, and thus the formulation is valid in any coordinate patch. A basis for the Lie algebra must be chosen, though the added complexity is minor. Once this is done, the observer is determined uniquely in terms of the structure constants of the Lie algebra. The observer is explicitly computed for two special cases of practical significance: the rotation group $SO(3)$ and the Euclidian motion group $SE(3)$. Simulations show excellent performance. The observer has also been used as the basis for closed-loop control design on an electrostatically-actuated MOEMS device, also with excellent simulation results [12].

Acknowledgements

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