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Adaptive Robust Fuzzy Control for Output Tracking [1]

Yu Tang[2]
Programa de Investigación en Matemáticas Aplicadas y Computación
Instituto Mexicano del Petroleo, Mexico City, MEXICO

***Abstract*— This paper gives an adaptive robust fuzzy control design for a class of nonlinear systems represented by input-output models to follow a reference trajectory in the presence of uncertainties. Direct approach to this problem by using fuzzy logic systems (FLS's) to approximate the unknown ideal control law (designed when the nonlinear system is known) is followed, and the semi-global stability is established based on the Lyapunov analysis. By combining the advantages of fuzzy logic reasoning, robust control and adaptive control techniques, the proposed control has the following features: ability to incorporate in a transparent way an existing control experience into the controller design, robustness to a wide rang of uncertainties, reduced *a priori* information on the nonlinear system and on-line computation load required for its implementation, and semi-global exponential output tracking to the reference signal up to a ultimately bounded error. The effectiveness of this control is demonstrated through simulations.**

Keywords: Fuzzy direct control, Lyapunov stability, output tracking, nonlinear systems, uncertainty.

I. INTRODUCTION

Since fuzzy logic systems (FLS's) [23] introduced into control designs [12], fuzzy logic controllers have had great successes in applications [22]. Due to their simplicity in design and implementation, inherent capability of dealing with uncertainties in dynamic systems and ability to incorporate easily expert experiences into the controller design, fuzzy control has been becoming one of the favorite choices for control engineers. From a practical engineering point of view, all available information should be utilized in the design of a control system. Usually, three important sources of information are available: numerical information about the measurements of variables provided by sensors, linguistic information about how the plant behaves or how to control the plant under certain conditions provided by a human expert, and structural information about the plant provided by mathematical modelling. Conventional controls can only make use of numerical and structural information and have difficulty of incorporating linguistic information. Fuzzy control may easily incorporate numerical and linguistic information into its design, but structural information is largely unexplored, specially in its early stage of development. As a consequence, a main drawback of fuzzy control systems designed based on expert experience is the lack of stability analysis, so it may not guarantee beforehand the stability and the performance of the closed-loop system. Also, the lack of a systematic design method often makes fuzzy controller design a tedious, time-consuming process, specially for complex dynamic systems.

In recent years, significant advances towards providing a systematic design of fuzzy controllers with guaranteed stability have been made by using analytical tools developed for nonlinear systems (see, *e.g.* [8] and the references cited therein), particularly those for robust control and adaptive control systems [21], [17] . By exploring structural information of the plant to be controlled, analytical approach to fuzzy control retains the salient characteristics of fuzzy control, and further can deal easily with problems that are difficult to handle with conventional nonlinear control methods, such as unmodeled dynamics, non-linearly parameterized systems and high relative-degree [6], [13], [7], [5]. Within this context, two common methods are used to design a fuzzy logic control: indirect and direct. In the former FLS's are used to approximate the unknown dynamic systems and then controllers are synthesized based on this approximation [21], [20], [17], [10], [19], while in the later controllers are directly synthesized using FLS's [21], [2], [17], [9]. Most of the results in controller designs obtained by means of FLS's are shared with those resulted from neural network control [15], [11], [14], [16], [24], [3], and in general with those control syntheses based on the so called universal approximators [18]. But in the class of approximators which are linear in the parameters, FLS's are much closer in spirit to human thinking and natural language. They provide an effective means of capturing the approximate, inexact nature of the real world, in particular, when large amount of uncertainties is present.

Although analytical methods give systematic designs of fuzzy controls with proved closed-loop stability, many results reported in the literature suffer from at least one of the following drawbacks: (1) lack of robustness to unmodeled dynamics and/or external perturbations due to only *asymptotic* convergence of the tracking error to a residual set of the origin is achieved, (2) requirement of the knowledge on the nonlinear systems, which may result difficult to obtain in practice, for controller implementation, (3) requirement of the bound on the norm of the optimal parameter vector of the universal approximator, or a compact set to which the optimal parameter vector of the universal approximator belongs, (4) heavy on-line computation burden due to updating the parameters of the universal approximator. These drawbacks create once again a gap between the control theory development and control engineering practice, because design methods that result in a high complexity, lack of robustness, or based on *a priori* information hard to obtain are rarely utilized in practice.

As an attempt to reduce this gap, this paper gives an adaptive robust fuzzy control design for a class of nonlinear systems represented by input-output models to follow a reference trajectory in the presence of uncertainties. Direct approach to this problem by using FLS's to approximate the unknown ideal control law is followed, and the semi-global stability is established based on the Lyapunov

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[2] On Leave from the National University of Mexico, P.O. Box 70-273, 04510 Mexico D.F., MEXICO, e-mail:tang@servidor.unam.mx

analysis. By combining the advantages of fuzzy logic reasoning, robust control and adaptive control techniques, the proposed control has the following features: ability to incorporate in a transparent way an existing control experience into the controller design, robustness to a wide rang of uncertainties, reduced *a priori* information on the nonlinear system and on-line computation load required for its implementation, and semi-global exponential output tracking to the reference signal up to a ultimately bounded error. The effectiveness of this control is demonstrated through simulations.

The rest of the paper is organized as follows: after the problem statement and control design for known plant given in Section 2, functional approximation using FLS's is briefly described in Section 3, which will be used in the control design in Section 4. In this Section, control design with state feedback is considered first. Next, the high-gain observer [7] is used to design an output feedback control. Section 5 gives a numerical example to illustrate the proposed control. Section 6 concludes the paper with some remarks.

II. PROBLEM STATEMENT AND CONTROL DESIGN FOR KNOWN NONLINEAR SYSTEMS

We consider in this paper the class of single-input-single-output nonlinear systems in the following form

$$\begin{aligned} \dot{\mathbf{x}} &= F(\mathbf{x}) + g(\mathbf{x})u, \\ y &= h(\mathbf{x}), \end{aligned} \tag{1}$$

where $\mathbf{x} \triangleq [x_1 \ x_2 \ ... \ x_n]^T \in \mathbb{R}^n$ is the state vector of the system, $u, \ y \in \mathbb{R}$ is the control input and measured output, respectively, $F, \ g : \mathbb{R}^n \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are unknown smooth functions of $\mathbf{x}$. In the design of the fuzzy controller, we assume the plant (1) to have the well defined relative order n. Therefore, through transformation $\mathbf{y} = T(\mathbf{x}) \triangleq [y \ y^{(1)} \ ...y^{(n-1)}]^T$, $T : \mathbb{R}^n \to \mathbb{R}^n$, being a diffeomorphism, (1) is expressed as

$$\dot{\mathbf{y}} = A_o\mathbf{y} + B_o[f(\mathbf{x}) + b(\mathbf{x})u], \tag{2}$$

with

$$f(\mathbf{x}) = L_F^n h(\mathbf{x}), \quad b(\mathbf{x}) = L_g L_F^{n-1} h(\mathbf{x}) \neq 0, \quad \forall \mathbf{x} \in \mathbb{R}^n, \tag{3}$$

$$A_o = \begin{bmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 0 \end{bmatrix}, \quad B_o = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

Assumption 1: $0 < b_0 \leq b(T^{-1}(\mathbf{y})) \leq b_1 q(\| \mathbf{y} \|)$, $\forall \mathbf{y} \in \mathbb{R}^n$, where b_0 and b_1 are unknown constants, and $q(\| \mathbf{y} \|) = 1 + \| \mathbf{y} \| + ... + \| \mathbf{y} \|^p$ with $p \geq 0$ a known integer. Throughout the paper $\| \ . \ \|$ denotes the Euclidean norm.

Let y_r be a given reference signal. We assume that y_r and its derivatives up to order n to be bounded, and $y_r^{(n)}$ to be piecewise continuous. The problem we consider in this paper is to design an output feedback control law for (1) to ensure the plant output y and its derivatives up to order $n-1$ to track the reference y_r and its corresponding derivatives within a ultimately bounded error, while maintaining all the signals bounded. Also the ultimate error bound should be made arbitrarily small by choosing appropriately controller parameters.

Let $\mathbf{y_r} \triangleq [y_r \ y_r^{(1)} \ ... \ y_r^{(n-1)}]^T \in \mathbb{R}^n$ be the reference signal vector and the tracking error be $\mathbf{e} \triangleq \mathbf{y} - \mathbf{y_r}$. Its dynamics are obtained from (2) as

$$\dot{\mathbf{e}} = A_o\mathbf{e} + B_o[f(\mathbf{x}) + b(\mathbf{x})u - y_r^{(n)}]. \tag{5}$$

It is easy to see that if the control law is chosen as

$$u = \frac{1}{b(\mathbf{x})}[K\mathbf{e} - f(\mathbf{x}) + y_r^{(n)}] \triangleq u^*(\mathbf{x}, \mathbf{e}, y_r^{(n)}), \tag{6}$$

where $K \in \mathbb{R}^{1\times n}$ is such that the matrix

$$A_c \triangleq A_o + B_o K, \tag{7}$$

is Hurwitz, *i.e*. all its eigenvalues have the negative real parts, then the tracking error $\mathbf{e}(t) = \exp(A_c t)\mathbf{e}(0) \to 0$ exponentially.

We will refer to the control law (6) as the ideal control. In the $\mathbf{y}$-coordinate, this control law can be expressed as $u^* = u^*(\mathbf{y}, \mathbf{y}_R)$ where $\mathbf{y}_R \triangleq [y_r \ y_r^{(1)} \ ... \ y_r^{(n-1)} \ y_r^{(n)}]^T \in \mathbb{R}^{n+1}$. Although can not be implemented because the unknown functions $F(\mathbf{x})$, $g(\mathbf{x})$ and $h(\mathbf{x})$ and therefore unknown $f(\mathbf{x})$ and $b(\mathbf{x})$, it is a continuous function of $\mathbf{y}$ and $\mathbf{y}_R$. So it can be approximated to any degree of accuracy in a compact set in $\mathbb{R}^n \times \mathbb{R}^{n+1}$ by a universal approximator. In the following, FLS's will be used for this purpose.

III. FUNCTIONAL APPROXIMATION USING FLS'S

Consider an n_i-inputs, single-output fuzzy logic system [21] with the product-inference rule, singleton fuzzifier, center average defuzzifier, and Gaussian membership function given by n_r fuzzy if-then rules

$$\begin{aligned} R^r \ : \quad & \text{if } \ x_1 \text{ is } A_1^r(x_1) \text{ and ... and } \ x_{n_i} \text{ is } A_{n_i}^r(x_{n_i}) \\ & \text{then } \ u = \ b^r, \end{aligned} \tag{8}$$

where R^r denotes the rth rule, $1 \leq r \leq n_r$, $\mathbf{x} = [x_1 \ ... \ x_{n_i}]^T \in X \subset \mathbb{R}^{n_i}$ and $u \in \mathbb{R}$ are the input and the output of the fuzzy logic system, respectively, with X a compact set. b^r is the fuzzy singleton for the output in the rth rule, and $A_1^r(x_1) \ ... \ A_{n_i}^r(x_{n_i})$ are fuzzy sets characterized by Gaussian membership functions

$$\mu_{A_j^r}(x_j) = exp\left\{-(\frac{x_j - c_j^r}{\sigma_j^r})^2\right\}, \tag{9}$$

where c_j^r is the center and σ_j^r the width of the Gaussian membership function. The output of the FLS is given by

$$u = \sum_{r=1}^{n_r} w^r(x) b^r, \tag{10}$$

$$w^r(\mathbf{x}) = \frac{\prod_{j=1}^{n_i} \mu_{A_j^r}(x_j)}{\sum_{j=1}^{n_r} \prod_{i=1}^{n_i} \mu_{A_i^j}(x_i)}, \quad r = 1, ..., n_r \tag{11}$$

If the membership functions (*i.e.*, $c_j^r, \ \sigma_j^r$) are fixed, the normalized firing strength (activation degree) of the rth rule w^r is a function of only $\mathbf{x}$. Therefore, the output of an FLS allows a linear parameterization in its consequence parameters b^r:

$$u(\mathbf{x}) = [b^1\ b^2\ \dots\ b^{n_r}][w^1\ w^2\ \dots\ w^{n_r}]^T \triangleq BW(\mathbf{x}). \quad (12)$$

In the rest of the paper, $B \in \mathbb{R}^{1\times n_r}$ will be referred to as the *parameter vector* of FLS's, and $W : X \rightarrow \mathbb{R}^{n_r}$ the *fuzzy basis functions*. The fuzzy rule set (8) is said to be *complete*, if for any $\mathbf{x} \in X$, there is at least one fuzzy rule fired, i.e., $\sum_{j=1}^{n_r}\prod_{i=1}^{n_i}\mu_{A_i^j}(x_i) > 0$. It is well known that FLS's (12) are universal approximator in the sense that given any real continuous function $f : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ in a compact set $X \subset \mathbb{R}^{n_i}$ and any $k > 0$ there exists an FLS (12) such that [21]

$$\sup_{\mathbf{x}\in X} |u(\mathbf{x}) - f(x)| < k. \quad (13)$$

In light of this result, the function $f(\mathbf{x})$ can be expressed as

$$f(\mathbf{x}) = B^*W(\mathbf{x}) + \Delta f(\mathbf{x}), \quad \forall \mathbf{x} \in X \subset \mathbb{R}^{n_i}, \quad (14)$$

where $\Delta f(\mathbf{x})$ is called *approximation error* satisfying

$$\sup_{\mathbf{x}\in X} |\Delta f(\mathbf{x})| < k, \quad (15)$$

and B^* is the *optimal parameter vector*

$$B^* \triangleq \arg \min_{B\in\mathbb{R}^{1\times n_i}} \left\{ \sup_{\mathbf{x}\in X} |BW(\mathbf{x}) - f(\mathbf{x})| \right\}. \quad (16)$$

In practice, the optimal parameter vector may be not unique or known. Several methods based on the gradient of an error function are available to estimate it (see, *e. g.* [21]). Also, when some part of an FLS (number of rules n_r, membership functions $\mu_{A_j^r}(x_j)$, or consequence parameters b^r) is fixed, the approximation error bound k is unknown.

IV. CONTROL DESIGN

In this section, we design a control for the plant (1) by approximating the ideal control law (6) by means of an FLS. We use the robust control technique [1] to design a signal to compensate for the parameter uncertainty arising from the unknown optimal weight matrix B^*, and the error arising from approximating the unknown ideal control law. Firstly, the control design with state feedback is considered. Next, the high-gain observer [7] is used to design an output feedback control.

A. Control law

Given $\mathbf{y}_R = [y_r\ y_r^{(1)}\ \dots\ y_r^{(n-1)}\ y_r^{(n)}]^T$, the reference and its derivatives up to the order n, we first choose the fuzzy sets $A_j^r(y_r^{(j)})$ for $0 \le j \le n$, and the corresponding membership functions $\mu_{A_j^r}(y_r^{(j)})$ as in (9). Accordingly, the fuzzy sets and the membership functions for $\mathbf{y}$ may chosen as $A_j^r(.)$, *i.e.*, $A_{n+j+1}^r(y^{(j)}) = A_j^r(y_r^{(j)})$, $0 \le j \le n-1$.

We propose the control law for the plant (1) as follows:

$$u = u_0 + u_c, \quad (17)$$

where u_0 is the nominal control representing the best available experiences on how to control the plant given by n_r fuzzy rules, $r = 1, 2, \dots, n_r$:

$$\begin{aligned} R^r: \quad & \text{if } y_r \text{ is } A_0^r(y_r) \text{ and } y_r^{(1)} \text{ is } A_1^r(y_r^{(1)}) \text{ and ... and} \\ & y_r^{(n)} \text{ is } A_n^r(y_r^{(n)}) \text{ and } y \text{ is } A_{n+1}^r(y) \text{ and ... and} \\ & y^{(n-1)} \text{ is } A_{2n}^r(y^{(n-1)}) \quad \text{then } u = b_0^r. \end{aligned} \quad (18)$$

The fuzzy rule set (18) is assumed to be complete. In occasions, an available control experience may give a fuzzy rule set that is not complete. In this case, a complete fuzzy rule set can be obtained by fulfilling the fuzzy space with fuzzy sets whose membership functions have consequence part set to zero.

This fuzzy rule set gives the fuzzy basis function $W(\mathbf{y}, \mathbf{y}_R)$ as in (11) with $n_i = 2n + 1$ and the nominal parameter vector $B^0 = [b_0^1\ \ b_0^2\ \dots\ b_0^{n_r}]$. These in turn give the nominal control as

$$u_0 = B^0W(\mathbf{y}, \mathbf{y}_R). \quad (19)$$

The component u_c is designed to compensate for the uncertainties resulting from the error between the nominal parameter and the optimal parameter as well as the approximation error

$$u_c = -\hat{\delta}^2 \frac{q^2(\| \mathbf{y} \|)p_n^T\mathbf{e}}{\hat{\delta}q(\| \mathbf{y} \|) \mid p_n^T\mathbf{e} \mid + \epsilon}, \quad (20)$$

$$\dot{\hat{\delta}} = -\sigma\hat{\delta} + \gamma \mid p_n^T\mathbf{e} \mid q(\| \mathbf{y} \|), \quad \hat{\delta}(0) > 0, \quad (21)$$

where $\epsilon > 0$, σ and $\gamma > 0$ are design parameters, $\hat{\delta}(t)$ is the estimate of δ (defined below (26)) at the instant $t \ge 0$, and $p_n \in \mathbb{R}^n$ is the last column of the matrix $P \in \mathbb{R}^{n\times n}$ resulted from

$$A_c^TP + PA_c = -Q, \quad (22)$$

for a given $0 < Q \in \mathbb{R}^{n\times n}$, where A_c is defined in (7).

B. Stability analysis

From the results in Section 2, the ideal control law (6) may be approximated by an FLS with the optimal parameter vector B^* and the fuzzy basis function $W(\mathbf{y}, \mathbf{y}_R)$ resulting an approximation error $\Delta u(\mathbf{y}, \mathbf{y}_R)$

$$u^* = B^*W(\mathbf{y}, \mathbf{y}_R) + \Delta u(\mathbf{y}, \mathbf{y}_R). \quad (23)$$

It follows from Assumption 1 that the parameter error vector $\tilde{B} \triangleq B^0 - B^*$ and the approximation error $\Delta u(\mathbf{y}, \mathbf{y}_R)$ satisfy

$$\begin{aligned} & \mid b(\mathbf{x})[\tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u(\mathbf{y}, \mathbf{y}_R)] \mid & (24) \\ \le\ & b_1q(\| \mathbf{y} \|)[\| \tilde{B} \|\| W(\mathbf{y}, \mathbf{y}_R) \| + \mid \Delta u \mid], & \\ \le\ & b_1q(\| \mathbf{y} \|)[\| \tilde{B} \| + \mid \Delta u \mid], & \\ \le\ & b_0\delta q(\| \mathbf{y} \|), & (25) \end{aligned}$$

$\forall \mathbf{y} \in Y \subset \mathbb{R}^n$ and $\mathbf{y}_R \in Y_R \subset \mathbb{R}^{n+1}$, where

$$\delta = \frac{b_1}{b_0}\rho, \quad (26)$$

for some constant $\rho \ge \| \tilde{B} \| + \mid \Delta u \mid$, Y_R is a compact set containing the reference signal and its derivatives up to order n, and Y a compact set whose definition will be clear from the subsequent analysis.

Rewrite the control law (17) as

$$u = u^* + \left\{ u_c + \tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u(\mathbf{y}, \mathbf{y}_R) \right\}, \quad (27)$$

therefore, the error dynamics in terms of the parameter uncertainty and approximation error are obtained by substituting (27) and (6) into (5)

$$\dot{\mathbf{e}} = A_c\mathbf{e} + B_o b(\mathbf{x}) \left\{ u_c + \tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u(\mathbf{y}, \mathbf{y}_R) \right\}. \quad (28)$$

Let $P = P^T > 0$ be the solution of the Lyapunov equation (22). Consider the following Lyapunov function candidate

$$V = \frac{1}{2}\mathbf{e}^T P\mathbf{e} + \frac{b_0}{2\gamma}\tilde{\delta}^2. \quad (29)$$

where $\tilde{\delta} \triangleq \hat{\delta} - \delta$. Taking the time derivative of V, it follows from (22) and (28) that

$$\begin{aligned} \dot{V} &= \frac{1}{2}\mathbf{e}^T(A_c^T P + PA_c)\mathbf{e} \quad (30) \\ &\quad + \mathbf{e}^T PB_o b(\mathbf{x}) \left\{ u_c + \tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u \right\} + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}} \\ &= -\frac{1}{2}\mathbf{e}^T Q\mathbf{e} + p_n^T\mathbf{e}b(\mathbf{x}) \left\{ u_c + \tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u \right\} \\ &\quad + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}}. \end{aligned}$$

In the next, we will bound the last two right-hand terms of (30) by using the compensation component u_c in (20)

$$\begin{aligned} & p_n^T\mathbf{e}b(\mathbf{x}) \left\{ u_c + \tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u \right\} + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}} \\ \leq \; & p_n^T\mathbf{e}b(\mathbf{x})u_c + \mid p_n^T\mathbf{e} \mid\mid b(\mathbf{x})[\tilde{B}W(\mathbf{y}, \mathbf{y}_R) - \Delta u] \mid \\ & + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}} \\ \leq \; & p_n^T\mathbf{e}b(\mathbf{x})u_c + b_0\delta \mid p_n^T\mathbf{e} \mid q(\parallel \mathbf{y} \parallel) + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}} \\ \leq \; & b_0 p_n^T\mathbf{e}u_c + b_0\delta \mid p_n^T\mathbf{e} \mid q(\parallel \mathbf{y} \parallel) + \frac{b_0}{\gamma}\tilde{\delta}\dot{\hat{\delta}} \\ = \; & b_0 \left\{ p_n^T\mathbf{e}u_c + \hat{\delta} \mid p_n^T\mathbf{e} \mid q(\parallel \mathbf{y} \parallel) \right\} \\ & + \frac{b_0}{\gamma}\tilde{\delta} \left\{ \dot{\hat{\delta}} - \gamma \mid p_n^T\mathbf{e} \mid q(\parallel \mathbf{y} \parallel) \right\} \\ \leq \; & \epsilon b_0 - \frac{b_0\sigma}{\gamma}\tilde{\delta}^2 + \frac{b_0\sigma}{\gamma}\delta^2. \quad (31) \end{aligned}$$

The second inequality follows from (25), the third inequality is because $0 < b_0 \leq b(\mathbf{x})$ and $p_n^T\mathbf{e}u_c \leq 0$, the last inequality is because that $p_n^T\mathbf{e}u_c + \hat{\delta} \mid p_n^T\mathbf{e} \mid q(\parallel \mathbf{y} \parallel) \leq \epsilon$ and $-\tilde{\delta}\hat{\delta} \leq -\tilde{\delta}^2 + \delta^2$, since $\hat{\delta}(t) \geq 0, \forall t \geq 0$.

It follows from (30)-(31) that

$$\dot{V} \leq -2\alpha V + \epsilon_e, \quad \forall t \geq 0, \quad (32)$$

where $\alpha \triangleq \min\left\{ \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)},\ \sigma \right\}$, and $\epsilon_e \triangleq b_0(\epsilon + \frac{\sigma}{\gamma}\delta^2)$.

Let $r_E \triangleq \max\left\{ \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \parallel \mathbf{e}(0) \parallel^2 + \frac{b_0\tilde{\delta}^2(0)}{\lambda_{\min}(P)\gamma},\ \frac{\epsilon_e}{\lambda_{\min}(P)\alpha} \right\}$, $E \triangleq \left\{ \mathbf{e} \in I\!\!R^n \mid\parallel \mathbf{e} \parallel \leq r_E^{1/2} \right\} \subset I\!\!R^n$ a compact set, and $Y \subset I\!\!R^n$ the corresponding compact set for a given $Y_r \subset I\!\!R^n$. Then E is invariant, *i.e.*, for any initial condition $\mathbf{e}(0) \in E \Rightarrow \mathbf{e}(t) \in E$, $\forall t \geq 0$. Therefore, $\mathbf{y}(t) \in Y$, $\forall t \geq 0$.

A ultimate error bound is given by $\frac{\epsilon_e}{\lambda_{\min}(P)\alpha}$, which can be made arbitrarily small by properly choosing the design parameters.

C. Output feedback

We now use the high-gain observer (HGO) [7], [8] to estimate the state $\mathbf{y}$. It is shown in [8] (pg. 622) that the design of such HGO satisfies the separation principle, provided that the state feedback control guarantees the semi-global boundedness of $\mathbf{y}$ and the observer gains are high enough. The HGO is given by

$$\begin{aligned} \dot{\hat{e}}_i &= \hat{e}_{i+1} + \frac{\alpha_i}{\epsilon_{ob}^i}(e_1 - \hat{e_1}), \quad 1 \leq i \leq n-1, \\ \dot{\hat{e}}_n &= \frac{\alpha_n}{\epsilon_{ob}^n}(e_1 - \hat{e}_1), \end{aligned} \quad (33)$$

where $0 < \epsilon_{ob} << 1$ is a design parameter, and $\alpha_i > 0$ are chosen such that the roots of $s^n + \alpha_1 s^{n-1} + ... + \alpha_{n-1}s + \alpha_n$ have negative real parts. The estimate of $\mathbf{y}$ is therefore

$$\hat{y}^{(i-1)} = \hat{e}_i + y_r^{(i-1)}, \quad 1 \leq i \leq n. \quad (34)$$

The controller is implemented by substituting $\mathbf{y}$ and $\mathbf{e}$ in (17)-(21) by their estimates. The control is saturated outside a compact region of interest to prevent the peaking introduced by the HGO [8], *i.e.*,

$$u = Ssat\left(\frac{u_0 + u_c}{S}\right), \quad (35)$$

where $sat(\cdot)$ is the saturation function and S the saturation limit, chosen to cover the region of interest. The overall stability is ensured by the semi-global stability provided by the state feedback and the separation principle [8].

D. Design Procedure and Discussions

1) Design Procedure: Summarizing the control design gives the following design procedure: Given a nonlinear system in (1), verify Assumption 1 for an integer $p \geq 0$.

For state feedback control:

Step 1. Design the membership functions according to a given reference signal and its derivatives up to the order n.

Step 2. Get the fuzzy basis functions $W(\mathbf{y}, \mathbf{y}_R)$ as in (11) with $n_i = 2n + 1$.

Step 3. If there is a previous control experience as in (18), incorporate it into u_0 through B^0. Otherwise set $B^0 = 0$.

Step 4. Implement the compensation component and the uncertainty bound estimator (20)-(21). Choose $K = [k_1\ k_2\ ...\ k_n]$ such that the eigenvalues of the matrix A_c in (7) have negative real parts, and a matrix $Q > 0$, solve the equation $A_c^T P + PA_c = -Q$ with $A_c \triangleq A_o + B_o K$ to get the last column of P, p_n. Choose the positive constants ϵ, σ, γ such that the ultimate bound on the tracking error is acceptable.

For output feedback control:

Step 5. Implement the HGO (33). Choose the positive constants $\epsilon_{ob} << 1$, and $\alpha_i > 0$ such that the roots of

$s^n + \alpha_1 s^{n-1} + ... + \alpha_{n-1}s + \alpha_n$ have negative real parts. Choose the saturation limit $S > 0$ to cover the region of interest.

2) Discussions: Remark 4.1: The only *a priori* information on the unknown plant (1) required for the controller implementation is a upper order p of the bounding polynomial $q(\| \mathbf{y} \|)$ in Assumption 1. The incorporation of a control experience is optional. If it is incorporated, lees control effort will be needed from the compensation component in the control signal, giving a smoother control signal. This is true specially in the transient period. A control experience may be obtained from a human expert as discussed previously, or from an existing controller by collecting its input-output data and then training a fuzzy controller to get the nominal vector B^0 by means of off-line training methods, *e. g.*, Anfis [4].

Remark 4.2: Compared to the adaptive control with a universal approximator (fuzzy logic systems or neural networks, see *e.g.*, [15], [21], [2], [22], [11], [14], [20], [17], [16], [24], [9] and [3]), the robust adaptive approach followed here alleviates the need of off-line training of the universal approximator, the knowledge of a compact set to which the optimal parameter vector of the universal approximator belongs, the bound on the norm of the optimal parameter of the universal approximator, or the knowledge on the nonlinear systems, such as a known upper bound on the unknown function $F(\mathbf{x})$ and $g(\mathbf{x})$ for controller implementation. Also the on-line computation load of updating the universal approximator parameter vector (whose dimension may exceeds sometimes over hundred) is reduced to estimating only the uncertainty bound. More importantly, semi-global exponential tracking up to a ultimately bounded error is achieved, which will provide robustness to unmodeled dynamics and/or external perturbations.

V. A NUMERICAL EXAMPLE

In this section, we will illustrate the proposed control by a numerical example, carried out in Matlab/Simulink. In the simulation we use the model of a physical system, namely inverted pendulum used in many similar works (*e. g.*, [20], [9]), to show how to design the proposed control.

In the simulation, the plant dynamics are of the form

$$\begin{aligned} x^{(2)} &= F(x, \dot{x}) + g(x)u, \\ y &= x, \end{aligned} \tag{36}$$

and the reference signal y_r is generated as the output of a low-pass filter

$$y_r = \frac{1000}{(s+10)^3}\left(\frac{\pi}{30}\sin t\right). \tag{37}$$

The reference signal and its derivatives up to order 2 are shown in Fig. 1. The parameters used in the controller are $K = [-2 \ \ -3]$. The control component (20) - (21) are implemented with design parameters $\epsilon = 0.1$, $\sigma = 10$ and $\gamma = 10000$, and the initial conditions $\hat{\delta}(0) = 10$. The bounding polynomial in Assumption 1 is $q(\| \mathbf{y} \|) = 1$. The matrix in the Lyapunov function is $Q = diag[1 \ 1]$, giving as the last column of the resulted solution $p_n^T = [0.25 \ 0.25]$.

The parameters used in the HGO (33) are $\epsilon_{ob} = 10^{-5}$, $S = 100$, $\alpha_1 = 2$, $\alpha_2 = 1$, and the initial conditions for the observer (33) were $\mathbf{e}(0) = [0 \ 0]^T$.

Two Gaussian membership functions for each variables (y_r, $\dot{y}_r$, $\ddot{y}_r$, y, $\dot{y}$) are used resulting in $n_r = 32$ fuzzy rules. The center and the width of these membership functions are chosen according to the reference signal and its derivatives up to the order $n = 2$ to cover a compact set of interests.

The dynamic equation of an inverted pendulum is (36) with

$$F(x, \dot{x}) = [\dot{x} \quad \frac{g(m_c+m)\sin x - ml\dot{x}^2 \cos x \sin x}{l[\frac{4}{3}(m_c+m) - m\cos^2 x]}]^T \tag{38}$$

$$g(x) = [0 \quad \frac{\cos x}{l[\frac{4}{3}(m_c+m) - m\cos^2 x]}]^T, \tag{39}$$

which is already in the form of (2). The parameters' meaning and their values (in SI units) used in the simulation are: $m = 0.1$ the mass of the pendulum, $m_c = 1$ the mass of the car, $l = 0.5$ the length of the pendulum, $g = 9.8$ the gravity acceleration. The initial condition for the pendulum is $\mathbf{x}(0) = [-\frac{\pi}{60}, \ 0]^T$. All the conditions for the simulation are the same as in [20].

For this example, an intuitive control based on the experience is

if y_r is $N(y_r)$ and y is $N(y)$ then $u = PS$
if y_r is $P(y_r)$ and y is $N(y)$ then $u = PL$
if y_r is $N(y_r)$ and y is $P(y)$ then $u = NL$
if y_r is $P(y_r)$ and y is $P(y)$ then $u = NS$

where N and P mean fuzzy set Negative and Positive, characterized by a Gaussian membership function (9) with $(center, \ width) = (-\frac{\pi}{30}, \frac{\pi}{60})$ and $(\frac{\pi}{30}, \frac{\pi}{60})$, respectively, and fuzzy singleton NS, PS, NL, PL stand for Negative-Small, Positive-Small, Negative-Large and Positive-Large, with values $PS = -NS = 0.5$, $PL = -NL = 5$, respectively. Representing this control experience into the form (18) gives the rth rule (super-index $1 \le r \le n_r = 32$)

R^r : if y_r is $A_0^r(y_r)$ and $\dot{y}_r$ is $A_1^r(\dot{y}_r)$
and $\ddot{y}_r$ is $A_2^r(\ddot{y}_r)$ and y is $A_3^r(y)$ and $\dot{y}$ is $A_4^r(\dot{y})$
then $u = b_0^r$,

where each fuzzy set A_j^r is either N or P. The corresponding singleton b_0^r is assigned a value in $\{NS, PS, NL, PL\}$ according to which fuzzy sets y_r and y belong to. With the fuzzy basis function given by this fuzzy rule set, the control (17) with (20) and (21) is applied, the results are shown in Fig.2.

VI. CONCLUSIONS

A direct robust fuzzy control has been studied for the control of a class of non-linear systems represented by input-output models to track a reference signal. Semi-global exponential tracking to the reference signal up to a ultimate error was achieved, and the ultimate error can be made small by properly choosing some controller parameters.

By combine the advantages of the fuzzy logic reasoning, robust control as well as adaptive control techniques, the proposed control design is capable of using available information – numerical, linguistic and structural to give a simple controller, yet robust to a wide rang of uncertainties. Numerical simulations are included to illustrate the proposed control scheme.

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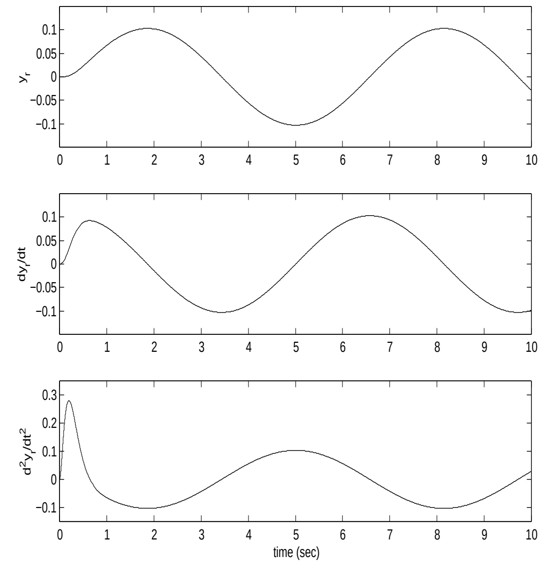

Fig. 1. The reference signal and its derivatives.

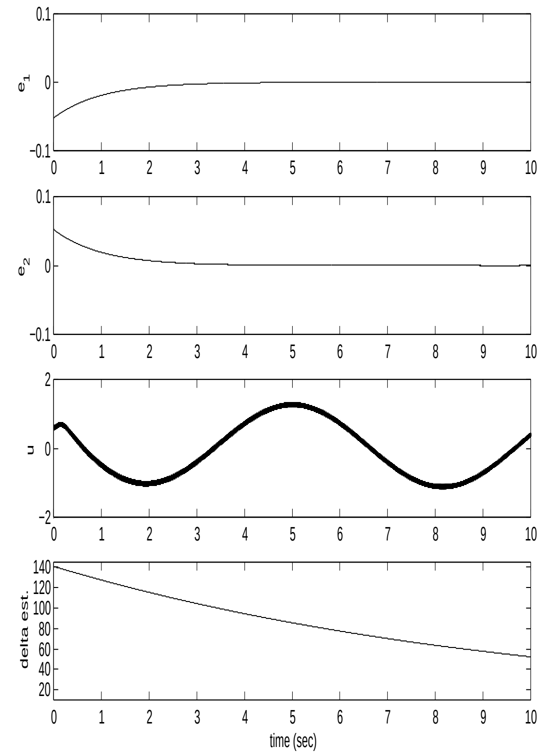

Fig. 2. Output feedback control of the inverted pendulum.