Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

On Integral Control in Backstepping: Analysis of Different Techniques

Roger Skjetne[1,2] Thor I. Fossen[1,2]

[1]*Centre for Ships and Ocean Structures*
Norwegian University of Science and Technology
NO-7491 Trondheim, Norway

[2]*Department of Engineering Cybernetics*
Norwegian University of Science and Technology
NO-7491 Trondheim, Norway

E-mail: skjetne@ieee.org E-mail: tif@itk.ntnu.no

***Abstract*— Including integral action in a nonlinear backstepping design is the topic of this paper. Two methods for adding integral feedback are proposed and analyzed. These are compared to the more traditional methods: 1) adaptive backstepping, and 2) plant augmentation that adds an extra relative degree and thus gives one extra step of backstepping. A test plant is used to compare the different control laws. Based on the theoretical analysis and the simulations, some interesting conclusions are made for each integral control strategy.**

I. Introduction

Integral control is one of the principle components in feedback control for industrial use. In theory it has the capacity to remove constant steady-state offsets in a closed-loop regulation system. In practice it is a robustifying part of the feedback controller, alleviating problems with unmodeled dynamics, parameter deviations, and slowly varying disturbances. Already one of the first commercially available feedback controllers, the ship autopilot *Metal-Mike* designed by Elmer Sperry in 1922, included integral action to remove offsets in the heading error; see [1], [2]. In fact, through the theoretical analysis of [3] this controller paved the way for what later is known as the PID controller. In modern state-of-the-art industrial control designs for ships, integral feedback is a requirement. See for instance [4] where an LQG feedback controller with integral action is designed and tested in full scale for dynamic positioning of a supply vessel. More recently, nonlinear control designs with integral action included, have been implemented for ship control. The paper [5] is an example where integral action by adaptation is used to counteract slowly varying environmental forces.

For linear SISO systems a single integrator can be augmented to the transfer function to compensate a single bias term. For MIMO systems with m inputs and m outputs, this generalizes to m integrators that can be augmented to compensate m bias terms. In nonlinear control systems, feedback laws are often designed by the aid of a control Lyapunov function (CLF). For mechanical systems, this often results in a nonlinear PD-type control law; see for instance [6]. However, awareness to the importance of integral effect is clearly made in [7], [8], [9], [5], [10].

The purpose of this paper is to present and analyze different methods for integral action in backstepping to robustly deal with a bias and to ensure zero steady-state tracking error. We therefore consider a class of nonlinear mechanical systems given by the vector relative degree 2 plant

$$\dot{x}_1 = f_1(x_1) + G_1(x_1)x_2 \tag{1a}$$
$$\dot{x}_2 = f_2(x_1, x_2) + G_2(x_1, x_2)u + b \tag{1b}$$

where x_1, x_2 are the states, u is the control input, the functions f_1, f_2, G_1, G_2 are smooth, and b is an unknown constant bias. The plant is called 'undisturbed' if $b = 0$. It is assumed that $G_1(x_1)$ and $G_2(x_1, x_2)$ are nonsingular for all x_1, x_2, and all variables are of the same dimension, that is, $x_1, x_2, u, b \in \mathbb{R}^m$. The plant is a strict feedback-form system representing a class of fully actuated m-degrees-of-freedom (m-DOF) mechanical systems. Examples are 3-DOF ocean surface vessels, 6-DOF autonomous underwater vehicles, or m-DOF robotic manipulators. For a mechanical system, (1a) typically represents the perfectly known kinematic equation, and (1b) represents the more uncertain dynamic equation. The bias b may represent constant (or slowly varying) environmental forces or perhaps steady-state offsets necessary to maintain the desired equilibrium.

The most common way to include integral action in backstepping is to use parameter adaptation [11]. Another method is to augment the plant dynamics with the integral state $\dot{\xi} = x_1 - x_d(t)$. Together with (1), the resulting system is still in strict feedback form; however, the vector relative degree is increased to 3 and three steps of backstepping is therefore necessary. Based on the complexity of the nonlinear functions f_1 and G_1 this may involve cumbersome differentiation of two virtual controls resulting in a complex nonlinear control law. The next section will present a design method which avoids a three step design. This results in a negative semi-definite CLF time derivative. Usually, this must be analyzed further by Krasowskii-LaSalle's principles or Barbalat's Lemma. Recently, a new version of Matrosov's Theorem [12] has been developed in [13]. This is a convenient tool used in this paper to directly guarantee *Uniform Global Asymptotic Stability* of the closed-loop systems. In fact, a predecessor to this new theorem is [10] where integral action is the main motivation.

Notation: In GS, LAS, LES, UGAS, UGES, etc., stands G for Global, L for Local, S for Stable, U for Uniform, A for Asymptotic, and E for Exponential. Total time derivatives of $x(t)$ are denoted $\dot{x}$, $\ddot{x}$, $x^{(3)}, \ldots, x^{(n)}$, while a superscript denotes partial differentiation: $f^t(x,y,t) := \frac{\partial f}{\partial t}$, $f^{x^2}(x,\theta,t) := \frac{\partial^2 f}{\partial x^2}$, and $\alpha^{y^n}(x,\theta,t) := \frac{\partial^n f}{\partial y^n}$, etc. The Euclidean vector norm is $|x| := (x^\top x)^{1/2}$, while $||x||$ denotes the $\operatorname{ess\,sup}\{|x(t)| : t \geq 0\}$.The induced norm of a matrix A is denoted $||A||$, and $\operatorname{col}(x,y)$ means the column vector of x and y stacked on top of each other.

A. Motivational Examples

Example 1: A time-varying closed-loop system

For the scalar plant

$$\dot{x} = (1+x^2)u - x^3 + b \tag{2}$$

let the task be to track the signal $x_d(t)$ without steady-state error regardless of an unknown constant bias b. Additionally, the monotone damping term $-x^3$ should not be cancelled directly since this may result in unwanted transients in the control law. Under the assumption that $b = 0$, the control law

$$u = \frac{1}{(1+x^2)}\left[-(x - x_d(t)) + x_d(t)^3 + \dot{x}_d(t)\right] \tag{3}$$

renders the equilibrium $e = x - x_d(t) = 0$ UGES. This is verified by the Lyapunov function $V_1(x,t) := \frac{1}{2}(x - x_d(t))^2$ for which the time derivative satisfy $\dot{V}_1 \leq -(x - x_d(t))^2$ by using the property $(x-y)(x^3-y^3) \geq 0, \forall x, y$. If on the other hand $b \neq 0$, then the closed-loop equilibrium is shifted with the result that zero tracking error cannot be achieved. The above control law consists of a proportional feedback (P) term $-(x - x_d(t))$ and a feed-forward (FF) term $x_d(t)^3 + \dot{x}_d(t)$. To ensure zero tracking error, the integral equation $\xi(t) = \int_0^t (x(\tau) - x_d(\tau))d\tau$ is introduced, and (3) is modified with the inclusion of the integral feedback (I) term $-k\xi$ where $k > 0$. This results in a nonlinear PI+FF control law. The closed-loop system

$$\begin{aligned}\dot{\xi} &= x - x_d(t)\\ \dot{x} &= -k\xi - (x - x_d(t)) - \left(x^3 - x_d(t)^3\right) + \dot{x}_d(t) + b\end{aligned}$$

has an invariant manifold $\mathcal{M}_t = \{(x,\xi) : x = x_d(t),\ \xi = \frac{1}{k}b\}$ on which the tracking error is zero. To analyze stability, we define $V_2(x,\xi,t) := V_1(x,t) + \frac{1}{2}k(\xi - \frac{1}{k}b)^2$. This gives

$$\dot{V}_2 \leq -(x - x_d(t))^2 \leq 0$$

which is only negative semi-definite. This shows directly that $\mathcal{M}_t$ is UGS, but not necessarily UGAS. Since the closed-loop system is time-varying, Krasovskii-LaSalle's principles cannot be applied to show convergence to $\mathcal{M}_t$. Instead, one must resort to Barbalat's Lemma [14] or some other theorem to prove UGAS. In the next section, application of Matrosov's Theorem [13] will prove UGAS of $\mathcal{M}_t$ directly.

Example 2: PID control by 3 steps of backstepping

Consider the relative degree 2 plant

$$\begin{aligned}\dot{x}_1 &= g(x_1)x_2 - x_1^3\\ \dot{x}_2 &= u + b\end{aligned} \tag{4}$$

where the function $g(x_1)$ is strictly nonzero. We can think of x_1 as a 'position' that we want to steer to a desired position $x_d(t)$, and x_2 as a 'velocity.' Accordingly, feedback from $x_1 - x_d(t)$ gives "proportional action" (P), while feedback from x_2 gives "derivative action" (D). We augment the state space with the integrator $\dot{\xi} = x_1 - x_d(t)$ so that the plant becomes a relative degree 3 strict feedback form system. Three steps of backstepping on the resulting system is the most common way to design a nonlinear PID controller; see for instance [15, Chapter 7.4.5]. A problem with this procedure is that it will attempt to drive the extra integrator state to zero. For $b = 0$ this will certainly solve the tracking objective. However, we study what happens in the closed-loop for $b \neq 0$.

Define $z_1 := x_1 - \alpha_1(\xi, x_1, t)$ and $z_2 := x_2 - \alpha_2(\xi, x_1, x_2, t)$ where α_1 and α_2 are virtual control functions to be specified. Letting

$$\begin{aligned}\alpha_1 &= -k\xi + x_d(t)\\ \alpha_2 &= \frac{1}{g(x_1)}\left[-\xi - c_1 z_1 + x_1^3 - k(x_1 - x_d(t)) + \dot{x}_d(t)\right]\end{aligned}$$

then

$$\begin{aligned}u = &-g(x_1)z_1 - c_2 z_2 + \alpha_2^{\xi}(\xi, x_1, x_2, t)\left[x_1 - x_d(t)\right]\\ &+\alpha_2^{x_1}(\xi, x_1, x_2, t)\left[(1+x_1^2)x_2 - x_1^3\right] + \alpha_2^{t}(\xi, x_1, x_2, t)\end{aligned}$$

is a nonlinear PID control law which renders the equilibrium $(\xi, z_1, z_2) = 0$ UGES for $b = 0$. With $z := \operatorname{col}(\xi, z_1, z_2)$ the closed-loop (time-varying) system becomes $\dot{z} = A(x_1(t))z + eb$ where $e = [0,0,1]^\top$ and

$$A\left(x_1(t)\right) = \begin{bmatrix} -k & 1 & 0\\ -1 & -c_1 & g(x_1(t))\\ 0 & -g(x_1(t)) & -c_2 \end{bmatrix}.$$

The stable equilibrium $z_{eq} = 0$ cease to exist for $b \neq 0$. However, since the unforced closed-loop system is UGES and b enters linearly, the system is input-to-state stable (ISS) with b as input and linear gain dependent on k, c_1, c_2; see [16], and thus the solutions will stay bounded. If the term $x_1(t) = z_1 - k\xi + x_d(t)$ is time-varying, the steady-state solution is time-varying. If, on the other hand, $x_d(t) = x_{ref}$ is constant then there exist a new constant equilibrium given by $z_{eq} = -A(x_{ref})^{-1}eb$. In this equilibrium, the tracking error $x_1 - x_{ref} = 0$ as desired, while the integral state ξ is driven to the non-zero value $bg(x_{ref})/(c_2 + kc_1c_2 + kg(x_{ref})^2)$.

II. Integral Control

Consider the plant (1) and let the control objective be to solve the tracking problem $\lim_{t\to\infty}[x_1(t) - x_d(t)] = 0$ where $x_d(t)$ is a bounded smooth reference signal. This should be achieved in presence of a constant bias $b \neq 0$. To counteract the effect of the constant unkown bias, we consider using integral action and investigate two methods next, denoted A and B. With the backstepping state transformation $z_1 := x_1 - x_d(t)$ and $z_2 := x_2 - \alpha_1$ where α_1 is a virtual control, Method A will add feedback from the integral term $\xi(t) = \int_0^t z_1(\tau)d\tau$ in the first step of backstepping. This is perhaps the most intuitive method. However, it will be shown that for a generic plant model this method cannot guarantee convergence of the tracking error, even when the reference signal is constant. To overcome this problem, Method B will instead use feedback from $\xi(t) = \int_0^t z_2(\tau)d\tau$ in the second step of backstepping. This ensures that ξ is matched with b in the closed-loop and can therefore asymptotically cancel it.

Method *A*, Step 1: For the plant

$$\dot{\xi} = z_1 \tag{5}$$

$$\dot{z}_1 = f_1(x_1) + G_1(x_1)z_2 + G_1(x_1)\alpha_1 - \dot{x}_d(t) \tag{6}$$

choose the first CLF as

$$V_{1A}(\xi, x_1, t) := \frac{1}{2}\xi^\top K_A \xi + \frac{1}{2} z_1^\top z_1 \tag{7}$$

where $K_A = K_A^\top > 0$. The time derivative becomes

$$\dot{V}_{1A} = z_1^\top [K_A\xi + f_1 + G_1\alpha_1 - \dot{x}_d] + z_1^\top G_1 z_2$$

and the virtual control are chosen as

$$\alpha_1(\xi, x_1, t) = G_1^{-1}\left[-K_A\xi - C_1 z_1 - f_1 + \dot{x}_d\right] \tag{8}$$

where $C_1 = C_1^\top > 0$. Recall that 'good' nonlinearities in f_1 can be exploited at this point when designing α_1, as was the case in Example 1. The above choice yields

$$\dot{V}_{1A} = -z_1^\top C_1 z_1 + z_1^\top G_1 z_2 \tag{9}$$

which for $z_2 = 0$ is only negative semidefinite. For now we allow this and continue the design.

Method *A*, Step 2: Differentiating z_2 with respect to time gives

$$\dot{z}_2 = f_2(x_1, x_2) + G_2(x_1, x_2)u - \dot{\alpha}_1 + b \tag{10}$$

where

$$\dot{\alpha}_1 = \alpha_1^\xi z_1 + \alpha_1^{x_1}\left[f_1 + G_1 x_2\right] + \alpha_1^t. \tag{11}$$

With the choice

$$V_{2A}(\xi, x_1, x_2, t) := V_{1A} + \frac{1}{2} z_2^\top z_2, \tag{12}$$

the derivative along the state solutions becomes

$$\begin{aligned}\dot{V}_{2A} = &-z_1^\top C_1 z_1 + z_2^\top b \\ &+ z_2^\top\left[G_1^\top z_1 + f_2 + G_2 u - \dot{\alpha}_1\right]\end{aligned} \tag{13}$$

and the control u is chosen as

$$u = G_2^{-1}\left[-G_1^\top z_1 - C_2 z_2 - f_2 + \dot{\alpha}_1\right] \tag{14}$$

where $C_2 = C_2^\top > 0$. To see that this is in fact a nonlinear PID+FF control law, we state it in the original coordinates as

$$\begin{aligned}u = &-K_I(x_1, x_2)\xi - K_P(x_1, x_2)\left[x_1 - x_d(t)\right] \\ &-K_D(x_2)\left[x_2 - G_1(x_1)^{-1}\left(\dot{x}_d - f_1(x_1)\right)\right] \\ &+F_F(\xi, x_1, x_2, t)\end{aligned} \tag{15}$$

where

$$\begin{aligned}K_I(x_1, x_2) &:= G_2^{-1} C_2 G_1^{-1} K_A \\ K_P(x_1, x_2) &:= G_2^{-1} G_1^\top + G_2^{-1} C_2 G_1^{-1} C_1 \\ &\quad + G_2^{-1} G_1^{-1} K_A \\ K_D(x_2) &:= G_2^{-1} C_2 \\ F_F(\xi, x_1, x_2, t) &:= -G_2^{-1} f_2 + G_2^{-1}\alpha_1^{x_1}\left[f_1 + G_1 x_2\right] \\ &\quad + G_2^{-1}\alpha_1^t.\end{aligned}$$

This control law yields

$$\dot{V}_{2A} = -z_1^\top C_1 z_1 - z_2^\top C_2 z_2 + z_2^\top b \tag{16}$$

and the closed-loop system

$$\begin{aligned}\dot{\xi} &= z_1 \\ \dot{z}_1 &= -K_A\xi - C_1 z_1 + G_1(x_1(t))z_2 \\ \dot{z}_2 &= -G_1(x_1(t))^\top z_1 - C_2 z_2 + b.\end{aligned} \tag{17}$$

Proposition 1: The equilibrium $(\xi, z_1, z_2) = 0$ of the closed-loop system (17) with $b = 0$ is UGAS.

Proof: To prove stability for the case $b = 0$ we apply Theorem 1 by [13]. The origin $(\xi, z_1, z_2) = 0$ is UGS by (12) and (16). Define $W_1 := V_{2A}$ and $W_2 := \xi^\top z_1$, and accordingly $Y_1 := -z_1^\top C_1 z_1 - z_2^\top C_2 z_2$ and $Y_2 := z_1^\top z_1 - \xi^\top K_A \xi - \xi^\top C_1 z_1 + \xi^\top G_1(x_1(t))z_2$. Then $\dot{W}_1 = Y_1$ and $\dot{W}_2 = Y_2$. From the boundedness of $x_d(t)$ and continuity of $G_1(x_1)$ we get that $\phi(z_1, t) := G_1(z_1 + x_d(t))$ and $W_i(\xi, z_1, z_2)$, $i = 1, 2$, are bounded for all bounded values of (ξ, z_1, z_2). Moreover, for $\xi \neq 0$ then $Y_1 = 0 \Rightarrow Y_2 < 0$ and $Y_1 = Y_2 = 0 \Rightarrow (\xi, z_1, z_2) = 0$. This proves that the origin $(\xi, z_1, z_2) = 0$ is UGAS for $b = 0$. ■

For the case $b \neq 0$, the closed-loop is a UGAS system forced by the constant input b. Such a constant input may destabilize the system; see for instance [14, p. 177]. Investigating (17) shows that if $x_d(t)$ is time-varying, then this system in general cannot settle at a constant equilibrium due to the time-varying term $G_1(x_1(t))$. As a result, convergence $z_1(t) \to 0$ must fail. If, on the other hand, $x_d(t) = x_{ref} =$'constant' then there exist a constant equilibrium for the system - which may or may not be UGAS.

Remark 1: In the case when G_1 is a constant matrix, then the closed-loop (17) becomes linear and can be written $\dot{z} = Az + eb$, where A is Hurwitz and $e = [0, 0, 1]^\top$. For $b \neq 0$, the new equilibrium becomes $z_0 = -A^{-1}eb$. In this equilibrium the tracking error is zero. By defining $w := z - z_0$ we get $\dot{w} = Aw$ which shows that z_0 is a UGES equilibrium. This conclusion also holds for the 3 step design in Example 2 if $g(x_1)$ is a constant.

Method *B*, Step 1: For the equation

$$\dot{z}_1 = f_1(x_1) + G_1(x_1)z_2 + G_1(x_1)\alpha_1 - \dot{x}_d(t), \tag{18}$$

choose the first CLF as

$$V_{1B}(x_1, t) := \frac{1}{2} z_1^\top z_1. \tag{19}$$

The time derivative becomes

$$\dot{V}_{1B} = z_1^\top [f_1 + G_1\alpha_1 - \dot{x}_d] + z_1^\top G_1 z_2$$

and the virtual control are chosen as

$$\alpha_1(x_1, t) = G_1^{-1}\left[-C_1 z_1 - f_1 + \dot{x}_d\right] \tag{20}$$

where $C_1 = C_1^\top > 0$. 'Good' nonlinearities in f_1 can be exploited at this point when designing α_1. The above choice yields

$$\dot{V}_{1B} = -z_1^\top C_1 z_1 + z_1^\top G_1 z_2 \tag{21}$$

which for $z_2 = 0$ is negative definite.

Method *B*, Step 2: Introducing the integral term and differentiating z_2 with respect to time gives

$$\dot{\xi} = z_2 \tag{22}$$

$$\dot{z}_2 = f_2(x_1, x_2) + G_2(x_1, x_2)u - \dot{\alpha}_1 + b \tag{23}$$

where

$$\dot{\alpha}_1 = \alpha_1^{x_1}\left[f_1 + G_1 x_2\right] + \alpha_1^t. \quad (24)$$

With the choice

$$V_{2B}(\xi, x_1, x_2, t) := V_{1B} + \frac{1}{2}\xi^\top K_B \xi + \frac{1}{2} z_2^\top z_2 \quad (25)$$

where $K_B = K_B^\top > 0$, and the control

$$u = G_2^{-1}\left[-G_1^\top z_1 - K_B\xi - C_2 z_2 - f_2 + \dot{\alpha}_1\right], \quad (26)$$

the derivative along the state solutions becomes

$$\dot{V}_{2B} = -z_1^\top C_1 z_1 - z_2^\top C_2 z_2 + z_2^\top b \quad (27)$$

where $C_2 = C_2^\top > 0$. By writing the control law in the original coordinates, one can again verify that it indeed is a nonlinear PID+FF control law. The closed-loop system becomes

$$\begin{aligned} \dot{z}_1 &= -C_1 z_1 + G_1(x_1(t)) z_2 \\ \dot{\xi} &= z_2 \\ \dot{z}_2 &= -G_1(x_1(t))^\top z_1 - K_B\xi - C_2 z_2 + b. \end{aligned} \quad (28)$$

Proposition 2: The equilibrium $(\xi, z_1, z_2) = 0$ of the closed-loop system (28) with $b = 0$ is UGAS.

Proof: Recall Matrosov's Theorem as stated in [13, Theorem 1]. For $b = 0$, the origin $(\xi, z_1, z_2) = 0$ is UGS by (25) and (16). Define $W_1 := V_{2B}$ and $W_2 := \xi^\top z_2$, and accordingly, $Y_1 := -z_1^\top C_1 z_1 - z_2^\top C_2 z_2$ and $Y_2 := z_2^\top z_2 - \xi^\top G_1(x_1(t))^\top z_1 - \xi^\top K_B \xi - \xi^\top C_2 z_2$. Then $\dot{W}_1 = Y_1$ and $\dot{W}_2 = Y_2$. From the boundedness of $x_d(t)$ and continuity of $G_1(x_1)$ we get that $\phi(z_1, t) := G_1(z_1 + x_d(t))$ and $W_i(\xi, z_1, z_2)$, $i = 1, 2$, are bounded for bounded (ξ, z_1, z_2). Moreover, for $\xi \neq 0$ then $Y_1 = 0 \Rightarrow Y_2 < 0$ and $Y_1 = Y_2 = 0 \Rightarrow (\xi, z_1, z_2) = 0$. This proves that the origin $(\xi, z_1, z_2) = 0$ is UGAS. ■

The advantage of this method appears for the case when $b \neq 0$. This gives the new (constant) equilibrium $(\xi, z_1, z_2) = (K_B^{-1} b, 0, 0)$. Letting $\tilde{\xi} = \xi - K_B^{-1} b$ yields

$$\begin{aligned} \dot{z}_1 &= -C_1 z_1 + G_1(x_1(t)) z_2 \\ \dot{\tilde{\xi}} &= z_2 \\ \dot{z}_2 &= -G_1(x_1(t))^\top z_1 - K_B\tilde{\xi} - C_2 z_2 \end{aligned} \quad (29)$$

which is the same as (28) for $b = 0$. By Proposition 2 this gives the following result:

Theorem 3: The equilibrium $(\xi, z_1, z_2) = (K_B^{-1} b, 0, 0)$ of the closed-loop system (28) is UGAS for any constant b.

In other words, this method guarantees zero tracking error for the generic plant (1), even when the reference is time-varying.

Remark 2: By designing an adaptive control law using adaptive backstepping [11], one obtains the same controller as in Method *B*. Indeed, let $\hat{b}$ be the bias estimate. Then the adaptive closed loop becomes equal to (28) by setting $\hat{b} = K_B \xi$.

III. A Comparison Between Different Designs

In this section a comparative study is performed on different integral control strategies for the plant

$$\begin{aligned} \dot{x}_1 &= g(x_1) x_2 - x_1^3 \\ \dot{x}_2 &= u + b. \end{aligned} \quad (30)$$

The control laws are called *'Method A,'* '*Method B,'* and *'3 Step'* according to the designs in Section II and in Example 2. In addition, an *'ISS-backstepping'* control law (see [17] and references therein) is implemented for comparison. Since the adaptive backstepping control law yields the exact same responses as for 'Method *B*,' this has been left out. However, advantages and drawbacks of the adaptive backstepping control law relative to the other control laws tested here will be discussed below.

The objective is for x_1 to asymptotically track a desired reference signal $x_d(t)$, and the resulting tracking control laws are shown in Table I.

TABLE I

Integral and ISS Control Laws for the benchmark testplant

Method *A*:
$\dot{\xi} = z_1$
$z_1 := x_1 - x_d, \quad z_2 := x_2 - \alpha_1$
$\alpha_1 = \frac{1}{g}\left[-k\xi - c_1 z_1 + x_d(t)^3 + \dot{x}_d(t)\right]$
$u = -g z_1 - c_2 z_2 + \alpha_1^\xi z_1 + \alpha_1^{x_1}\left(g x_2 - x_1^3\right) + \alpha_1^t$
Method *B*: (equivalent to adaptive backstepping)
$\dot{\xi} = z_2$
$z_1 := x_1 - x_d, \quad z_2 := x_2 - \alpha_1$
$\alpha_1 = \frac{1}{g}[-c_1 z_1 + x_d(t)^3 + \dot{x}_d(t)]$
$u = -k\xi - g z_1 - c_2 z_2 + \alpha_1^{x_1}\left(g x_2 - x_1^3\right) + \alpha_1^t$
3 Step:
$\dot{\xi} = x_1 - x_d(t)$
$z_1 := x_1 - \alpha_1, \quad z_2 := x_2 - \alpha_2$
$\alpha_1 = -k\xi + x_d(t)$
$\alpha_2 = \frac{1}{g}[-\xi - c_1 z_1 + x_1^3 + \dot{x}_d(t) + \alpha_1^\xi (x_1 - x_d(t))]$
$u = -g z_1 - c_2 z_2 + \alpha_2^\xi (x_1 - x_d(t))$ $+\alpha_2^{x_1}\left(g x_2 - x_1^3\right) + \alpha_2^t$
ISS backstepping:
$z_1 := x_1 - x_d, \quad z_2 := x_2 - \alpha_1$
$\alpha_1 = \frac{1}{g}\left[-c_1 z_1 + x_d(t)^3 + \dot{x}_d(t)\right]$
$u = -g z_1 - c_2 z_2 - k z_2 + \alpha_1^{x_1}\left(g x_2 - x_1^3\right) + \alpha_1^t$

A. Simulation Results

In the following simulations, the feedback gains was set to $k = 0.25$, $c_1 = 1.5$, and $c_2 = 0.5$. The closed-loops were tested for a constant reference $x_d = x_{ref} = 1$ and also for a sinusoidal reference $x_d(t) = 1 + 0.5\sin(0.2\pi t)$ fed through a reference filter to produce the necessary derivatives.

The first test is a step response ($x_{ref} = 1$) using $g(x_1) = 1 + x_1^2$ and $b = 0$. The responses for $x_1(t)$ are shown in Figure 1. It is seen that the 'ISS-backstepping' design has a superior response in this case. The other control loops experience a transient in the integral term that requires time to converge. Notice also the indication that $x_1(t)$ for 'Method *A'* converges very slowly to the reference.

In the second test, the bias is turned on, $b = 2$, and it is clearly seen in Figure 2 that this is detrimental to the 'Method *A'* and '3 Step' closed loops. While the '3 Step' rapidly enters a bounded oscillation, the 'Method *B'* are trying to converge before it also suddenly starts oscillating. The 'ISS-backstepping' design obtains a steady-state

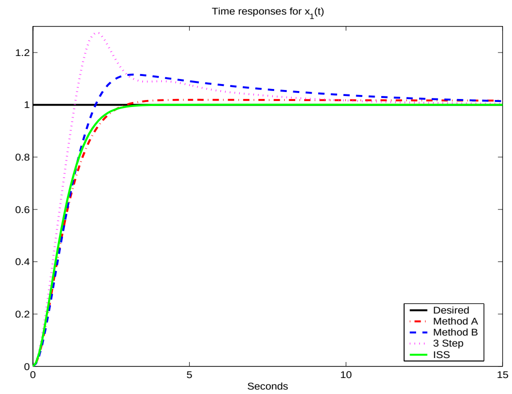

Fig. 1. Step responses ($x_{ref} = 1$) for the closed-loop systems with $g(x_1) = 1 + x_1^2$ and $b = 0$.

offset, as expected, but this is preferable to the oscillating responses. Only 'Method *B*' satisfies the control objective with success.

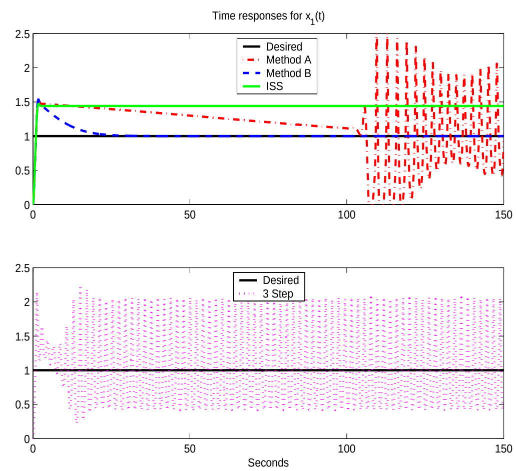

Fig. 2. Step responses ($x_{ref} = 1$) for $g(x_1) = 1 + x_1^2$ and $b = 2$. Upper plot: Method A, B, and ISS. Lower plot: 3 Sted Design.

When the reference is a sinusoid, we observe that the 'Method *B*' response still converges nicely as shown in Figure 3. The other closed-loops clearly fails. On the other hand, when the function $g(x_1)$ is constant, then Figure 4 verifies the stability and convergence properties of all designed closed loops; see Remark 1.

Remark 3: Simulations have shown that the incapacity of 'Method *A*' and '3 Step' to counteract a bias, even when the reference is constant, is gain dependent. Adjusting the positive feedback gains k, c_1, and c_2 (usually increasing them) will eventually give stability and convergence. A quantitative analysis of this is future work.

B. Discussion

The only integral control design presented here that for a generic plant model (1) guarantees UGAS with convergence

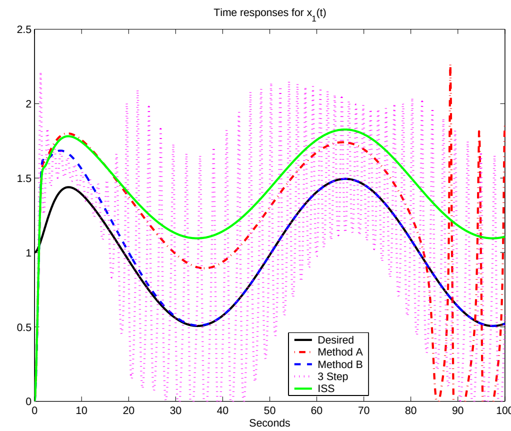

Fig. 3. Responses to $x_d(t) = 1 + 0.5\sin(0.2\pi t)$ with $g(x_1) = 1 + x_1^2$ and $b = 2$.

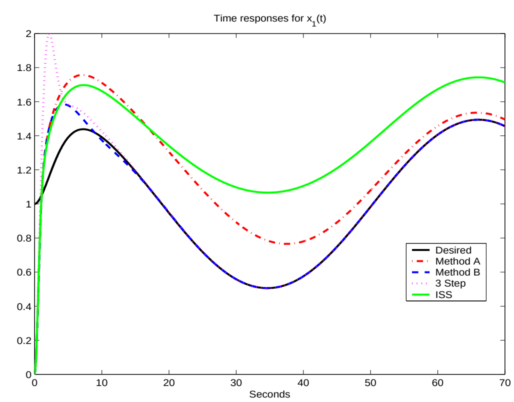

Fig. 4. Responses to $x_d(t) = 1 + 0.5\sin(0.2\pi t)$ with $g(x_1) = 1$ and $b = 2$.

of the tracking error in presence of a constant bias b is 'Method *B*,' which is equivalent to the feedback law obtained by adaptive backstepping [11]. In the plant considered, only one bias is included with constant unity regressor, that is, $b = 1b$. Adaptive backstepping is a much stronger design methodology since it can handle unknown biases in all state equations, and also more general regressors, for instance, $\varphi(x)^\top b$. However, adding pure integral action in an already available nonlinear control law is often sought for since it is believed to be a robustifying term that can handle disturbances and dynamics that have not been included in the original design model. Doing so has shown to be applicable by the help of Matrosov's Theorem.

However, as illustrated in the above simulations, such designs must be performed with care. In special cases (for instance when $G_1(x_1)$ is constant) the integral designs will robustly deal with constant disturbances. However, for other plant models (for instance when $G_1(x_1) = 1 + x_1^2$) then UGAS of the undisturbed loop does not imply stability and convergence in presence of constant disturbances. In fact, certain feedback gains that guarantee UGAS in the undisturbed case may give large oscillations for nonzero but small bias terms. This is the weakness of control laws that only guarantee a UGAS undisturbed closed loop: UGAS do not imply ISS.

The properties of each integral control method are summarized below:

Method *A*:

- In general UGAS closed-loop for $b = 0$, but no robustness is guaranteed for $b \neq 0$.
- When $G_1(x_1)$ is constant it gives a UGES closed-loop for any constant b.
- When x_d is constant it seems to provide stability and convergence for higher feedback gains (no proof).
- No convergence in general when $x_d = x_d(t)$.
- Can be mixed with other designs, for instance, adaptive control to give "adaptive PID control" or ISS backstepping to give "ISS PID control."

Method *B*:

- Guarantees UGAS and convergence of tracking error in presence of any constant b in all considered cases.
- Adaptive backstepping by tuning function design gives the same controller.
- Specialized for the plant (1), that is, matched uncertainty with constant regressor.

3 Step:

- A 3 step backstepping design is more cumbersome.
- In general UGES closed-loop for $b = 0$, and robust with respect to b since the closed-loop is ISS.
- When $G_1(x_1)$ is constant it gives a UGES closed-loop for any constant b.
- When x_d is constant it seems to provide stability and convergence for higher feedback gains (no proof).
- No convergence in general when $x_d = x_d(t)$.

Adaptive-backstepping:

- The most general method since it guarantees stability and tracking error convergence for multiple biases and regressor/bias structures.
- Involves in general a rather complex nonlinear design, and it is specialized to deal with unknown constant model parameters.
- It has otherwise the same conclusions as for Method *B*.

ISS-backstepping:

- Guarantees bounded solutions with linear gain from b to $x_1 - x_d(t)$.
- Increasing the disturbance damping may increase control effort.
- Superior performance in undisturbed case, but results in steady-state error in general.

IV. Conclusion

This paper has elaborated on several designs for integral action in a nonlinear backstepping design. A class of vector relative degree 2 systems with a matched bias was considered. For such systems, integral action can be included in the design either by adaptive backstepping or by a three step design on an augmented plant model. This paper has shown that integral action can be included at any convenient location in the closed-loop by the help of a control Lyapunov function for which the time derivative only becomes negative semidefinite. Matrosov's Theorem is then applied to show UGAS of the overall closed-loop.

What variable to take integral feedback from (here z_1 or z_2) must be decided in case by case. In this paper, two such *feedbacks* (Methods *A* and *B*) were proposed and analyzed. Though derived differently, Method *B* was shown to be equivalent to adaptive backstepping for the considered plant. While Method *B* solved the control objective with success in the simulation case, Method *A* experienced undesirable behavior. This illustrated that care must be taken when designing integral feedback since in worst case it may hurt more than it helps.

References

[1] S. Bennet, *A History of Control Engineering 1800-1930*. Peter Peregrinus Ltd., 1979.

[2] Sperry Marine, "A short history of sperry marine." Internet, accessed Aug. 2003. <http://www.litton-marine.com/company-information_sperry-history.asp>.

[3] N. Minorsky, "Directional stability of automatically steered bodies," *J. Am. Soc. Naval Eng.*, vol. 34, p. 284, 1922.

[4] A. J. Sørensen, S. I. Sagatun, and T. I. Fossen, "Design of a dynamic positioning system using model-based control," *Contr. Eng. Practice*, vol. 4, no. 3, pp. 359–368, 1996.

[5] T. I. Fossen and J. P. Strand, "Nonlinear passive weather optimal positioning control (wopc) system for ships and rigs: experimental results," *Automatica*, vol. 37, no. 5, pp. 701–715, 2001.

[6] B. Paden and R. Panja, "Globally asymptotically stable 'PD+' controller for robot manipulators," *Int. J. of Control*, vol. 47, no. 6, pp. 1697–1712, 1988.

[7] I. Kanellakopoulos and P. T. Krein, "Integral-action nonlinear control of induction motors," in *Proc. 12th IFAC World Congress Automatic Control*, (Sydney, Australia), pp. 251–254, IFAC, July 1993.

[8] Y. Tan, J. Chang, H. Tan, and J. Hu, "Integral backstepping control and experimental implementation for motion system," in *Proc. IEEE Int. Conf. Contr. Appl.*, (Anchorage, Alaska, USA), pp. 367–372, IEEE, Sept. 2000.

[9] R. Ortega, A. Loría, P. J. Nicklasson, and H. Sira-Ramírez, *Passivity-based Control of Euler-Lagrange Systems*. London: Springer-Verlag, 1998.

[10] T. I. Fossen, A. Loría, and A. R. Teel, "A theorem for UGAS and ULES of (passive) nonautonomous systems: robust control of mechanical systems and ships," *Int. J. Robust Nonlinear Contr.*, vol. 11, pp. 95–108, 2001.

[11] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*. New York: John Wiley & Sons Ltd, 1995.

[12] V. M. Matrosov, "On the stability of motion," *J. Appl. Math. Mech.*, vol. 26, no. 5, pp. 1337–1353, 1962.

[13] A. Loría, E. Panteley, D. Popović, and A. R. Teel, "An extension of Matrosov's theorem with application to stabilization of nonholonomic control systems," in *Proc. 41st IEEE Conf. Decision & Control*, (Las Vegas, NV, USA), pp. 1528–1533, IEEE, Dec. 2002.

[14] H. K. Khalil, *Nonlinear Systems*. New Jersey: Prentice-Hall, Inc, 3 ed., 2002.

[15] T. I. Fossen, *Marine Control Systems: Guidance, Navigation, and Control of Ships, Rigs and Underwater Vehicles*. Trondheim, Norway: Marine Cybernetics, 2002.

[16] E. D. Sontag and Y. Wang, "On characterizations of the input-to-state stability property," *Systems Contr. Lett.*, vol. 24, no. 5, pp. 351–359, 1995.

[17] R. Freeman and P. Kokotović, *Robust nonlinear control design*. Systems & Control: Foundations & Applications, Boston, MA: Birkhäuser Boston Inc., 1996. State-space and Lyapunov techniques.