Optimization of a joint sensor placement and robust estimation scheme for distributed parameter processes subject to worst case spatial disturbance distributions

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Abstract

In this paper, we consider the development of a systematic optimization process that can be used to locate sensors in distributed parameter systems. We focus on two components in this process: developing a performance metric, and invoking optimization algorithms to find the best location of sensors according to this metric. In choosing a metric, we incorporate the notion of a worst-case, spatially distributed disturbance. The included numerical study uses a variety of second-order parabolic problems.

1. Introduction

A practical engineering concern in the design of control systems is the location of actuators and sensors. Actuator placement has been the subject of numerous studies (see, e.g., the survey papers by van de Wal and de Jager [11] and Kubrusly and Malebranche [10]). Optimization techniques have been proposed for optimal actuator placement by the authors in [7], [2] and [6]. The dual problem of optimal sensor location has been studied for distributed parameter systems through the use of functional gains [3, 9, 1]. These gains are the kernel of the feedback operator. The philosophy is that regions where these kernel functions have the most "support" are the regions where sensors need to be located. This idea is formally presented in [8] where the domain is subdivided according to weights associated with the gains. These subdivisions suggest sensor locations.

Our approach constructs a performance measure based on either the observability grammian, the solution to the filter Riccati equation or a transfer function in the state estimation problem. The latter allows us to consider worst-case distributed disturbances. In all of these cases, the value of the performance measure depends on the location of the sensor. Although we plot the entire performance indicator for our one-dimensional example problems, in practical applications, one would rely on an optimization algorithm to determine the best place to locate the sensors.

The problem is presented in Section 2 along with a mathematical description of the dynamics that describe the class of dynamical systems under consideration. The main results are summarized in Section 3 and numerical results are presented in Sections 4 and 5. Conclusions and future work follow in Section 6.

2. Problem statement

We consider a class of parabolic partial differential equations expressed by

$$\begin{aligned} \frac{\partial x}{\partial t}(t,\xi) &= A(\xi)x(t,\xi)+b(\xi)u(t)+d(\xi)\nu(t) \\ y(t) &= \int_{\Omega} c(\xi)x(t,\xi)\,d\xi + w(t), \end{aligned} \tag{1}$$

where $A(\xi)$ is a strongly elliptic operator [5], $\xi \in \Omega \subset \mathbb{R}^n$ ($n = 1,2,3$) denotes the spatial variable, $x(t,\xi)$ is the solution, $u(t)$ the control signal, $b(\xi)$ the distribution function of the actuator, $d(\xi)$ the distribution function of the disturbance entering the system and $w(t)$ the temporal component of the disturbance. It is assumed that partial state measurements are possible at regions of the spatial domain via the output $y(t)$ with $c(\xi)$ denoting the output distribution function that describes the type and location of the sensing devices. Associated with the above are the corresponding boundary conditions (Dirichlet, Neumann or mixed) and given initial conditions $x(0,\xi) = x_0(\xi)$. The above distributed parameter system may be placed in an abstract framework [4] via the general evolution equation

$$\begin{aligned}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}_1\nu(t) + \mathcal{B}_2 u(t)\\ z(t) &= \mathcal{C}(\theta)x(t) + w(t),\end{aligned} \tag{2}$$

where $x \in \mathcal{X}$ ($= L^2(\Omega)$) is the state of the system, $\mathcal{X}$ the state space (Hilbert space), u denotes the control signal, ν denotes the system disturbance, $y \in \mathcal{Y}$ the measured output signal and w the measurement noise. The output operator $\mathcal{C}(\cdot)$ is parameterized by the candidate sensor locations $\theta \in \Theta$ to emphasize the dependence of the output operator on the sensor location. The set of candidate sensor locations Θ is defined via

$$\Theta = \Big\{\theta \in \Omega : (\mathcal{C}(\theta), \mathcal{A}) \text{ is approx. observable}\Big\}. \tag{3}$$

We are now in a position to define the problem under consideration.

Problem statement: *The problem at hand is to choose the sensor location from the set Θ such that is produces the "best" possible state estimator against all possible spatial distributions of disturbances $d(\xi)$ and which provides an estimator with a desired spatial robustness.*

3. Main Results

We follow different approaches for the sensor location that either enhance the observability properties of the system or improve its ability to provide a better state estimator.

3.1. Enhanced observability

The resulting "optimal" sensor locations should be such that they at least provide enhanced observability. Thus one may search in the set Θ for the sensor positions that yield the locations that make the system "more" observable.

Using enhanced observability measures, one may choose the "optimal" sensor locations θ from the set Θ that maximize the bound $\alpha = \alpha(\theta)$ in

$$\langle \mathcal{W}_{ob}(\theta)\phi, \phi\rangle_{\mathcal{X}} \geq \alpha\|\phi\|^2_{\mathcal{X}}, \qquad \phi \in \mathcal{X}, \tag{4}$$

where $\mathcal{W}_{ob}(\theta)$ denotes the θ-parameterized *observabilty Gramian operator* defined via

$$\begin{aligned}\langle \mathcal{W}_{ob}(\theta)\phi_1, \mathcal{A}\phi_2\rangle_{\mathcal{X}} + \langle \mathcal{A}\phi_1, \mathcal{W}_{ob}(\theta)\phi_2\rangle_{\mathcal{X}} &=\\ -\langle \mathcal{C}(\theta)\phi_1, \mathcal{C}(\theta)\phi_2\rangle_{\mathcal{X}}, &\quad \theta \in \Theta,\end{aligned} \tag{5}$$

for $\phi_1, \phi_2 \in \mathcal{D}(\mathcal{A})$. In other words, the "best" sensor locations are given via

$$\boxed{\theta^{opt} = \arg\sup_{\theta\in\Theta} \alpha(\theta)}, \tag{6}$$

where $\alpha(\theta)$ is given via (1), (2). It should be noted that under conditions on self-adjointness and boundedness of the Gramian operator, one may replace the coercivity bound by the trace of the operator.

3.2. Optimal state estimator

Since the immediate goal is the construction of an optimal state estimator, it seems then a natural progression to require that the sensor location results in the "best" possible state estimator. Towards that end, we propose a sensor location-parameterized optimal (Kalman) estimator

$$\dot{\hat{x}}(t) = \Big(\mathcal{A} - \mathcal{L}(\theta)\mathcal{C}(\theta)\Big)\hat{x}(t) + \mathcal{B}_2 u(t) + \mathcal{L}(\theta)y(t), \tag{7}$$

where the filter operator $\mathcal{L}(\theta) \triangleq \mathcal{S}(\theta)\mathcal{C}^*(\theta)W^{-1}$, and $\mathcal{S}(\theta)$ is the sensor-parameterized positive operator solution to the filter Riccati operator equation

$$\begin{aligned}&\langle \mathcal{A}^*\phi, \mathcal{S}(\theta)\psi\rangle_{\mathcal{X}} + \langle \mathcal{S}(\theta)\phi, \mathcal{A}^*\psi\rangle_{\mathcal{X}} + \langle \phi, \mathcal{B}_1\mathcal{B}_1^*\psi\rangle_{\mathcal{X}}\\ &-\langle \phi, \mathcal{S}(\theta)\mathcal{C}^*(\theta)W^{-1}\mathcal{C}(\theta)\mathcal{S}(\theta)\psi\rangle_{\mathcal{X}} = 0, \quad \theta \in \Theta,\end{aligned} \tag{8}$$

for $\phi, \psi \in \mathcal{D}(\mathcal{A}^*)$. Therefore, the best optimal sensor location is found as the one that minimizes the θ-parameterized mean reconstruction error and which is expressed in terms of the trace of the variance operator

$$\mathcal{E}\Big[\langle x(t) - \hat{x}(t), x(t) - \hat{x}(t)\rangle^2\Big] = \text{trace}\Big[\mathcal{S}(\theta)\Big]. \tag{9}$$

This optimal sensor scheme is given below via

$$\boxed{\theta^{opt} = \arg\inf_{\theta\in\Theta} \text{trace}\Big[\mathcal{S}(\theta)\Big].} \tag{10}$$

3.3. Robustness of I/O map

We now consider frequency criteria for the selection of the optimal sensor location as it pertains to the optimal filter. Towards that end, we consider the *state estimation error* $e(t) \triangleq x(t) - \hat{x}(t)$ given by

$$\dot{e}(t) = \Big(\mathcal{A} - \mathcal{L}(\theta)\mathcal{C}(\theta)\Big)e(t) + \mathcal{B}_1\nu(t) - \mathcal{L}(\theta)w(t), \tag{11}$$

or alternatively via

$$e(t) = \mathcal{T}(t,t_0;\theta)e(0) + \int_0^t \mathcal{T}(t,\tau;\theta)\Big(\mathcal{B}_1\nu(\tau) - \mathcal{L}(\theta)w(\tau)\Big)d\tau,$$

where $\mathcal{T}(t,t_0;\theta)$ denotes the θ-parameterized C_0 semigroup generated by $\mathcal{A} - \mathcal{L}(\theta)\mathcal{C}(\theta)$. The goal here is to select the sensor location so that the effects of $\nu(t)$ on the state error $e(t)$ are minimized. A measure of this may be taken as the transfer function T_{ev} from $\nu(t)$ to $e(t)$, parameterized by the sensor location, and given by

$$T_{ev}(s;\theta) = I\Big(sI - (\mathcal{A} - \mathcal{L}(\theta)\mathcal{C}(\theta))\Big)\mathcal{B}_1. \tag{12}$$

In a similar fashion, we propose the following for the optimal sensor location

$$\boxed{\xi^{opt} = \arg\min_{\xi\in\Theta} \|T_{ev}(s;\theta)\|_2.} \tag{13}$$

4. Finite dimensional implementation and numerical considerations

The above scheme is approximated via an exponential detectability-preserving scheme. The corresponding sensor location optimal measures are given below.

4.1. Enhanced observability

One now solves for the θ-parameterized Gramian $W_{ob}(\theta)$ given by

$$A^T W_{ob}(\theta) + W_{ob}(\theta)A = -C^T(\theta)C(\theta) \tag{14}$$

and the associated optimization is

$$\boxed{\theta^{opt} = \arg\max_{\theta\in\Theta} \text{trace}\left[W_{ob}(\theta)\right].} \tag{15}$$

4.2. Optimal state estimator

The associated filter matrix Riccati equation is

$$A\Sigma(\theta) + \Sigma(\theta)A^T - \Sigma(\theta)C^T(\theta)W^{-1}C(\theta)\Sigma(\theta) + B_1B_1^T = 0 \tag{16}$$

and the sensor optimization is given by

$$\boxed{\theta^{opt} = \arg\min_{\theta\in\Theta} \text{trace}\left[\Sigma(\theta)\right].} \tag{17}$$

4.3. Robustness of I/O map

The finite dimensional state error equation is now given by

$$\dot{e} = \Big(A - L(\theta)C(\theta)\Big)e + B_1\nu(t) - L(\theta)w(t) \tag{18}$$

and the $\mathcal{H}^2$ norm of the associated transfer function is given by

$$\|T_{ev}(s;\theta)\|_2 = \sqrt{\text{trace } (W_c(\theta))}, \tag{19}$$

where $W_c(\theta)$ is the controllability gramian of the pair $(A - L(\theta)C(\theta), B_1)$ and is found via the solution to the Lyapunov equation

$$\Big(A - L(\theta)C(\theta)\Big)W_c(\theta) + W_c(\theta)\Big(A - L(\theta)C(\theta)\Big)^T = -B_1B_1^T. \tag{20}$$

Therefore, the best sensor is found via

$$\boxed{\theta^{opt} = \arg\min_{\theta\in\Theta} \left[W_c(\theta)\right].} \tag{21}$$

Remark 4.1 *It should be noted that one may use the $\mathcal{H}^\infty$ norm in* (13) *or* (19),(21) *instead of the $\mathcal{H}^2$, to arrive at*

$$\boxed{\theta^{opt} = \arg\min_{\theta\in\Theta} \|T_{ev}(s;\theta)\|_\infty.}$$

The latter is more attractive from a computational point of view as it is computationally easier to calculate it.

5. Numerical results

We report results of a study in which a suboptimal procedure was used in the 1-D heat equation with Dirichlet boundary conditions. Specifically, a sensor location-parameterized state estimator was considered and subsequently the $\mathcal{H}^\infty$ and/or $\mathcal{H}^2$ norms of the transfer function $T_{ev}(s)$ were considered for a range of distributions $d(\xi)$ that include distributions of the eigenfunctions. The details of this study are as follows: Specifically, we consider the partial differential equation

$$\begin{aligned}
&\frac{\partial x}{\partial t}(t,\xi) = 0.01\frac{\partial^2 x}{\partial \xi^2}(t,\xi) + b(\xi)u(t) + d(\xi)\nu(t)\\
&x(t,0) = x(t,L),\\
&y(t) = \int_0^L \delta(\xi-\theta)x(t,\xi)\,d\xi + w(t) = x(t,\theta) + w(t)
\end{aligned}$$

which has eigenfunctions and eigenvalues given by $\phi_k(\xi) = \sin\left(\frac{k\pi\xi}{L}\right)$, $\lambda_k = -k^2\pi^2$, $k \geq 1$. Then, using a Galerkin scheme, we approximate the above PDE to arrive at the finite dimensional system

$$\begin{aligned}
\dot{x} &= Ax + B_1w + B_2u\\
z &= C(\theta)x.
\end{aligned}$$

Various disturbance distribution functions that have the same L_2 norm, i.e.

$$\int_\Omega |d(\xi)|^2\,d\Omega = 1,$$

are used to find the sensor locations. Therefore, the set of admissible distribution functions for disturbances to be considered, is be chosen as

$$\mathcal{D} = \left\{d \in \mathcal{C}(\Omega) : \int_\Omega |d(\xi)|^2\,d\Omega = 1\right\}.$$

Six different distribution functions $d(\xi)$ are used in this study to demonstrate the effects of spatial variability on es-

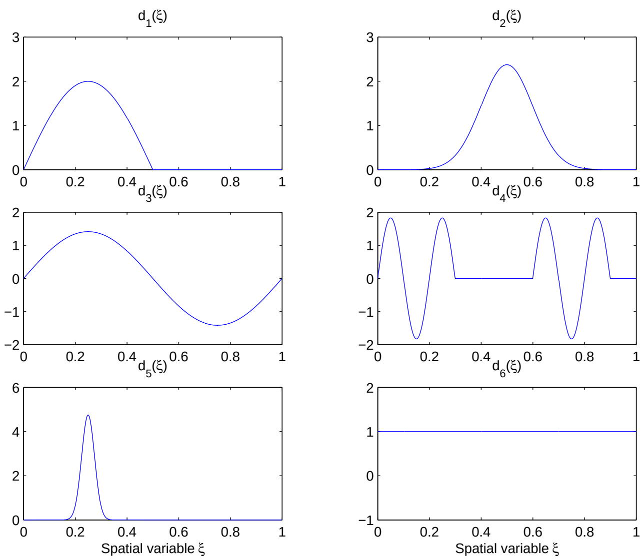

Figure 1. Distributions of disturbances normalized with respect to $\int_0^L |d_i(\xi)|^2 \, d\xi = 1$.

timator robustness and sensor location

$$
\begin{aligned}
d_1(\xi) &= \chi_{[0,0.5L]}(\xi)\sin(2\pi\xi/L) \\
&= \chi_{[0,0.5L]}(\xi)\phi_2(\xi), \\
d_2(\xi) &= \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left(-\frac{1}{2}\left(\frac{(\xi-\mu_2)}{\sigma_2}\right)^2\right), \\
d_3(\xi) &= \sin(2\pi\xi/L) \\
&= \phi_2(\xi), \\
d_4(\xi) &= \left(\chi_{[0,0.3L]}(\xi)+\chi_{[0.6L,0.9L]}(\xi)\right)\sin(10\pi\xi/L) \\
&= \left(\chi_{[0,0.3L]}(\xi)+\chi_{[0.6L,0.9L]}(\xi)\right)\phi_{10}(\xi), \\
d_5(\xi) &= \frac{1}{\sqrt{2\pi}\sigma_5}\exp\left(-\frac{1}{2}\left(\frac{(\xi-\mu_5)}{\sigma_5}\right)^2\right), \\
d_6(\xi) &= 1,
\end{aligned}
$$

where $d(\xi)$ are related to the disturbance operator via

$$\langle \mathcal{B}_1 \nu(t), \phi \rangle_X = \int_0^L d(\xi)\phi(\xi)\, d\xi \; \nu(t).$$

These normalized disturbance distributions are depicted in Figure 1.

5.1. Time domain criteria

In the first part of our investigation, we consider an optimal estimator measure which minimizes the sensor location-parameterized Gramian (14). As expected, it predicts the midpoint as the optimal sensor location. Similarly, the filter covariance in (17) in Figure 3 considers this for all 6 disturbance distributions. In this case, one observes that the optimal location is different for different disturbance distributions. For example, distribution $d_2(\xi)$ (Fig. 3b) predicts the midpoint as the optimal location whereas $d_3(\xi)$ (Fig. 3c) considers this location as the worst possible location.

5.2. Frequency domain criteria

The two norms $\mathcal{H}^2$ and $\mathcal{H}^\infty$ of $T_{ev}(s;\theta)$ are considered here for sensor placement. Figure 4 depicts the $\mathcal{H}^2$ norm of the θ-parameterized transfer function (19) and Figure 5 depicts the corresponding $\mathcal{H}^\infty$ norm. Both provide similar results for the optimal sensor locations.

5.3. Estimator time evolution

To further demonstrate the effects of sensor location on the state error, we considered $d(\xi) = d_3(\xi)$ and simulated the plant and the associated state estimator with a sensor placed in the optimal location for d_3, which according to Figure 5 is at $\xi = 0.25L$, and with a sensor placed at $\xi = 0.5L$. The latter is indeed an optimal location for either a uniform distribution or a Gaussian centered at the middle of the spatial domain, *but* it is a non-optimal location for $d_3(\xi)$. Figure 6 depicts the evolution of the state error, where it is observed that the optimal sensor location results in a faster convergence than a non optimal one.

6. Conclusions and future work

This work studies the problem of optimal sensor locations. For the problem of developing a sensor for the heat equation, the measure based on the observability grammian ($\text{trace}[W_{ob}(\theta)]$) found the best location in the center of the domain. However, this was independent of any distribution of disturbances. When spatially distributed disturbances are taken into account, the measures ($\text{trace}[\Sigma(\theta)]$) or ($\|T_{ev}(s;\theta)\|$) give more practical information. In our last experiment, we show the impact that an optimally placed sensor can have on the closed-loop system. For future work (to be included in the final version of this paper), we intend to study the use of gradient-based optimization algorithms to find the minima of these performance measures. In addition, we will look at systems in higher dimension.

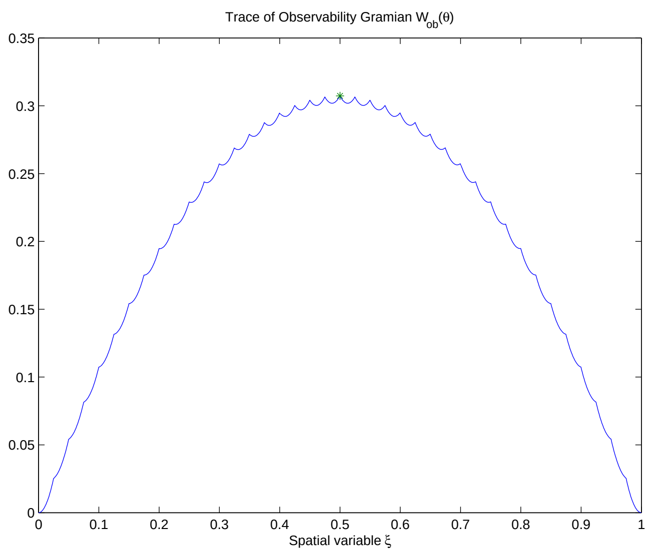

Figure 2. Observability Gramian as a function of sensor location.

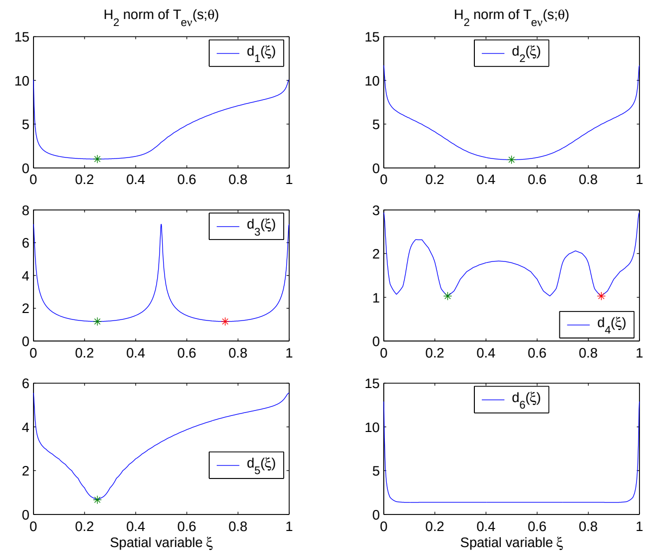

Figure 4. $\mathcal{H}^2$ norm of $T_{ev}(s;\theta)$ as a function of sensor location θ for various disturbance distributions.

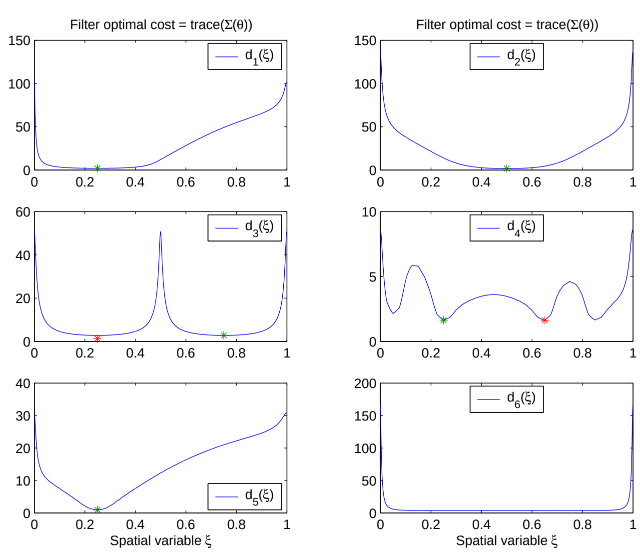

Figure 3. Optimal Kalman filter cost (trace$[\Sigma(\theta)]$) as a function of actuator location.

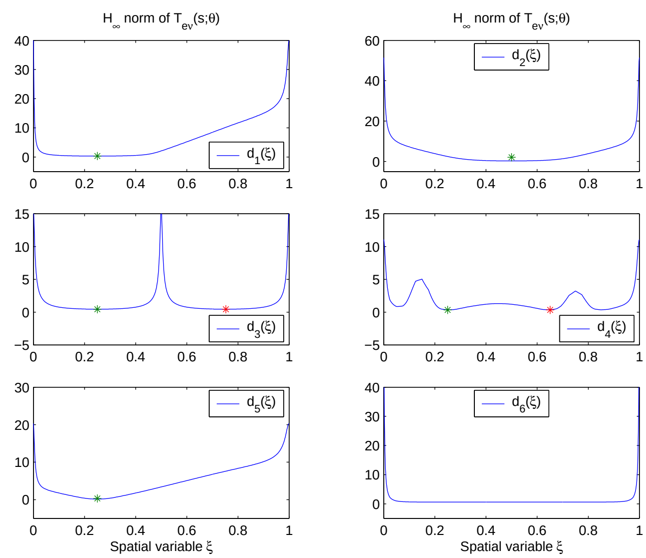

Figure 5. H^∞ norm of $T_{ev}(s;\theta)$ as a function of sensor location θ for various disturbance distributions.

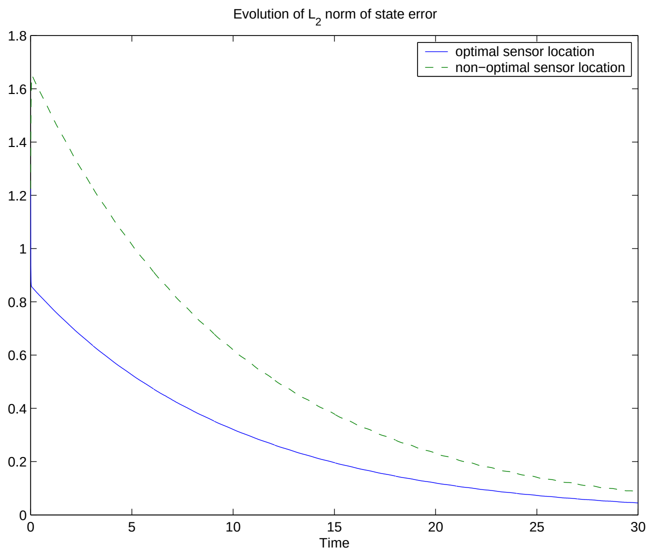

Figure 6. Evolution of L_2 of state error using optimal and non-optimal sensor locations.

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