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Stability and Safety of Multi-Vehicle Systems

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Abstract— **In this paper we study the safety control of multiple vehicles moving along a straight line. Each vehicle implements a linear feedback law on the positions and velocities of the adjacent vehicles. We assume the number of vehicles is large so by taking to the continuum limit the model is described by a hyperbolic partial differential equation, namely the wave equation. We analyze the stability of the system when each vehicle is subject to finite bandwidths of sensing and actuation. We also investigate the dynamic response of the system to disturbances. Finally, we specify the safety conditions of the system and safety control is formulated as a robust control problem.**

I. Introduction

Continuum representation of the dynamics of solids and fluids has existed for a long time, though strictly speaking nothing in the physical world is a continuum. Nonetheless, the continuum models widely used in physics are also suitable for spatially discrete systems such as traffic flows (see [18], [13]), which consist discrete vehicles interacting with their neighboring vehicles. Lighthill and Whitham are the first to use continuum wave equations to describe the behavior of cars that are dense enough on a highway, where disturbances of densities of cars are modelled as waves travelling with respect to the traffic. Recent work on control of traffic flows has used this model extensively (see [11], [14]).

Much of recent research effort is to control systems with a platoon of land vehicles ([12], [16]), multiple aerial vehicles ([5], [19]), spacecraft ([8], [10], [15]). General theoretic framework has been developed to control interconnected systems ([1], [6], [17]). In [6], a general theory of robust control analysis and synthesis of interconnected systems is established, using the framework of linear matrix inequalities (LMIs).

On the other hand, there is also a rich theory in control of linear distributed parameter systems (see [3], [4]). In this paper we consider the continuum limit of decentralized control of multiple vehicles, where the emphasis is on the effects actuator bandwidth and sensor bandwidth on the stability and safety of the vehicles. In particular, the stability of the dynamics of different length scales are studied in the presence of actuator and sensor bandwidths. The effects of disturbances from the vehicles on the dynamic responses of different modes are also studied. The choice of control gains must keep a trade off between robust stability and the performance of disturbance rejection. Numerical calculations are provided for parametric studies on the effects of feedback gains on robust stability and robust performance.

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II. Dynamic model and stability

In this section we approximate the dynamics of multiple vehicles using partial differential equations. We study the effects of actuator and sensor bandwidth on the stability of the system.

Consider n vehicles moving along the straight line with uniform speed v, and equally spaced. Suppose each vehicle is to remain in the middle between the vehicle in front of it and that behind it. The goal is to characterize the stability and dynamic response of different vehicles. Assume the coordinate is moving with the vehicles. Suppose the states of the i-th vehicle are the position and velocity and are denoted by (p_i, v_i). Also suppose the input for each vehicle (except the first one and the last one) is the positions of its two adjacent vehicles and the feedback control law is linear. The equations of motion of each vehicle are given by the following ordinary differential equations (ODEs):

$$\begin{aligned} \dot{p}_i &= v_i, \\ \dot{v}_i &= K_i \frac{2}{\Delta^2}\left(\frac{p_{i+1}+p_{i-1}}{2} - p_i\right), \end{aligned}$$

for $i = 2, 3, \cdots, n-1$. Here Δ is the spacing between the vehicles when there is no disturbances, and $K_i/\Delta^2 > 0$ is the feedback gain. The positions of the first and the last vehicle are given by $p_1(t)$ and $p_n(t)$. By taking the limit $n \longrightarrow \infty$, and $\Delta \longrightarrow 0$, the above ordinary differential equations can be approximated by the following partial differential equation (PDE):

$$\frac{\partial^2 p}{\partial t^2} = K(x)\frac{\partial^2 p}{\partial x^2}, \tag{1}$$

with the boundary conditions

$$p(0,t) = p_0(t), \quad p(L,t) = p_L(t). \tag{2}$$

The above equation is the linear wave equation, with $\sqrt{K(x)}$ being the local speed of the disturbances. Suppose $K(x)$ is constant and $K = a^2$, then the solution of the equation for the homogeneous boundary condition $p(0,t) = p_L(t) = 0$ is given by

$$p(x,t) = \sum_{m=1}^{\infty} \sin\frac{m\pi x}{L}\left(A_m \sin\frac{m\pi a t}{L} + B_m \cos\frac{m\pi a t}{L}\right),$$

where A_m and B_m are determined by initial conditions. It is easy to see that the damping of the disturbances is zero and

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and each mode of is neutrally stable. We can use velocity feedback to add damping to the system. The ODEs become

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= K_1\frac{2}{\Delta^2}\left(\frac{p_{i+1}+p_{i-1}}{2}-p_i\right)\\ &\quad +K_2\frac{2}{\Delta^2}\left(\frac{v_{i+1}+v_{i-1}}{2}-v_i\right),\end{aligned}$$

where $2 \leq i \leq n-1$. Since $v_i = \dot{p}_i$, in the limiting case of $\Delta \longrightarrow 0$ and $n \longrightarrow \infty$, the ODEs can be approximated by the following PDE:

$$\frac{\partial^2 p}{\partial t^2} = K_1\frac{\partial^2 p}{\partial x^2} + K_2\frac{\partial^3 p}{\partial t \partial x^2}, \tag{3}$$

It is easy to see that the PDE is stable if K_1 and K_2 are positive. It can be argued as follows. For the homogeneous boundary condition $p(0,t) = p(L,t) = 0$, the spatial modes of the PDE are given by $\sin\frac{m\pi x}{L}$ for $m = 1, 2, \cdots$, so the solutions are given by

$$p(x,t) = \sum_{m=1}^{\infty} A_m(t)\sin\frac{m\pi x}{L},$$

where the amplitude $A_m(t)$ satisfies the following ODE:

$$\ddot{A}_m + \left(\frac{m\pi}{L}\right)^2 K_2\dot{A}_m + \left(\frac{m\pi}{L}\right)^2 K_1 A_m = 0.$$

It is clear that the amplitude equation is stable if K_1 and K_2 are positive. This implies that all the modes are asymptotically stable.

In reality, the sensors and the actuators of the vehicles have bandwidths associated with them. Suppose the both the sensor dynamics and the actuator dynamics are described as first order differential equations, then the dynamics of the vehicles is given by

$$\begin{aligned}\dot{p}_i &= v_i,\\ \dot{v}_i &= u_i,\\ \dot{u}_i &= \frac{1}{\tau_a}\left[K_1\frac{2}{\Delta^2}\left(\frac{p^s_{i+1}+p^s_{i-1}}{2}-p^s_i\right)\right.\\ &\quad \left.+K_2\frac{2}{\Delta^2}\left(\frac{v^s_{i+1}+v^s_{i-1}}{2}-v^s_i\right)-u_i\right],\\ \dot{p}^s_i &= \frac{p_i-p^s_i}{\tau_s},\end{aligned}$$

where τ_a is the time constant of the actuator, and τ_s is the time constant of the sensor. By taking the limit, we get a PDE coupled with an ODE

$$\tau_a\frac{\partial^3 p}{\partial t^3} + \frac{\partial^2 p}{\partial t^2} = K_1\frac{\partial^2 p^s}{\partial x^2} + K_2\frac{\partial^3 p^s}{\partial t\partial x^2}, \tag{4}$$

$$\frac{\partial p^s}{\partial t} = \frac{p-p^s}{\tau_s}, \tag{5}$$

with boundary conditions

$$p(0,t) = p_0(t), \quad p(L,t) = p_L(t). \tag{6}$$

We first consider the homogeneous boundary conditions, and study the stability property of the system. By letting

$$p(x,t) = A(t)e^{ikx}, \quad p^s(x,t) = B(t)e^{ikx},$$

where k is the wave number, we get the ODEs for amplitude of the k-th mode,

$$\begin{aligned}\tau_a\frac{d^3A}{dt^3} + \frac{d^2A}{dt^2} &= -k^2\left(K_1B + K_2\frac{dB}{dt}\right),\\ \frac{dB}{dt} &= \frac{A-B}{\tau_s}.\end{aligned}$$

Using Laplace transform, we get the characteristic equation of the above ODEs:

$$\tau_a\tau_s s^4 + (\tau_a+\tau_s)s^3 + s^2 + k^2K_2 s + k^2K_1 = 0.$$

By using the Routh-Hurwitz criterion, the necessary and sufficient condition for the stability of the polynomial

$$\alpha_1 s^4 + \alpha_2 s^3 + \alpha_3 s^2 + \alpha_4 s + \alpha_5 = 0, \ \ \alpha_1 > 0,$$

is $\alpha_j > 0$ $(j = 1,2,3,4)$, $\alpha_2\alpha_3 > \alpha_1\alpha_4$, and $\alpha_2\alpha_3\alpha_4 > \alpha_1\alpha_4^2 + \alpha_2^2\alpha_5$. So the stability of the k-th mode is given by

$$\begin{aligned}K_1 &< \frac{K_2}{\tau_a+\tau_s} - \frac{k^2\tau_a\tau_s K_2^2}{(\tau_a+\tau_s)^2},\\ K_2 &< \frac{1}{k^2}\left(\frac{1}{\tau_a}+\frac{1}{\tau_s}\right).\end{aligned}$$

It is easy to see from the second inequality that high damping gain K_2 tend to destabilize small spatial modes (k large), especially when the bandwidth of the sensors and actuators are small (large τ_a and τ_s). From the first inequality it is clear that low bandwidth and high wave number also limits the gain K_1. An illustration of the stability region is given in Figure 1.

III. Dynamic response to disturbances

Suppose the feedback gains K_1 and K_2 are selected such that closed loop the system is stable, it is also important to study the dynamic response of the system of vehicles to the disturbance of the perturbations of the positions of the leading and trailing vehicles. This is equivalent to analyze the dynamic response of different wave numbers to the boundary conditions. For the nonhomogeneous boundary conditions, by letting

$$\begin{aligned}p(x,t) &= P(x,t) + \left(1-\frac{x}{L}\right)p_0(t) + \frac{x}{L}p_L(t),\\ p^s(x,t) &= P^s(x,t) + \left(1-\frac{x}{L}\right)p_0(t) + \frac{x}{L}p_L(t),\end{aligned}$$

then the PDE (4), the ODE (5), and the boundary conditions (6) are transformed to the following system

$$\tau_a\frac{\partial^3 P}{\partial t^3} + \frac{\partial^2 P}{\partial t^2} = K_1\frac{\partial^2 P^s}{\partial x^2} + K_2\frac{\partial^3 P^s}{\partial t\partial x^2} + F(x,t), \tag{7}$$

$$\frac{\partial P^s}{\partial t} = \frac{P-P^s}{\tau_s}, \tag{8}$$

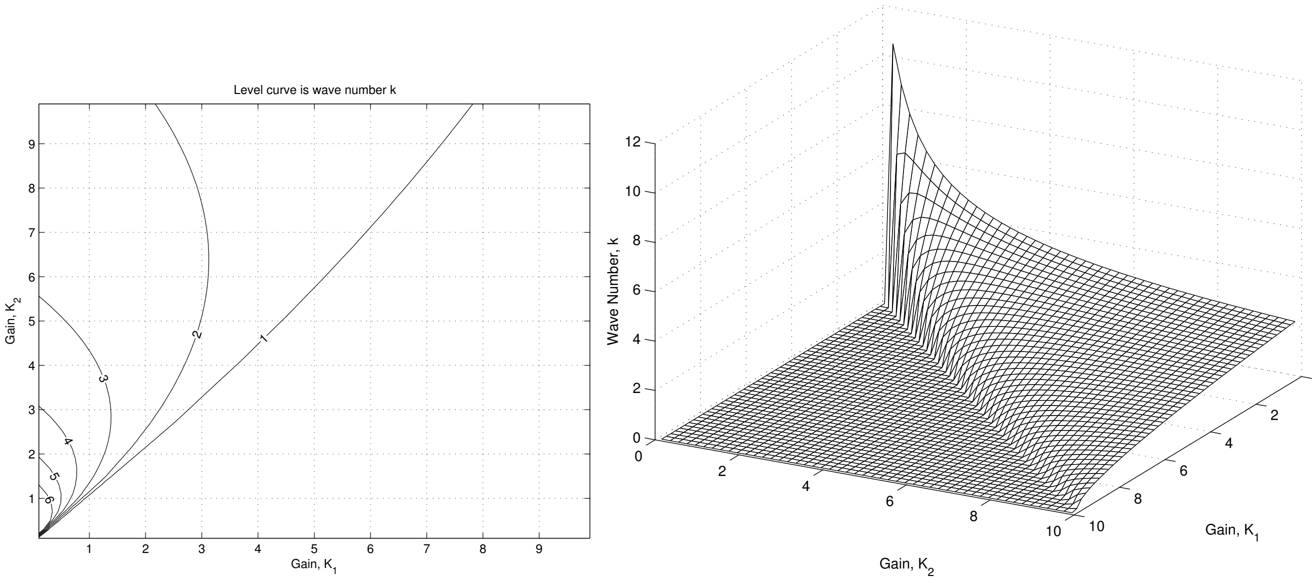

Fig. 1. Region of stability in the K_1-K_2-k space. The region of stability is given by the enclosed region between the parabolas and the K_2-axis for different wave numbers in the first figure. In the second figure, the region of stability is below the surface. The parameters: $\tau_a = 1.0$, $\tau_s = 0.02$.

where

$$\begin{aligned} F(x,t) &= -\left(1-\frac{x}{L}\right)\left(\tau_a \frac{d^3 p_0}{dt^3}(t) + \frac{d^2 p_0}{dt^2}(t)\right) \\ &\quad -\frac{x}{L}\left(\tau_a \frac{d^3 p_L}{dt^3}(t) + \frac{d^2 p_L}{dt^2}(t)\right) \end{aligned}$$

with homogeneous boundary conditions $P(0,t) = P(L,t) = 0$. Now letting

$$\begin{aligned} P(x,t) &= \sum_{m=1}^{\infty} A_m(t) \sin k_m x, \\ P^s(x,t) &= \sum_{m=1}^{\infty} B_m(t) \sin k_m x, \end{aligned} \tag{9}$$

where $k_m = \frac{m\pi}{L}$. By substituting the mode representation (9) into the PDE (7) and the ODE (8), multiplying the PDE with $\sin k_n x$, and integrating from 0 to L, we get

$$\begin{aligned} \tau_a \frac{d^3 A_n}{dt^3} + \frac{d^2 A_n}{dt^2} &= -k_n^2 \left(K_1 B_n + K_2 \frac{dB_n}{dt}\right) \\ &+(-1)^{n+1} \frac{2}{n\pi}\left(\tau_a \frac{d^3 p_0}{dt^3}(t) + \frac{d^2 p_0}{dt^2}(t)\right) \\ &+(-1)^{n} \frac{2}{n\pi}\left(\tau_a \frac{d^3 p_L}{dt^3}(t) + \frac{d^2 p_L}{dt^2}(t)\right), \end{aligned} \tag{10}$$

$$\frac{dB_n}{dt} = \frac{1}{\tau_s}(A_n - B_n), \tag{11}$$

where $n = 1, 2, \cdots, \infty$. Now, applying the Laplace transform to the ODEs (10) and (11), we get

$$\begin{aligned} A_n(s) &= \frac{2(-1)^n}{n\pi} \frac{s^2(\tau_a s + 1)(\tau_s s + 1)}{a_n(s)} \\ &\quad (p_L(s) - p_0(s)), \end{aligned}$$

where $a_n(s)$ is the characteristic polynomial and is given by

$$a_n(s) = \tau_a \tau_s s^4 + (\tau_a + \tau_s)s^3 + s^2 + k^2 K_2 s + k^2 K_1. \tag{12}$$

The sensitivity of the amplitude of the n-th mode to the boundary disturbances is given by

$$\begin{aligned} \|G_n(s)\|_\infty &= \left\| \frac{A_n(s)}{p_0(s)} \right\|_\infty \\ &= \max_{\omega \in \mathbb{R}} \frac{2}{n\pi} \frac{\omega^2 |1 + i\omega\tau_a||1 + i\omega\tau_s|}{|a(i\omega)|}, \end{aligned}$$

and

$$\|H_n(s)\|_\infty := \left\| \frac{A_n(s)}{p_L(s)} \right\|_\infty = \left\| \frac{A_n(s)}{p_0(s)} \right\|_\infty.$$

The goal is to design K_1, K_2 such that the maximum frequency response is minimized, i.e.,

$$\min_{K_1, K_2 \in \mathcal{S}_n} \|G_n(s)\|_\infty = \min_{K_1, K_2 \in \mathcal{S}_n} \max_{\omega \in \mathbb{R}} |G_n(i\omega)|,$$

where $\mathcal{S}_n \subset \mathbb{R}^2$ is the set of gains such that the n-th mode $\sin \frac{n\pi x}{L}$ is asymptotically stable, and is given by

$$\mathcal{S}_n = \left\{ (K_1, K_2) \,\middle|\, 0 < K_1 < \frac{K_2}{\tau_a + \tau_s} - \frac{k^2 \tau_a \tau_s K_2^2}{(\tau_a + \tau_s)^2}, \right. \\ \left. 0 < K_2 < \frac{1}{k^2}\left(\frac{1}{\tau_a} + \frac{1}{\tau_s}\right) \right\}$$

Numerical calculation of $\max_{\omega \in \mathbb{R}} |G_n(i\omega)|$ for different K_1 and K_2 are shown in Figure 2(a). It can be seen from the figure that the minimum γ_n is achieved at $K_1 = K_2 = 0$, but the n-th mode is not asymptotically stable when $K_1 = K_2 = 0$. By comparing with Figure 1, the design of performance has to trade-off with robust stability.

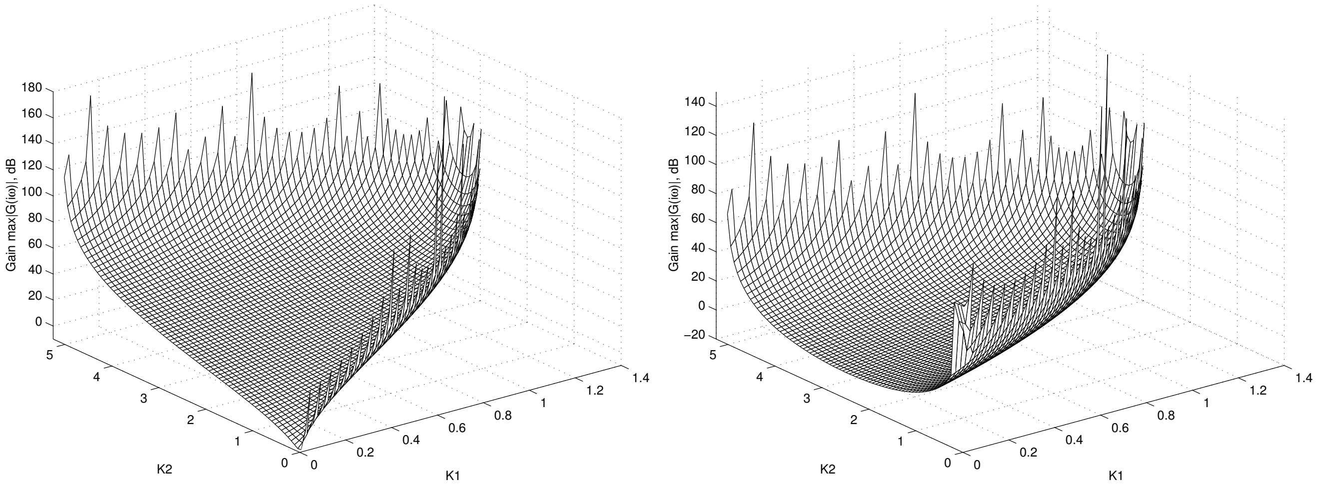

Fig. 2. Frequency response of the first mode amplitude to disturbances. Figure (a), frequency response to the boundary disturbances; Figure (b), frequency response to internal disturbances. The parameters: $\tau_a = 1.0$, $\tau_s = 0.02$.

Suppose the vehicle system is subjected to the internal disturbances, which are generated from all the vehicles except the leading one and the trailing one. The equations are given by

$$\tau_a \frac{\partial^3 p}{\partial t^3} + \frac{\partial^2 p}{\partial t^2} = K_1 \frac{\partial^2 p^s}{\partial x^2} + K_2 \frac{\partial^3 p^s}{\partial t \partial x^2} + W(x,t),$$
$$\frac{\partial p^s}{\partial t} = \frac{p - p^s}{\tau_s},$$

with homogeneous boundary conditions $p(0,t) = p(L,t) = 0$. Assume the disturbance $W(xt)$ can be written as follows,

$$W(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin k_n x + \sum_{n=1}^{\infty} z_n(t) \cos k_n x,$$

where $w(t)$ and $z_n(t)$ are white noises. Then the projection of the PDE to the n-th spatial mode is given by

$$\tau_a \frac{d^3 A_n}{dt^3} + \frac{d^2 A_n}{dt^2} = - k_n^2 \left(K_1 B_n + K_2 \frac{dB_n}{dt} \right) + w_n(t),$$
$$\frac{dB_n}{dt} = \frac{1}{\tau_s}(B_n - A_n).$$

So the frequency response of the amplitude of the n-th mode to the disturbance $w_n(t)$ is given by

$$g_n(s) = \frac{A_n(s)}{w_n(s)} = \frac{\tau_s s + 1}{a_n(s)},$$

where $a_n(s)$ is given by (12). The goal of control design is to find $K_1, K_2 \in \mathcal{S}_n$, such that the maximum frequency response is minimized, i.e.,

$$\min_{K_1, K_2 \in \mathcal{S}_n} \|g_n(s)\|_\infty = \min_{K_1, K_2 \in \mathcal{S}_n} \max_{\omega \in \mathbb{R}} |g_n(i\omega)|.$$

Numerical calculation of $\max_{\omega \in \mathbb{R}} |g_n(i\omega)|$ for different K_1 and K_2 is shown in Figure 2(b). The minimum is again achieved on the boundary of stability on which $K_1 = 0$.

IV. Specification of conditions of safety

The safety condition of the vehicles is that any two adjacent vehicles must not collide with each other at any time. In other words, we have $|p_i - p_{i+1}| < \Delta$, i.e.,

$$\left| \frac{p_{i+1} - p_i}{\Delta} \right| < 1.$$

By letting $\Delta \longrightarrow 0$, we get

$$\left| \frac{\partial p}{\partial x}(x,t) \right| < 1,$$

for any $x \in [0, L]$ and $t \in \mathbb{R}$. By projecting the solutions to the PDE onto the spatial modes, we get

$$p(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi x}{L}.$$

So the safety condition is given by

$$\left| \frac{\partial p}{\partial x}(x,t) \right| \leq \sum_{n=1}^{\infty} A_n(t) \left| \frac{n\pi}{L} \sin \frac{n\pi x}{L} \right| < \nu,$$

where $0 \leq \nu < 1$ is a constant. Hence a sufficient condition of safety is given by

$$\sum_{n=1}^{\infty} n \|A_n(t)\|_\infty \leq \frac{\nu L}{\pi},$$

where the ∞-norm of the signal $A(t)$ is given by

$$\|A_n(t)\|_\infty := \max_{t \in \mathbb{R}} |A_n(t)|.$$

Assume $w_n(t)$ is the projection of the disturbances onto the n-th mode, then we have

$$A_n(t) = (G_n \star w_n)(t),$$

where G_n is the impulse response and $\star$ denotes the convolution operator. Since we have

$$\|A_n\|_\infty = \|G_n\|_1 \|w_n\|_\infty,$$

the safety condition can be written as the $\mathcal{L}^1$ design problem

$$\sum_{n=1}^{\infty} n\|G_n\|_1\|w_n\|_\infty \leq \frac{\nu L}{\pi}. \qquad (13)$$

A necessary condition that satisfies (13) is

$$\|G_n\|_1 \leq \frac{\nu L}{n\pi}\frac{1}{\|w_n\|_\infty},$$

i.e., the higher the wave number is, the more stringent the $\mathcal{L}^1$-norm response is, since the $\mathcal{L}_1$-gain of the transfer functions is inversely proportional to the mode number. It is also clear that the requirement of the $\mathcal{L}_1$-gain is inversely proportional to the infinity norm of the disturbance $w(t)$. A sufficient condition that satisfies (13) is

$$\|G_n\|_1 \leq \frac{6\nu L}{(n\pi)^3}\frac{1}{\|w_n\|_\infty}.$$

It is clear from the sufficient condition that the requirement for the $\mathcal{L}_1$-gain is inversely proportional to the third power of the mode number.

V. Conclusions

We have modelled the dynamical behavior of multi-vehicles by using the continuum limit. We have studied the stability of the system for finite bandwidth of actuators and sensors, where the parametric relations between the sensor/actuator bandwidth and the region of stability has been explicitly derived. We have also analyzed the responses of vehicles to boundary and internal disturbances. Safety is specified as the $\mathcal{L}_1$ norm of the operators between the disturbances and the waves of different length scales. It is concluded that the regions of stability for modes with high wave numbers are small compared with those of the low wave numbers, and the system is more sensitive to the small scale disturbances.

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