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Controller and Observer Design for Lipschitz Nonlinear Systems

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***Abstract*— We consider three related problems for stabilization of a class of Lipschitz nonlinear systems: (1) full-state feedback controller design, (2) observer design, and (3) output feedback controller design. Sufficient conditions are developed for the design of an exponentially stable linear full-state feedback controller and an exponentially stable nonlinear observer. Given that the sufficient conditions of the controller and observer problem are satisfied, we show that the proposed controller with estimated state feedback from the proposed observer will achieve exponential stabilization. Simulation results on an example are given to numerically verify the proposed designs.**

I. Introduction

The output feedback control problem for nonlinear systems has received, and continues to receive, considerable attention in the literature due to its importance in many practical applications where measurement of all the state variables is not possible. Output feedback control design usually involves two related problems: observer design and controller design which uses estimated state and output as feedback. Unlike linear systems, separation principle does not generally hold for nonlinear systems. Therefore, the output feedback control problem for nonlinear systems is much more challenging than stabilization using full-state feedback. It is well known that the observer design problem for nonlinear systems by itself is quite challenging. One has to often consider special classes of nonlinear systems to solve the observer design problem as well as the output feedback control problem. Due to their practical significance, two special classes of systems that were often considered in the literature are nonlinear systems with a triangular structure and Lipschitz nonlinear systems.

A systematic approach to the development of observers for nonlinear systems was given in [1]; a nonlinear coordinate transformation was used to transform the original nonlinear system to a linear system with the addition of an output injection term. The nonlinear state transformations were also employed in [2], [3], [4] to obtain linear canonical forms that can be used for observer design. A comparative study of four techniques that appeared in the 1980's for observing the states of nonlinear systems was given in [5]. In [6], a new approach was given for the nonlinear observer design problem; a general set of necessary and sufficient conditions was derived using the Lyapunov's auxiliary theorem.

In [7], counterexamples were given to discuss the problem of global asymptotic stabilization by output feedback; a phenomenon called "unboundedness unobservability" was defined; it means that some unmeasured state components may escape in finite time whereas the measurements remain bounded. Recent research has focused on considering a selective class of nonlinear systems by placing some structural conditions on the nonlinearities to solve the output feedback problem. Global stabilization by dynamic output feedback of nonlinear systems which can be transformed to the output feedback form was given in [8]. Output feedback control of nonlinear systems in triangular form with nonlinearities satisfying certain growth conditions was considered in [9], [10]. In [11], it was shown that global stabilization of nonlinear systems is possible using linear feedback for a class of systems which have triangular structure and nonlinearities satisfy certain norm bounded growth conditions. A backstepping design procedure for dynamic feedback stabilization for a class of triangular Lipschitz nonlinear systems with unknown time-varying parameters was given in [12]. Output feedback control of nonlinear systems has been extensively studied in recent literature [13], [14], [15], [16].

Observer design techniques for Lipschitz nonlinear systems were considered in [17], [18], [19], [20], [21]. The observer design techniques proposed in these papers are based on quadratic Lyapunov functions and thus depend on the existence of a positive definite solution to an algebraic Ricatti equation. In [19], insights into the complexity of designing observers for Lipschitz nonlinear systems were given; it was discussed that in addition to choosing the observer gain in their nonlinear Luenberger-like observer, one has to make sure that the eigenvectors of the closed-loop observer system matrix must also be well-conditioned to ensure asymptotic stability. The existence of a stable observer for Lipschitz nonlinear systems was addressed in [20]; a sufficient condition was given on the Lipschitz constant. Some of the results of [20] were recently corrected by [21]. For the nonlinear observer of [20], it was shown in [21] that two sufficient conditions are required to guarantee that the observer is exponentially stable.

In this paper, we provide a solution to the output feedback control problem for Lipschitz nonlinear systems under some sufficient conditions on the Lipschitz constant. First, we design a linear full-state feedback controller and derive a sufficient condition under which exponential stabilization is achieved with full-state feedback. Second, we propose a Luenberger-like observer, which is shown to be an exponentially stable observer under only one sufficient condition. Given that the sufficient conditions of the controller and observer problem are satisfied, we show that the proposed controller with estimated state feedback from the proposed observer will achieve exponential stabilization, that is, the proposed controller and observer designs satisfy the separation principle.

The rest of the paper is organized as follows. In Section II, we give the class of Lipschitz nonlinear systems, the assumptions, the notation used, and some prior results that will be useful for the developments in the paper. The full-state feedback control problem, the observer design problem, and the output feedback control problem are considered in Sections III, IV, and V, respectively. Section VI gives an algorithmic procedure for computing the controller and observer gains while satisfying the sufficient conditions. An illustrative example is given in Section VII. Section VIII gives conclusions and some relevant future research.

II. Preliminaries

We consider the problem of controller and observer design for the following class of Lipschitz nonlinear systems:

$$\dot{x} = Ax + Bu + \Phi(x,u), \tag{1a}$$

$$y = Cx \tag{1b}$$

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where $x \in \mathbb{R}^n, u \in \mathbb{R}^p$, and $y \in \mathbb{R}^q$ are the system state, input, and output, respectively. We assume, without loss of generality, that $x = 0$ is the equilibrium point of the system.

We use the following notation throughout the paper. $\|M\|$ denotes the Euclidean norm of the matrix or vector M. M^H denotes the complex conjugate transpose of the matrix M. $\sigma_{min}(M)$ denotes the smallest singular value of the matrix M. The square of the singular values of the matrix M are the eigenvalues of the matrix $M^H M$. $\det(M)$ represents the determinant of the matrix M.

Assumption A1: $\Phi(x,u)$ is Lipschitz with respect to the state x, uniformly in the control u, that is, there exists a constant γ such that

$$\|\Phi(x_1,u)-\Phi(x_2,u)\| \le \gamma \|x_1 - x_2\|, \; x_1, x_2 \in \mathbb{R}^n, u \in \mathbb{R}^p. \quad (2)$$

Assumption A2: $\|\Phi(x,u)\| \le \gamma \|x\|, \; \forall u \in \mathbb{R}^p$.

Assumption A3: The pair (A,B) is controllable.

Assumption A4: The pair (C,A) is observable.

Definition D1: The number $\delta(M,N)$ is defined as

$$\delta(M,N) = \min_{\omega \in \mathbb{R}} \sigma_{min} \begin{pmatrix} i\omega I - M \\ N \end{pmatrix} \quad (3)$$

where $i = \sqrt{-1}$ and I is an identity matrix with appropriate dimension. See [21] for a discussion of the number δ.

Lemma 1: [21] Consider the Algebraic Ricatti Equation

$$A^\top P + PA + PRP + Q = 0. \quad (4)$$

If $R = R^\top \ge 0$, $Q = Q^\top > 0$, A is Hurwitz, and the associated Hamiltonian matrix $H = \begin{bmatrix} A & R \\ -Q & -A^\top \end{bmatrix}$ is hyperbolic, i.e., H has no eigenvalues on the imaginary axis, then there exists a unique $P = P^\top > 0$ to be the solution of (4).

Lemma 2: [20], [21] Let $\gamma \ge 0$ and define

$$H_\gamma = \begin{bmatrix} A & I \\ C^\top C - \gamma^2 I & -A^\top \end{bmatrix}.$$

Then $\gamma < \delta(A,C)$ if and only if H_γ is hyperbolic.

III. Full-state feedback controller design

In this section we consider the regulation problem for the system (1a) with full-state linear feedback under the assumptions A2 and A3. Consider the control input:

$$u = -Kx/\|B\|^2 - K_1 x \quad (5)$$

where K_1 is the pre-feedback gain matrix and is chosen such that $A_{nc} \triangleq A - BK_1$ is stable and K is the feedback gain matrix to be determined later. With this control input, the closed-loop dynamics of (1a) is

$$\dot{x} = \left(A_{nc} - BK/\|B\|^2\right)x + \Phi(x,u) \triangleq \bar{A}_c x + \Phi(x,u). \quad (6)$$

Consider the following Lyapunov function candidate

$$V_c(x) = x^\top P_c x, \quad P_c = P_c^\top > 0. \quad (7)$$

The time derivative of $V_c(x)$ along the trajectories of (6) is

$$\begin{aligned} \dot{V}_c(x) &= x^\top \left(\bar{A}_c^\top P_c + P_c \bar{A}_c\right)x + 2x^\top P_c \Phi(x,u) \\ &\le x^\top \left(\bar{A}_c^\top P_c + P_c \bar{A}_c\right)x + 2\gamma \|P_c x\| \|x\| \\ &\le x^\top \left(\bar{A}_c^\top P_c + P_c \bar{A}_c + P_c P_c + \gamma^2 I\right)x \end{aligned} \quad (8)$$

where the first inequality is a consequence of assumption A2 and the second inequality is obtained by completing squares on the term $2\gamma \|P_c x\| \|x\|$. For any $\eta_c > 0$, $\dot{V}_c \le -\eta_c x^\top x$, if

$$\bar{A}_c^\top P_c + P_c \bar{A}_c + P_c P_c + \gamma^2 I = -\eta_c I. \quad (9)$$

The choice of the control gain matrix in (9)

$$K = B^\top P_c / 2 \quad (10)$$

results in the following ARE:

$$A_{nc}^\top P_c + P_c A_{nc} + P_c \left(I - BB^\top/\|B\|^2\right) P_c + (\gamma^2 + \eta_c) I = 0. \quad (11)$$

Now we consider the problem of the existence of a symmetric positive definite matrix P_c, which is the solution to the ARE (11). Since A_{nc} is Hurwitz, $(I - BB^\top/\|B\|^2) \ge 0$, and $(\gamma^2 + \eta_c)I > 0$, by Lemma 1, the problem reduces to showing that the associated Hamiltonian matrix

$$H_c = \begin{bmatrix} A_{nc} & I - BB^\top/\|B\|^2 \\ -(\gamma^2 + \eta_c) I & -A_{nc}^\top \end{bmatrix} \quad (12)$$

is hyperbolic.

Lemma 3: H_c is hyperbolic if and only if

$$\sqrt{\gamma^2 + \eta_c} < \delta\left(A_{nc}^\top, \sqrt{\gamma^2 + \eta_c} B^\top / \|B\|\right). \quad (13)$$

Proof: The determinant of the matrix $(-i\omega I - H_c)$ is

$$\begin{aligned} \det(-i\omega I - H_c) &= \det \begin{bmatrix} -i\omega I - A_{nc} & -I + BB^\top/\|B\|^2 \\ (\gamma^2+\eta_c) I & -i\omega I + A_{nc}^\top \end{bmatrix} \\ &= (-1)^n \det \begin{bmatrix} (\gamma^2+\eta_c) I & -i\omega I + A_{nc}^\top \\ -i\omega I - A_{nc} & -I + BB^\top/\|B\|^2 \end{bmatrix} \\ &= (-1)^n \det \left((\gamma^2+\eta_c)(-I + BB^\top/\|B\|^2) - (i\omega I + A_{nc})(i\omega I - A_{nc}^\top) \right) \\ &= (-1)^n \det \left(G^H(i\omega) G(i\omega) - (\gamma^2 + \eta_c) I \right) \end{aligned}$$

where $G(i\omega) \triangleq \begin{pmatrix} i\omega I - A_{nc}^\top \\ \sqrt{\gamma^2 + \eta_c} \, B^\top / \|B\| \end{pmatrix}$, and the third equality is obtained by using the formula for determinant of block matrices [22, p. 650] because $(\gamma^2 + \eta_c)I$ is non-singular. Using the definition D1, it is seen that, if (13) holds, $\det(-i\omega I - H_c) \ne 0$ which implies that $j\omega, \forall \omega \in \mathbb{R}$ is not an eigenvalue of H_c, which in turn implies that H_c is hyperbolic. The necessary part of the proof is similar to that of Lemma 2; we refer the readers to [21] for details. ■

Theorem 1: For the nonlinear system given by (1), with the assumptions A2, A3, and with the control input given by (5), the equilibrium $x = 0$ is exponentially stable if the condition given by (13) is satisfied.

IV. Observer design

Consider the following observer for the system (1):

$$\dot{\hat{x}} = A\hat{x} + Bu + \Phi(\hat{x},u) + \frac{\gamma^2 + \varepsilon_o}{\|C\|^2} L(y - C\hat{x}) + L_1(y - C\hat{x}) \quad (14)$$

where $\varepsilon_o \ge -\gamma^2$, L_1 is chosen such that $A_{no} \triangleq A - L_1 C$ is Hurwitz, and L is the observer gain matrix.

Let the estimation error $\tilde{x} \triangleq x - \hat{x}$. The error dynamics is

$$\begin{aligned} \dot{\tilde{x}} &= \left(A_{no} - \frac{\gamma^2 + \varepsilon_o}{\|c\|^2} LC\right) \tilde{x} + \Phi(x,u) - \Phi(\hat{x},u), \\ &\triangleq \bar{A}_o \tilde{x} + \Phi(x,u) - \Phi(\hat{x},u). \end{aligned} \quad (15)$$

Consider the following Lyapunov function candidate

$$V_o(\widetilde{x}) = \widetilde{x}^\top P_o \widetilde{x}. \quad (16)$$

The time derivative of $V_o(\widetilde{x})$ along the trajectory of (15) is

$$\dot{V}_o(\widetilde{x}) = \widetilde{x}^\top \left(\bar{A}_o^\top P_o + P_o \bar{A}_o\right)\widetilde{x} + 2\widetilde{x}^\top P_o\left(\Phi(x,u) - \Phi(\widehat{x},u)\right). \quad (17)$$

Choosing

$$L = P_o^{-1} C^T/2, \quad (18)$$

using Assumption A1, and simplifying along the same lines as done in the full-state feedback controller case, we have

$$\dot{V}_o(\widetilde{x}) \le -\eta_o \widetilde{x}^\top \widetilde{x}, \quad (19)$$

if $P_o = P_o^\top > 0$ satisfies

$$A_{no}^\top P_o + P_o A_{no} + P_o P_o + (\gamma^2 + \eta_o)I - (\gamma^2 + \varepsilon_o)C^\top C/\|C\|^2 = 0 \quad (20)$$

for some $\eta_o > \max(\varepsilon_0, 0)$. Since A_{no} is Hurwitz and the matrix $(\gamma^2+\eta_o)I - ((\gamma^2+\varepsilon_o)/\|C\|^2)C^\top C > 0$, by lemma 1, the above ARE has a unique $P_o = P_o^\top > 0$ as the solution if the following associated Hamiltonian is hyperbolic:

$$H_o = \begin{bmatrix} A_{no} & I \\ -(\gamma^2+\eta_o)I + (\gamma^2+\varepsilon_o)C^\top C/\|C\|^2 & -A_{no}^\top \end{bmatrix}. \quad (21)$$

Lemma 4: H_o is hyperbolic if and only if

$$\sqrt{\gamma^2+\eta_o} < \delta\left(A_{no}, \sqrt{\gamma^2+\varepsilon_o}\, C/\|C\|\right). \quad (22)$$

Proof: Similar to lemma 3. ■

The following theorem summarizes the results of this section.

Theorem 2: For the nonlinear system given by 1, with the assumptions A1 and A4, if the condition given by (22) is satisfied, (14) is an exponentially stable observer for the system.

Remark 1: Notice that the proposed observer, (14), requires only one sufficient condition, (22), as opposed to the two sufficient conditions for the observer given in [21]. The two conditions are required in [21] because: (1) the observer structure does not guarantee that the "Q" matrix in the ARE (4) is positive definite and (2) the associated Hamiltonian matrix must be hyperbolic. The proposed observer, (14), guarantees that the "Q" matrix in the ARE (4) is positive definite.

V. Output feedback controller design

Combining the full-state feedback control design of Section III and the observer design of Section IV, we design an output feedback controller for the system (1).

Theorem 3: Consider the system (1) with assumptions A1, A2, A3, and A4. If conditions (13) and (22) hold, then the equilibrium $x = 0$ of the system (1) is exponentially stable, with

$$u = -K\widehat{x}/\|B\|^2 - K_1\widehat{x} \quad (23)$$

where $\widehat{x}$ is the estimate of x generated by (14), K is the gain matrix given by (10), Further, the observation error, $\widetilde{x} = x - \widehat{x}$, exponentially converges to zero.

Proof: Substituting the output feedback control law given by (23) in (1) and simplifying we obtain

$$\dot{x} = \left(A_{nc} - BB^\top P_c/2\|B\|^2\right)x + \Phi(x,u) + BB^\top P_c\widetilde{x}/2\|B\|^2 + BK_1\widetilde{x}. \quad (24)$$

The time derivative of the Lyapunov function candidate $V_c(x)$ given by (7) along the trajectories of (24) is

$$\begin{aligned}\dot{V}_c(x) \le\ & x^\top\left(A_{nc}^\top P_c + P_c A_{nc} + P_c\left(I - BB^\top/\|B\|^2\right)P_c + \gamma^2 I\right)x \\ & + x^\top P_c BB^\top P_c \widetilde{x}/\|B\|^2 + 2x^\top P_c BK_1\widetilde{x}. \quad (25)\end{aligned}$$

Since P_c is the solution to the ARE (11), we have

$$\dot{V}_c(x) \le -\eta_c x^\top x + \zeta_c\|x\|\|\widetilde{x}\| \quad (26)$$

where $\zeta_c = \|P_cP_c\| + \|2P_cBK_1\|$.

Now consider the function

$$W(x,\widetilde{x}) = \zeta V_c(x) + V_o(\widetilde{x}) \quad (27)$$

where $\zeta > 0$ and $V_o(\widetilde{x})$ is as given by (16). The time derivative of $W(x,\widetilde{x})$ is given by

$$\dot{W}(x,\widetilde{x}) \le -\zeta\eta_c\|x\|^2 + \zeta\zeta_c\|x\|\|\widetilde{x}\| - \eta_o\|\widetilde{x}\|^2. \quad (28)$$

Choosing $\zeta = \eta_c\eta_o/\zeta_c^2$ results in

$$\dot{W}(x,\widetilde{x}) \le -\zeta\eta_c\|x\|^2/2 - \eta_o\|\widetilde{x}\|^2/2. \quad (29)$$

Therefore, x and $\widetilde{x}$ exponentially converge to zero. ■

Remark 2: The number δ is realization dependent, that is, its value depends on A, B, C. If A is unstable to begin with, then any preliminary control used to stabilize A will affect δ. Since δ and γ depend on the realization, appropriate coordinate transformations as discussed in [20], in some cases, can be used to increase δ and reduce γ.

Remark 3: The bisection algorithm given in [21] can be used to compute δ; it was suggested that 0 and $\|A\|$ be used as the initial guess for the lower and upper bounds, respectively, for $\delta(A,C)$. It is possible that the value of δ may be greater than $\|A\|$. The upper bound must be changed to $\sigma_{min}\begin{pmatrix} A \\ C\end{pmatrix}$ because

$$\delta(A,C) = \min_{\omega\in\mathbb{R}} \sigma_{min}\begin{pmatrix} i\omega I - A \\ C\end{pmatrix} \le \sigma_{min}\begin{pmatrix} -A \\ C\end{pmatrix} = \sigma_{min}\begin{pmatrix} A \\ C\end{pmatrix}.$$

Remark 4: Since $\delta\left(A_{nc}^\top, \gamma B^\top/\|B\|\right)$ is a continuous function of γ, $f(\gamma) \stackrel{\triangle}{=} \gamma - \delta\left(A_{nc}^\top, \gamma B^\top/\|B\|\right)$ is also a continuous function of γ. Therefore, if $f(\gamma) < 0$, then there exists a $\gamma_1 > \gamma$ such that $f(\gamma_1) < 0$. Hence, if $f(\gamma) < 0$, then there exists an $\eta_c > 0$ such that $f(\sqrt{\gamma^2+\eta_c}) < 0$, that is, (13) holds. Same arguments hold for condition (22) with $\varepsilon_o = 0$. Hence, instead of checking conditions given by (13) and (22), one can respectively check the following two conditions

$$\gamma < \delta\left(A_{nc}^\top, \gamma B^\top/\|B\|\right), \ \gamma < \delta\left(A_{no}, \gamma C/\|C\|\right). \quad (30)$$

Notice that the conditions given by (30) guarantees the existence of $\eta_c > 0$ and $\eta_o > 0$, but not their values. Conditions (13) and (22) with specified η_c and η_o give the rate of convergence of controller and observer, respectively.

Remark 5: The three results given by theorems 1, 2, and 3 will be applicable locally or globally depending on whether $\Phi(x,u)$ is locally or globally Lipschitz.

VI. Implementation procedure

In the following, we give a systematic procedure to compute the observer and controller gain matrices with respect to the original system (1) in the event of the use of the preliminary control and coordinate transformations.

A. Observer gain matrix

1) **Pole placement**

Rewrite (1) in the following form

$$\dot{x} = (A - L_1C)x + Bu + L_1y + \Phi(x,u), \quad (31a)$$

$$y = Cx \quad (31b)$$

where L_1 is chosen such that $(A - L_1C)$ is stable.

2) **Similarity transformation**

Let $x = T_o x'$, (31) becomes

$$\begin{aligned} \dot{x}' &= T_o^{-1}(A - L_1C)T_o x' + T_o^{-1}(Bu + L_1 y) + T_o^{-1}\Phi(T_o x', u) \\ &\triangleq A'x' + B'u + T_o^{-1}L_1 y + T_o^{-1}\Phi(T_o x', u), \quad (32a) \end{aligned}$$

$$y = CT_o x' \triangleq C'x' \quad (32b)$$

where $T_o \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. The new Lipschitz gain γ' is obtained from the following inequality

$$\|T_o^{-1}\Phi(T_o x_1, u) - T_o^{-1}\Phi(T_o x_2, u)\| \le \gamma' \|x_1 - x_2\| \quad (33)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$.

The observer for (32) is given by

$$\begin{aligned} \dot{\widehat{x'}} &= A'\widehat{x'} + B'u + T_o^{-1}L_1 y + T_o^{-1}\Phi(T_o\widehat{x'}, u) \\ &\quad + (\gamma'^2 + \varepsilon_o)L'(y - C'\widehat{x'})/\|C'\|^2, \quad (34a) \end{aligned}$$

$$\widehat{y} = C'\widehat{x'}. \quad (34b)$$

After choosing $\varepsilon_o \ge -\gamma'^2$ and $\eta_o > \max(\varepsilon_o, 0)$, check the condition

$$\gamma'^2 + \eta_o < \delta^2\left(A', \sqrt{\gamma'^2 + \varepsilon_o}\, C'/\|C'\|\right) \quad (35)$$

for the existence of the solution P_o to the ARE

$$A'^{\top}P_o + P_oA' + P_oP_o + (\gamma'^2 + \eta_o)I - (\gamma'^2 + \varepsilon_o)C'^{\top}C'/\|C'\|^2 = 0. \quad (36)$$

If (35) is satisfied, the observer gain is chosen to be $L' = P_o^{-1}C'^{\top}/2$ where P_o is the solution of the above ARE.

Notice that if one defines $\widehat{x} = T_o\widehat{x'}$ as the estimate of x, the system (34) can be rewritten in terms of $\widehat{x}$ by the following equations.

$$\dot{\widehat{x}} = A\widehat{x} + Bu + \Phi(\widehat{x}, u) + L(y - C\widehat{x}), \quad (37a)$$

$$\widehat{y} = C\widehat{x} \quad (37b)$$

where
$$L = L_1 + (\gamma'^2 + \varepsilon_o)T_oL'/\|CT_o\|^2. \quad (38)$$

B. Controller gain matrix

1) **Pole placement**

Rewrite (1) in the following form

$$\dot{x} = (A - BK_1)x + B(u + K_1x) + \Phi(x, u) \quad (39)$$

where K_1 is chosen such that $(A - BK_1)$ is stable.

2) **Similarity transformation**

Let $x = T_c x'$, (39) becomes

$$\begin{aligned} \dot{x}' &= T_c^{-1}(A - BK_1)T_c x' + T_c^{-1}B(u + K_1 T x') + T_c^{-1}\Phi(T_c x', u) \\ &\triangleq A'x' + B'(u + K_1T_c x') + T_c^{-1}\Phi(T_c x', u) \quad (40) \end{aligned}$$

where $T_c \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. The new Lipschitz gain γ' is obtained from the following inequality

$$\|T_c^{-1}\Phi(T_c x_1, u) - T_c^{-1}\Phi(T_c x_2, u)\| \le \gamma'\|x_1 - x_2\| \quad (41)$$

for all $x_1, x_2 \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$. Choosing $u = -K_1T_c x' - K'x'/\|B'\|^2$ results in the following closed-loop system

$$\dot{x}' = (A' - B'K'/\|B'\|^2)x' + T_c^{-1}\Phi(T_c x', u). \quad (42)$$

Choose $\eta_c > 0$ and check the condition

$$\gamma'^2 + \eta_c < \delta^2\left(A'^{\top}, \sqrt{\gamma'^2 + \eta_c}\, B'^{\top}/\|B'\|\right) \quad (43)$$

for the existence of the solution P_c to the ARE

$$A'^{\top}P_c + P_cA' + P_c\left(I - B'B'^{\top}/\|B'\|^2\right)P_c + \left(\gamma'^2 + \eta_c\right)I = 0. \quad (44)$$

If (43) is satisfied, the control gain is chosen to be $K' = B'^{\top}P_c/2$ where $P_c = P_c^{\top} > 0$ is the solution of (44).

The gain matrix used in the full-state feedback controller or output feedback controller is

$$K = K_1 + K'T_c^{-1}/\|T_c^{-1}B\|^2. \quad (45)$$

VII. An illustrative example: a flexible link robot

In this section, we consider the observer and controller design for a flexible link robot [18], [20], [21], [23]. The dynamics of the robot is described by the following state space representation:

$$\dot{x} = Ax + bu + \Phi(x, u), \quad (46a)$$

$$y = Cx \quad (46b)$$

where

$$x = \begin{bmatrix} \theta_m \\ \omega_m \\ \theta_1 \\ \omega_1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\ \Phi(x, u) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3.33\sin(x_3) \end{bmatrix},$$

and θ_m is the angular position of the motor; ω_m is the angular velocity of the motor; θ_1 is the angular position of the link; and ω_1 is the angular velocity of the link.

Observer design: Since A is not stable, we design a preliminary gain L_1 such that $(A - L_1C)$ is stable with poles at $-9.3275, -8.9203, -9.6711$ and -4.7722. The gain L_1 is

$$L_1 = \begin{bmatrix} 9.3275 & 1.0000 \\ -48.7804 & 22.1136 \\ -0.0524 & 3.1994 \\ 19.4066 & -0.9032 \end{bmatrix}.$$

The Lipschitz constant of $\Phi(x, u)$ with respect to x is $\gamma = 3.33$. Using the similarity transformation $x = T_o x'$, transform the system (46) with $T_o = \mathrm{diag}(\begin{bmatrix} 1 & 1 & 1 & 10 \end{bmatrix})$. The new Lipschitz constant is $\gamma' = 0.333$. Choose constants $\varepsilon_o = 0.1111$ and $\eta_o = 0.1211$, and check the condition given by (35). It is computed that $\delta\left(A', \sqrt{\gamma'^2 + \varepsilon_o}\, C'/\|C'\|\right) = 0.8389$, so (35) is satisfied. Solving the ARE (36) results in

$$P_o = \begin{bmatrix} 18.6546 & -0.0234 & 0.0396 & 0.0012 \\ -0.0234 & 5.9522 & -12.5731 & 1.9503 \\ 0.0396 & -12.5731 & 30.8320 & -8.8656 \\ 0.0012 & 1.9503 & -8.8656 & 9.7302 \end{bmatrix}.$$

$$L' = \frac{1}{2}P_o^{-1}C'^{\top} = \begin{bmatrix} 0.0268 & 0.0003 \\ 0.0003 & 1.1392 \\ 0.0001 & 0.5405 \\ 0.0000 & 0.2641 \end{bmatrix}.$$

The observer for the flexible link robot (46) is in the form of (37) with

$$L = \begin{bmatrix} 9.3334 & 1.0001 \\ -48.7804 & 22.3665 \\ -0.0524 & 3.3194 \\ 19.4066 & -0.3167 \end{bmatrix}$$

where (38) is used.

The simulation results of the observer, (46), are shown in Figures 1 and 2. In the simulation, the initial value of x, $x(0)$, is chosen to be $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^\top$; the initial value of $\widehat{x}$, $\widehat{x}(0)$, is chosen to be $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top$. The system is assumed to be under no control, that is, $u = 0$. Fig. 1 shows the motor angular position, motor angular velocity, and their estimates. Fig. 2 shows the link angular position, the link angular velocity and their estimates. From both the figures, one can see that the estimates converge to the true states.

Output feedback control: As done in the observer design case, we first use a preliminary control to make $(A - BK_1)$ stable with poles at -5.8989, -5.6390, -4.9245 and -8.9109. The gain K_1 is found to be

$$K_1 = \begin{bmatrix} 7.8092 & 1.1168 & -4.3436 & 1.12 \end{bmatrix}.$$

Then, a similarity transformation, $x = T_c x'$, is used to reduce the Lipschitz gain with $T_c = \mathrm{diag}(\begin{bmatrix} 1 & 1 & 1 & 10 \end{bmatrix})$. The new Lipschitz constant is $\gamma\prime = 0.333$. Choose constants $\eta_c = 3.7947(10^{-4})$, and check the condition given by (43). It is computed that $\delta\left(A'^\top, \sqrt{\gamma\prime^2 + \varepsilon_c}\, B'^\top / \|B'\|\right) = 0.3552$, so (43) is satisfied. Solving the ARE (44) results in

$$P_c = \begin{bmatrix} 13.4725 & 1.4496 & -8.8421 & 17.387 \\ 1.4496 & 0.18736 & -0.99393 & 1.8462 \\ -8.8421 & -0.99393 & 6.1806 & -11.1047 \\ 17.387 & 1.8462 & -11.1047 & 26.2607 \end{bmatrix},$$

which in turn results in

$$K' = \frac{1}{2} B'^\top P_c = \begin{bmatrix} 15.6553 & 2.0235 & -10.7345 & 19.9385 \end{bmatrix}.$$

The control input for the flexible link robot (46) is $u = -K\widehat{x}$ with

$$K = \begin{bmatrix} 7.8428 & 1.1212 & -4.3666 & 1.1243 \end{bmatrix}$$

where (45) is used.

The simulation results for regulating the states of the flexible robot (46) to zero are shown in Figures 3 and 4. In this simulation, the initial values of x and $\widehat{x}$ are chosen to be the same as those in the simulation for the observer in the previous simulation. Fig. 3 shows the motor angular position, motor angular velocity, and their estimates. Fig. 4 shows the link angular position, the link angular velocity and their estimates. Comparing Figures 3 and 4 with Figures 1 and 2, it is clearly seen that, under the output feedback control, four states of the robot ($\theta_m, \omega_m, \theta_1$ and ω_1) converge to zero rapidly; whereas, without control, the states converge to zero very slowly. Also, the convergence of the estimated states to their true values is observed.

VIII. Conclusions

In this paper, we considered the full-state feedback control problem, the observer design problem, and the output feedback control problem for a class of Lipschitz nonlinear systems. We proposed a linear full-state feedback controller and a nonlinear observer and gave sufficient conditions under which exponential stability is achieved. Generally, for nonlinear systems, stabilization by state feedback plus observability does not imply stabilization by output feedback, that is, separation principle usually does not hold for nonlinear systems. However, for the class of nonlinear systems considered in this paper, by using the proposed full-state linear feedback controller and the proposed nonlinear observer, we show that the separation principle holds; that is, the same gain matrix which was obtained in the design of the full-state linear feedback controller can be used with the estimated state, where the estimates are obtained from the proposed observer.

Systems with Lipschitz nonlinearity are common in many practical applications. Many nonlinear systems satisfy the Lipschitz property at least locally by representing them by a linear part plus a Lipschitz nonlinearity around their equilibrium points. Hence, the class of systems considered in this paper cover a fairly large number of systems in practice.

There are some challenging problems that need to be addressed in the future. It is clear that the number δ is realization dependent. So, a natural question to ask is which realization gives the maximum value for δ and further, how does one transform the system given in any arbitrary form to this particular realization. Moreover, it is also not clear as to how one can, in general, find transformations that increase δ and decrease γ simultaneously.

It is also emphasized here that the conditions for both full-state feedback and output feedback stabilization are sufficient conditions; how to satisfy these two sufficient conditions is a challenging problem which needs to be investigated in the future.

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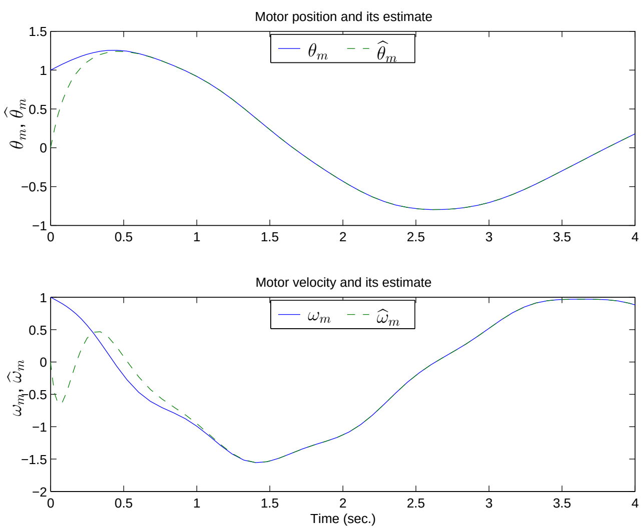

Fig. 1. Motor angular position θ_m, motor angular velocity ω_m, and their estimates $\widehat{\theta}_m$ and $\widehat{\omega}_m$ are shown. The control is $u = 0$.

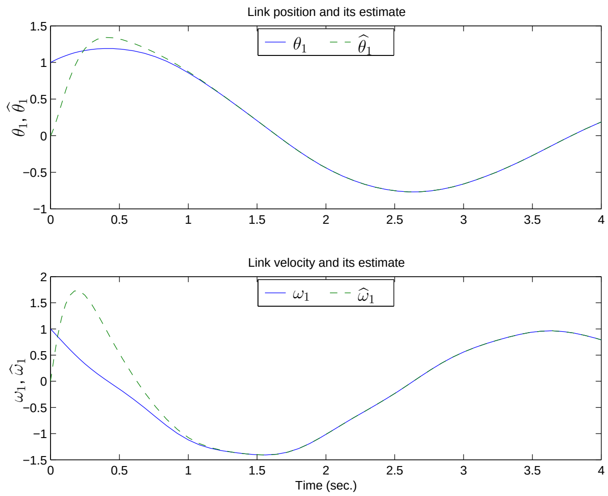

Fig. 2. Link angular position θ_1, link angular velocity ω_1, and their estimates $\widehat{\theta}_1$ and $\widehat{\omega}_1$ are shown. The control is $u = 0$.

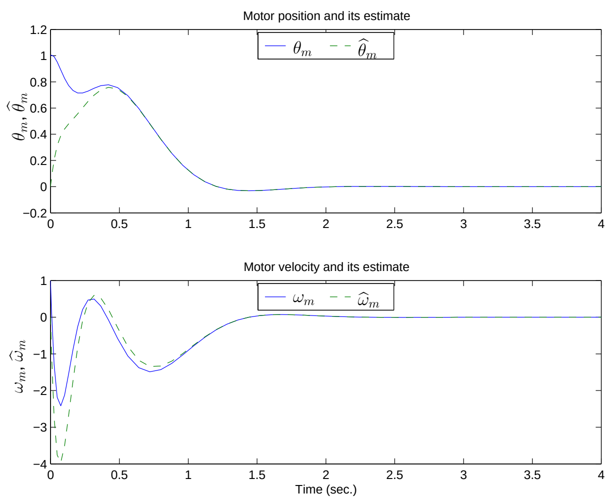

Fig. 3. Motor angular position θ_m, motor angular velocity ω_m, and their estimates $\widehat{\theta}_m$ and $\widehat{\omega}_m$ are shown. The control is $u = -K\widehat{x}$.

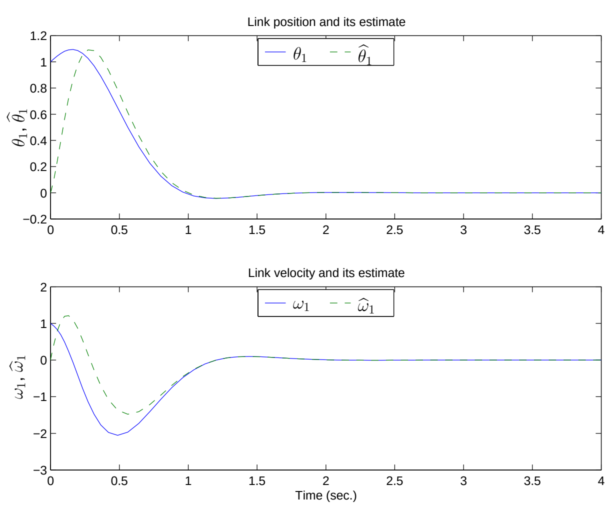

Fig. 4. Link angular position θ_1, link angular velocity ω_1, and their estimates $\widehat{\theta}_1$ and $\widehat{\omega}_1$ are shown. The control is $u = -K\widehat{x}$.