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On the Stability of Jump-Linear Systems Driven by Finite-State Machines with Markovian Inputs

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***Abstract*— This paper presents two mean-square stability tests for a jump-linear system driven by a finite-state machine with a first-order Markovian input process. The first test is based on conventional Markov jump-linear theory and avoids the use of any higher-order statistics. The second test is developed directly using the higher-order statistics of the machine's output process. The two approaches are illustrated with a simple model for a recoverable computer control system.**

I. INTRODUCTION

Safety critical computer control systems such as digital fly-by-wire aircraft are required to operate reliably in harsh environments. Typical fault tolerant techniques like modular redundancy and error correcting codes may not be tolerant to common-mode faults that affect near simultaneously more than one fault containment region. A technique that is being investigated to deal with transient or soft common-mode faults is error recovery with multiple dual-lock-step processors together with new fault tolerant architectures and communication subsystems [9], [10]. NASA Langley Research Center's Recoverable Computer System (RCS) is such an example where error recovery is being studied when transient faults are introduced into flight controllers by high intensity electromagnetic radiation [12], [13] and atmospheric neutrons [8]. During error recovery, different control laws come into effect which can significantly alter the dynamics of the closed-loop system. Stability analysis for such system was first considered in [4]–[6], [16] under the simplifying assumption that system faults were rare events. Then in [7] a new class of jump-linear models was introduced that captures the essential behavior of the closed-loop system under more general conditions. This model consisted of three distinct parts: a Markovian exosystem, a finite-state machine and a jump-linear dynamical system. Specifically, as shown in Fig. 1, the input process $\boldsymbol{\nu}(k)$ is a

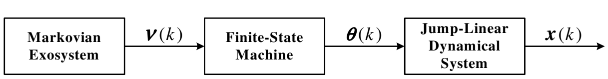

Fig. 1. The jump-linear model under consideration.

homogeneous, finite-state, first-order Markov chain whose state takes on symbols from the set $\Sigma_I = \{\eta_1, \eta_2, \ldots, \eta_M\}$ according to a probability transition matrix Π_I. (By convention the column sums of Π_I are assumed to be unity.) In turn, the Markov chain drives a finite-state machine $\mathcal{M} = (\Sigma_I, \Sigma_S, \Sigma_O, \delta, \omega)$. The state of the machine, $\boldsymbol{z}(k)$, takes on values from the set $\Sigma_S = \{e_1, e_2, \ldots, e_N\}$, which is simply the collection of elementary vectors $e_j = [0 \cdots \underbrace{0\ 1\ 0}_{j\text{-th position}} \cdots 0]^T$. The next state function $\delta : \Sigma_I \times \Sigma_S \mapsto \Sigma_S$ is a mapping of the form

$$\boldsymbol{z}(k+1) = S_{\boldsymbol{\nu}(k)}\boldsymbol{z}(k),$$

where each matrix S_η, $\eta \in \Sigma_I$, is a deterministic transition matrix, i.e., a matrix where each column contains exactly a single one and $N-1$ zeros. The output function $\omega : \Sigma_S \mapsto \Sigma_O$ is uniquely specified by the isomorphism

$$\omega(e_j) = \xi_j, \quad j = 1, \ldots, N.$$

It assigns to each state in Σ_S a unique output symbol from the set $\Sigma_O = \{\xi_1, \xi_2, \ldots, \xi_N\}$. Finally, the output from the machine, $\boldsymbol{\theta}(k) = \omega(\boldsymbol{z}(k))$, is used to drive an n-dimensional jump-linear dynamical system

$$\boldsymbol{x}(k+1) = A_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k).$$

The matrices A_ξ, $\xi \in \Sigma_O$, are completely arbitrary matrices in $\mathbb{R}^{n\times n}$. A necessary and sufficient condition for mean-square stability of such a system was developed in [7] under the assumption that the finite-state machine produced an output process which was first-order Markov. But, in general, this is not always the case. So in this paper, a more sophisticated stability analysis of the model class is developed. It involves characterizing the precise nature of random processes that are generated from finite-state machines driven by Markov processes. This topic has been addressed for more general classes of input processes in [2], [11], and more recently for Markov processes in [14] and [15]. Employing this literature, the main results of the paper are two completely general mean-square stability tests for jump-linear systems driven by finite-state machines with first-order Markovian inputs. The first test is based on conventional Markov jump-linear theory like that which appears in [3]. It is developed exclusively in the context of first-order Markov processes even though the output process may be higher-order. The second test, on the other hand, is derived directly using the higher-order statistics of $\boldsymbol{\theta}(k)$.

The paper is organized as follows. In the next section, a preliminary theorem is first given which characterizes the Markovian nature of a process $\boldsymbol{\rho}$ formed by the direct product of the input and output processes of a machine $\mathcal{M}$, i.e., $\boldsymbol{\rho} = (\boldsymbol{\nu}, \boldsymbol{\theta})$. Then a mean-square stability analysis is

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done for a jump-linear system driven by $\boldsymbol{\rho}$. These results can be viewed as being more general than the originally stated problem. Section 3 begins with a fundamental theorem characterizing the Markovian nature of the output process $\boldsymbol{\theta}$ for a class of finite-state machines known as *unifilar*. A stability analysis is then done for this class of systems. Next, it is shown that every finite-state machine has a unifilar representation with equivalent mean-square stability characteristics. This leads to the second general test for mean-square stability. In Section 4 the stability tests are demonstrated using a simple example of a finite-state machine frequently used to model recoverable computer control systems. The paper's conclusions are given in the final section.

II. STABILITY ANALYSIS OF A JUMP-LINEAR SYSTEM DRIVEN BY THE PROCESS $\boldsymbol{\rho}$

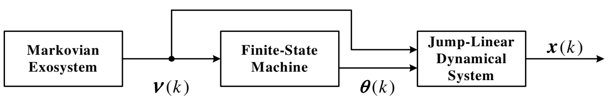

Fig. 2. A jump-linear system driven by the process $\boldsymbol{\rho}$.

Fix an underlying probability space $(\Omega, \mathcal{F}, P)$ and consider a jump-linear system as shown in Fig. 2. Here the driving process $\boldsymbol{\rho} = (\boldsymbol{\nu}, \boldsymbol{\theta})$ is comprised of an r-th order Markov process $\boldsymbol{\nu}$ and the corresponding output process $\boldsymbol{\theta}$ from a finite-state machine $\mathcal{M}$ driven by $\boldsymbol{\nu}$. The Markovian nature of $\boldsymbol{\rho}$ was characterized in [14] by the following theorem. A proof is given here to help establish a fundamental connection to the stability analysis that appears in the next section.

Theorem 2.1: [14] Consider a process $\boldsymbol{\rho} = (\boldsymbol{\nu}, \boldsymbol{\theta})$, where $\boldsymbol{\nu}$ and $\boldsymbol{\theta}$ are, respectively, the input and output of a finite-state machine $\mathcal{M} = (\Sigma_I, \Sigma_S, \Sigma_O, \delta, \omega)$. If $\boldsymbol{\nu}$ is an r-th order Markov process independent of the initial state of the machine, $\boldsymbol{z}(0)$, then $\boldsymbol{\rho}$ is also an r-th order Markov process.

Proof: Consider an event in $\mathcal{F}$ denoted by

$$\{\boldsymbol{\rho}(k), \boldsymbol{\rho}(k-1), \ldots, \boldsymbol{\rho}(k-m)\},$$

where $r \leq m \leq k$. Precisely, this denotes the set of all outcomes in Ω that produce a fixed but arbitrary sequence $\rho(k), \rho(k-1), \ldots, \rho(k-m)$. Now in the case where

$$\delta(\boldsymbol{\nu}(i), \omega^{-1}(\boldsymbol{\theta}(i))) = \omega^{-1}(\boldsymbol{\theta}(i{+}1)), \;\; \forall i = k{-}m, \ldots, k{-}1,$$

it follows immediately that

$$\begin{aligned} &P\{\boldsymbol{\rho}(k), \boldsymbol{\rho}(k-1), \ldots, \boldsymbol{\rho}(k-m)\} \\ &\quad = \; P\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m), \boldsymbol{\theta}(k-m)\}. \end{aligned}$$

All other such events are impossible. From the assumption that $\boldsymbol{\nu}$ is r-th order Markov and independent of $\boldsymbol{z}(0)$, it follows that $\boldsymbol{\nu}(k)$ is independent of $\boldsymbol{\theta}(k-m)$ for all $r \leq m \leq k$. Therefore,

$$\begin{aligned} &P\{\boldsymbol{\rho}(k), \boldsymbol{\rho}(k-1), \ldots, \boldsymbol{\rho}(k-m)\} \\ &\quad = \; P\{\boldsymbol{\nu}(k), \boldsymbol{\theta}(k-m) | \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} \cdot \\ &\qquad\quad P\{\boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} \\ &\quad = \; P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} \cdot \\ &\qquad\quad P\{\boldsymbol{\theta}(k-m) | \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} \cdot \\ &\qquad\quad P\{\boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} \\ &\quad = \; \frac{P\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\}}{P\{\boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\}} \cdot \\ &\qquad\quad P\{\boldsymbol{\theta}(k-m), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\}. \end{aligned} \tag{1}$$

Using a similar argument it follows that

$$\begin{aligned} &P\{\boldsymbol{\rho}(k-1), \boldsymbol{\rho}(k-2), \ldots, \boldsymbol{\rho}(k-m)\} \\ &\quad = P\{\boldsymbol{\nu}(k-1), \boldsymbol{\nu}(k-2), \ldots, \boldsymbol{\nu}(k-m), \boldsymbol{\theta}(k-m)\}. \end{aligned} \tag{2}$$

Dividing (1) by (2) gives

$$\begin{aligned} &P\{\boldsymbol{\rho}(k) | \boldsymbol{\rho}(k-1), \ldots, \boldsymbol{\rho}(k-m)\} \\ &\quad = \; P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1), \boldsymbol{\nu}(k-2), \ldots, \boldsymbol{\nu}(k-m)\}. \end{aligned}$$

Finally, since again the input process $\boldsymbol{\nu}$ is assumed to be r-th order Markov, for any $m \geq r$:

$$\begin{aligned} &P\{\boldsymbol{\rho}(k) | \boldsymbol{\rho}(k-1), \ldots, \boldsymbol{\rho}(k-m)\} \\ &\quad = \; P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1), \boldsymbol{\nu}(k-2), \ldots, \boldsymbol{\nu}(k-r)\} \\ &\quad = \; P\{\boldsymbol{\rho}(k) | \boldsymbol{\rho}(k-1), \boldsymbol{\rho}(k-2), \ldots, \boldsymbol{\rho}(k-r)\}, \end{aligned} \tag{3}$$

and hence the proof. ■

Note that in light of (3), when $r = 1$, the transition matrix for $\boldsymbol{\rho}$ can be written in terms of the transition matrix for $\boldsymbol{\nu}$ and the next state map for $\mathcal{M}$ as

$$\begin{aligned} &P\{\boldsymbol{\rho}(k+1) = (\eta_i, \xi_j) | \boldsymbol{\rho}(k) = (\eta_\ell, \xi_m)\} \\ &\quad = \begin{cases} P\{\boldsymbol{\nu}(k+1) = \eta_i | \boldsymbol{\nu}(k) = \eta_\ell\} & : \delta(\eta_\ell, e_m) = e_j \\ 0 & : otherwise. \end{cases} \end{aligned} \tag{4}$$

A more compact matrix form for this expression will be described shortly. Also, the case where $\boldsymbol{\nu}$ is i.i.d. (formally, $r = 0$) can be handled in a similar fashion. In this case, however, $\boldsymbol{\rho}$ is always first order Markov.

Next the mean-square stability of the jump-linear system

$$\boldsymbol{x}(k+1) = A_{\boldsymbol{\rho}(k)} \boldsymbol{x}(k) \tag{5}$$

is considered when $r = 1$. The following definition describes the exact notion of stability under consideration. The subsequent theorem gives the desired stability test.

Definition 2.1: The jump-linear system (5) is **mean-square stable** if for any initial condition $\boldsymbol{x}(0)$ and for any initial state probability for $\boldsymbol{\nu}(0)$ it follows that

$$\|Q(k)\| \to 0 \text{ as } k \to \infty,$$

where $Q(k) := E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\}$, or equivalently,

$$\hat{Q}(k) \to 0 \text{ as } k \to \infty,$$

where $\hat{Q}(k) := E\{\|\boldsymbol{x}(k)\|^2\}$ (cf. [1], [16]).

Theorem 2.2: The jump-linear system (5) is mean-square stable when $\boldsymbol{\nu}$ is a first-order Markov process if and only if the matrix

$$\mathcal{A}_1 := (\Pi_{I/O} \otimes I_{n^2}) \cdot \\ diag(A_{\eta_1,\xi_1} \otimes A_{\eta_1,\xi_1}, \ldots, A_{\eta_M,\xi_N} \otimes A_{\eta_M,\xi_N})$$

has a spectral radius less than one, where

$$\Pi_{I/O} = (\Pi_I \otimes I_N)\, diag(S_{\eta_1}, \ldots, S_{\eta_M}).$$

Proof: The claim follows directly from well known results in [3]. The transition probability matrix for the process $\boldsymbol{\rho}$, namely $\Pi_{I/O}$, is obtained from (4). ∎

Note that for the special case where $A_{\eta_i,\xi} = A_{\eta_j,\xi}$ for all $\eta_i, \eta_j \in \Sigma_I$, the jump-linear system is effectively being driven only by $\boldsymbol{\theta}$. In the next section, this situation is addressed in an alternative fashion.

III. STABILITY ANALYSIS OF A JUMP-LINEAR SYSTEM DRIVEN BY THE PROCESS $\boldsymbol{\theta}$

The first issue addressed in this section is the Markovian nature of the output process $\boldsymbol{\theta}$. The starting point is the special case where $\boldsymbol{\theta}$ is generated using a *unifilar* machine as described in [15]. After the definition, the key theorem characterizing $\boldsymbol{\theta}$ is given. A proof is provided for completeness. In particular, note that this result shows that higher-order Markov processes are in general always present in the jump-linear system shown in Fig. 1.

Definition 3.1: [15] A finite-state machine $\tilde{\mathcal{M}} = (\Sigma_I, \tilde{\Sigma}_S, \tilde{\Sigma}_O, \tilde{\delta}, \tilde{\omega})$ is **unifilar** if for any fixed state $e_m \in \Sigma_S$, $\tilde{\delta}(\eta_i, e_m) \neq \tilde{\delta}(\eta_j, e_m)$ whenever $\eta_i \neq \eta_j$.

Theorem 3.1: [15] Consider a unifilar finite-state machine $\tilde{\mathcal{M}} = (\Sigma_I, \tilde{\Sigma}_S, \tilde{\Sigma}_O, \tilde{\delta}, \tilde{\omega})$ with input $\boldsymbol{\nu}$ and output $\tilde{\boldsymbol{\theta}}$. If $\boldsymbol{\nu}$ is an r-th order Markov process, which is independent of the initial machine state $\tilde{\boldsymbol{z}}(0)$, then $\tilde{\boldsymbol{\theta}}$ is an $(r+1)$-st order Markov process.

Proof : Consider an event of the form $\{\tilde{\boldsymbol{\theta}}(k+1), \tilde{\boldsymbol{\theta}}(k), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\}$ where $r \leq m \leq k$. Since $\tilde{\mathcal{M}}$ is assumed to be unifilar and the output function $\tilde{\omega}$ is an isomorphism, this event is equivalent to the event $\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m), \tilde{\boldsymbol{\theta}}(k-m)\}$. Therefore,

$$P\{\tilde{\boldsymbol{\theta}}(k+1), \tilde{\boldsymbol{\theta}}(k), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\} \\ = P\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m), \tilde{\boldsymbol{\theta}}(k-m)\}$$

and similarly,

$$P\{\tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\} \\ = P\{\boldsymbol{\nu}(k-1), \boldsymbol{\nu}(k-2), \ldots, \boldsymbol{\nu}(k-m), \tilde{\boldsymbol{\theta}}(k-m)\}.$$

The assumption that $\boldsymbol{\nu}$ is an r-th order Markov process independent of $\tilde{\boldsymbol{z}}(0)$ implies that $\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\}$ is independent of $\tilde{\boldsymbol{\theta}}(k-m)$ when $r \leq m \leq k$. Thus,

$$\frac{P\{\tilde{\boldsymbol{\theta}}(k+1), \tilde{\boldsymbol{\theta}}(k), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\}}{P\{\tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\}} \\ = \frac{P\{\boldsymbol{\nu}(k), \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} P\{\tilde{\boldsymbol{\theta}}(k-m)\}}{P\{\boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\} P\{\tilde{\boldsymbol{\theta}}(k-m)\}},$$

or equivalently,

$$P\{\tilde{\boldsymbol{\theta}}(k+1) | \tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\} \\ = P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-m)\}. \quad (6)$$

Finally, since $\boldsymbol{\nu}$ is Markov of order r, for every $m \geq r$:

$$P\{\tilde{\boldsymbol{\theta}}(k+1) | \tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1), \ldots, \tilde{\boldsymbol{\theta}}(k-m)\} \\ = P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1), \ldots, \boldsymbol{\nu}(k-r)\} \\ = P\{\tilde{\boldsymbol{\theta}}(k+1) | \tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1), \ldots, \tilde{\boldsymbol{\theta}}(k-r)\},$$

and hence the proof. ∎

This result can also be shown to hold when $r = 0$, i.e., when $\boldsymbol{\nu}$ is i.i.d., independent of whether $\tilde{M}$ is unifilar or not. In addition, when $r = m = 1$ then (3) and (6) give, respectively,

$$P\{\boldsymbol{\rho}(k) | \boldsymbol{\rho}(k-1)\} = P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1)\} \\ P\{\tilde{\boldsymbol{\theta}}(k+1) | \tilde{\boldsymbol{\theta}}(k), \tilde{\boldsymbol{\theta}}(k-1)\} = P\{\boldsymbol{\nu}(k) | \boldsymbol{\nu}(k-1)\}.$$

Therefore, $\boldsymbol{\rho} = (\boldsymbol{\nu}, \tilde{\boldsymbol{\theta}})$ is a first-order Markov process, and $\tilde{\boldsymbol{\theta}}$ is a second-order Markov process, but their associated transition probabilities are clearly equivalent. (Of course in some special cases, the output process may be first-order, but then every first-order process is trivially second-order.) This observation is really at the center of the following development. First a stability theorem is given for jump-linear systems driven by unifilar machines.

Theorem 3.2: Consider a jump-linear system

$$\boldsymbol{x}(k+1) = A_{\tilde{\boldsymbol{\theta}}(k)} \boldsymbol{x}(k), \quad (7)$$

where $\tilde{\boldsymbol{\theta}}$ is generated by a unifilar finite-state machine $\tilde{\mathcal{M}} = (\Sigma_I, \tilde{\Sigma}_O, \tilde{\Sigma}_S, \tilde{\delta}, \tilde{\omega})$ with a first-order Markov input process, which is independent of the initial machine state $\tilde{\boldsymbol{z}}(0)$. The system is mean-square stable if and only if the matrix

$$\mathcal{B}_2 = (\Pi_{O,2} \otimes I_{n^2}) \cdot \\ diag(I_{\tilde{N}} \otimes (A_{\xi_1} \otimes A_{\xi_1}), \ldots, I_{\tilde{N}} \otimes (A_{\xi_{\tilde{N}}} \otimes A_{\xi_{\tilde{N}}})) \quad (8)$$

has a spectral radius less than one, where $\Pi_{O,2}$ is a block matrix composed of $\tilde{N} \times \tilde{N}$ matrices with the (J, I) block matrix having components

$$[[\Pi_{O,2}]_{JI}]_{ji} = \begin{cases} P\{\xi_J | \xi_i, \xi_I\} = P\{\eta_m | \eta_l\} & : I = j, \\ \quad \tilde{\delta}(\eta_l, e_I) = e_i,\ \tilde{\delta}(\eta_m, e_i) = e_J & \\ 0 & : otherwise. \end{cases}$$

(Here $\tilde{N} = card(\tilde{\Sigma}_O)$ and, for brevity, probabilities like $P\{\boldsymbol{\nu}(k+1) = \eta_m | \boldsymbol{\nu}(k) = \eta_l\}$ are written as $P\{\eta_m | \eta_l\}$.)

Proof: By Theorem 3.1, under the stated conditions, $\tilde{\boldsymbol{\theta}}$ is a second-order Markov process. Observe that

$$Q(k) = E\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\} = \sum_{i,j} Q_{ij}(k),$$

where

$$Q_{ij}(k) := E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_i\}} \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k-1)=\xi_j\}}\right\},$$

and $\mathbf{1}_{\{\cdot\}}$ denotes the Dirac function. Therefore,

$$
\begin{aligned}
&Q_{ij}(k+1)\\
&= A_{\xi_j} E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k+1)=\xi_i\}}\mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_j\}}\right\} A_{\xi_j}^T\\
&= A_{\xi_j} E\left\{E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k+1)=\xi_i\}}\mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_j\}}\middle| \mathcal{F}_k\right\}\right\}\cdot\\
&\quad A_{\xi_j}^T\\
&= A_{\xi_j} E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_j\}} P\{\xi_i|\xi_j, \tilde{\boldsymbol{\theta}}(k-1)\}\right\} A_{\xi_j}^T,
\end{aligned}
$$

where $\mathcal{F}_k$ denotes the σ-field generated by the random variables $\{\tilde{\boldsymbol{\theta}}(l), \boldsymbol{x}(l) : l = 0, 1, \ldots, k\}$. In addition,

$$
\begin{aligned}
&E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_j\}} P\{\xi_i|\xi_j, \tilde{\boldsymbol{\theta}}(k-1)\}\right\}\\
&\quad = \sum_l E\left\{\boldsymbol{x}(k)\boldsymbol{x}^T(k)\, \mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k)=\xi_j\}}\mathbf{1}_{\{\tilde{\boldsymbol{\theta}}(k-1)=\xi_l\}}\cdot\right.\\
&\qquad\quad \left. P\{\xi_i|\xi_j, \xi_l\}\right\}.
\end{aligned}
$$

Therefore,

$$
Q_{ij}(k+1) = A_{\xi_j}\left(\sum_l P\{\xi_i|\xi_j, \xi_l\} Q_{jl}(k)\right) A_{\xi_j}^T.
$$

Now apply the column stacking operator *vec* to produce

$$
\begin{aligned}
\vec{Q}_{ij}(k+1) \quad &:= \quad vec\left(A_{\xi_j}\sum_l P\{\xi_i|\xi_j, \xi_l\} Q_{jl}(k) A_{\xi_j}^T\right)\\
&= \quad (A_{\xi_j}\otimes A_{\xi_j})\sum_l P\{\xi_i|\xi_j, \xi_l\}\vec{Q}_{jl}(k).
\end{aligned}
$$

Then it can be shown that

$$
\vec{Q}(k+1) = \mathcal{B}_2\vec{Q}(k), \tag{9}
$$

where

$$
\vec{Q}(k) := \left[\, \vec{Q}_{11}^T(k)\ \cdots\ \vec{Q}_{1N}^T(k)\ \cdots\ \vec{Q}_{N1}^T(k)\ \cdots\ \vec{Q}_{NN}^T(k)\,\right]^T,
$$

and $\mathcal{B}_2$ is as given in (8). Finally, the mean-square stability test for system (7) follows directly from the given spectral radius condition for the linear system (9). ∎

In many applications, like the one discussed in the next section, the underlying machine is not unifilar. In order to perform stability analysis for such a system, additional tools are needed.

Definition 3.2: Two jump-linear systems

$$
\boldsymbol{x}(k+1) = A_{\boldsymbol{\theta}(k)}\boldsymbol{x}(k)
$$

and

$$
\bar{\boldsymbol{x}}(k+1) = \bar{A}_{\bar{\boldsymbol{\theta}}(k)}\bar{\boldsymbol{x}}(k),
$$

driven by machines $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, with the same Markov input process $\boldsymbol{\nu}$, are said to be ***A*- equivalent** if

$$
A_{\boldsymbol{\theta}(k)} = \bar{A}_{\bar{\boldsymbol{\theta}}(k)}, \quad k \geq 0.
$$

Note that A-equivalence does not imply that any A_i necessarily be equal to any $\bar{A}_j$, or that the processes $\boldsymbol{\theta}$ and $\bar{\boldsymbol{\theta}}$ even take on the same symbols. But from a state evolution point of view, $\boldsymbol{x}(k) = \bar{\boldsymbol{x}}(k)$ for all $k > 0$ if $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}(0)$. This makes the following claim obvious.

Theorem 3.3: Let $(A, \boldsymbol{\theta})$ and $(\bar{A}, \bar{\boldsymbol{\theta}})$ be two A-equivalent jump-linear systems. Then $(A, \boldsymbol{\theta})$ is mean-square stable if and only if $(\bar{A}, \bar{\boldsymbol{\theta}})$ is mean-square stable.

The next theorem shows that any finite-state machine has a unifilar companion machine that produces an A-equivalent jump-linear system. Therefore, Theorem 3.2 can be applied to this new system in order to determine the mean-square stability of the original system. There is, however, some *overhead* involved in determining the unifilar equivalent system as described in the proof.

Theorem 3.4: Let $\mathcal{M} = (\Sigma_I, \Sigma_S, \Sigma_O, \delta, \omega)$ be an arbitrary non-unifilar finite-state machine with input $\boldsymbol{\nu}$ and output $\boldsymbol{\theta}$. For any corresponding jump-linear system $(A, \boldsymbol{\theta})$, there always exists an A-equivalent system $(\tilde{A}, \tilde{\boldsymbol{\theta}})$, where $\tilde{\theta}$ is generated by a unifilar finite-state machine $\tilde{\mathcal{M}} = (\Sigma_I, \tilde{\Sigma}_S, \tilde{\Sigma}_O, \tilde{\delta}, \tilde{\omega})$ with input $\boldsymbol{\nu}$.

Proof : The proof is constructive. Since $\mathcal{M}$ is non-unifilar, there exists a pair of inputs symbols η_i and η_j and a machine state e_m such that $\delta(\eta_i, e_m) = \delta(\eta_j, e_m)$. Therefore, augment the machine's state space, Σ_S, with a new state $\tilde{e}_m$ and define a corresponding transition map $\tilde{\delta}$, which is identical to δ except now: $\tilde{\delta}(\eta_i, e_m) = \delta(\eta_i, e_m)$, $\tilde{\delta}(\eta_j, e_m) = \tilde{e}_m$, and $\tilde{\delta}(\eta, \tilde{e}_m) = \delta(\eta, e_m)$ for all input symbols η. Similarly, redefine the output function $\tilde{\omega}$ to be identical to ω except $\tilde{\omega}$ maps the new state $\tilde{e}_m$ to a new output symbol $\tilde{\xi}_m$ (so that the isomorphism between states and output is preserved). Next define for the new output symbol the transition matrix $\tilde{A}_{\tilde{\xi}_m} = A_{\xi_m}$, while in the other cases just set $\tilde{A}_{\xi_i} = A_{\xi_i}$. It can then be verified that the jump-linear systems $(A, \boldsymbol{\theta})$ and $(\tilde{A}, \tilde{\boldsymbol{\theta}})$ are A-equivalent, i.e.,

$$
\tilde{A}_{\tilde{\omega}(\boldsymbol{\nu}(k), \tilde{\boldsymbol{z}}(k))} = A_{\omega(\boldsymbol{\nu}(k), \boldsymbol{z}(k))}.
$$

It is also clear that the states and output symbols of the new machine can be re-indexed by the integers $1, 2, \ldots, N+1$, and the whole process above repeated if the new machine is not unifilar. Since the number of input symbols and machine states is finite, this procedure need only be repeated a finite number of times before a unifilar machine is produced. ∎

The final result of the paper establishes a connection between the main stability theorem of the previous section, Theorem 2.2, and the stability analysis described here.

Theorem 3.5: The jump-linear system (5) with $A_{\eta_i,\xi} = A_{\eta_j,\xi} =: A_\xi$ for all input symbols η_i, η_j is A-equivalent to a jump-linear system driven only by the output process $\boldsymbol{\theta}$. Therefore, $(A, \boldsymbol{\rho})$ is mean-square stable if and only if $(A, \boldsymbol{\theta})$ is mean-square stable.

Proof: The proof is just an application of Theorem 3.3. ∎

IV. A Simple Model for a Recoverable Computer Control System

In this section, a simple example is given to illustrate the mean-square stability tests developed in the previous sections. Consider a closed-loop control system implemented on a recoverable computer. Perhaps the simplest model for such a system is shown in Fig. 3. As long as there is no computer upset, the system operates in the *normal* mode, as per the dynamics

$$x(k+1) = A_0 x(k).$$

As soon as an upset occurs, the recovery system places the computer in the *recovery* mode for a fixed duration $M_R = 2$ clock cycles. During the recovery process the system dynamics are described by

$$x(k+1) = A_1 x(k).$$

After this duration, the nominal dynamics are restored. Here the counter, $c(k)$, keeps track of the lapse-time for each recovery process. To model this system as a finite-state machine, a new machine state must be introduced for each possible counter value. The state diagram for such a machine is shown in Fig. 4. Specifically, $\mathcal{M} = (\Sigma_I, \Sigma_S, \Sigma_O, \delta, \omega)$ with $\Sigma_I = \{\eta_1, \eta_2\}$, $\Sigma_S = \{e_1, e_2, e_3\}$, $\Sigma_O = \{\xi_1, \xi_2, \xi_3\}$, δ defined by

$$S_{\eta_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; S_{\eta_2} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and $\omega(e_j) = \xi_j$. The corresponding jump-linear system is specified by setting $A_{\xi_1} = A_0$ and $A_{\xi_2} = A_{\xi_3} = A_1$.

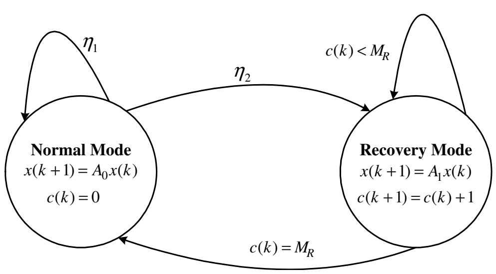

Fig. 3. A simple model for a recoverable computer control system.

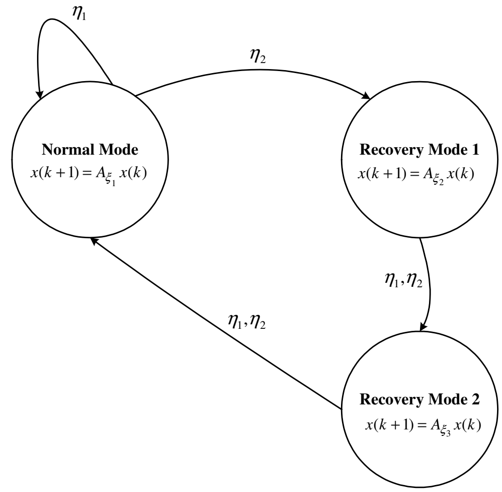

Fig. 4. A finite-state machine representation of a recoverable computer control system.

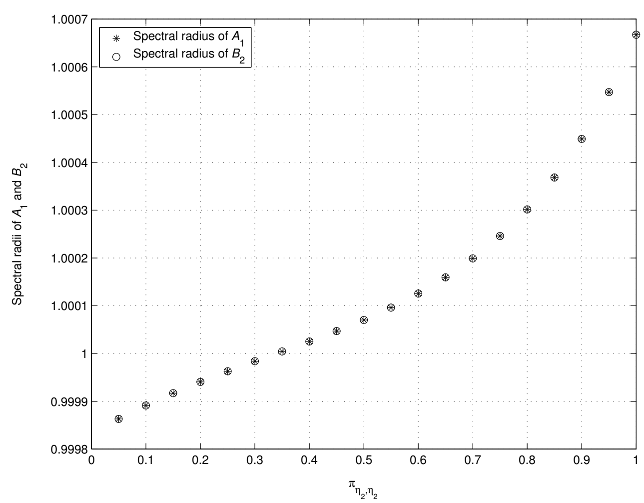

Fig. 5. A plot of the spectral radii of $\mathcal{A}_1$ and $\mathcal{B}_2$ versus π_{η_2,η_2} when $\pi_{\eta_1,\eta_1} = 0.45$.

The finite-state machine in Fig. 4 is clearly not unifilar. But the stability Theorem 2.2 can be applied directly by setting $A_{\eta,\xi_1} = A_0$ and $A_{\eta,\xi_2} = A_{\eta,\xi_3} = A_1$ for all $\eta \in \Sigma_I$. Suppose, for example, π_{η_1,η_1}=0.45. The spectral radius of $\mathcal{A}_1$ is plotted in Fig. 5 as a function of π_{η_2,η_2}. The system goes from being mean-square stable to unstable as π_{η_2,η_2} increases.

Now to illustrate the alternate approach given by Theorem 3.2, an A-equivalent unifilar finite-state machine is constructed as shown in Fig. 6. For the new states ξ_4 and ξ_5, let $A_{\xi_4} = A_{\xi_3} = A_1$ and $A_{\xi_5} = A_{\xi_1} = A_0$. Setting π_{η_1,η_1}=0.45, a plot of the spectral radius of $\mathcal{B}_2$ is also shown in Fig. 5 as a function of π_{η_2,η_2}. As expected, the two plots cross the stability boundary at exactly the same value $\pi_{\eta_2,\eta_2} = 0.345$. But more than that, the plots coincide exactly for all other values of π_{η_2,η_2}. In this example, the stability matrices $\mathcal{A}_1$ and $\mathcal{B}_2$ are distinct, in fact, they do not even have the same dimensions. So this observation does not follow immediately from the theory presented here.

V. CONCLUSIONS AND FUTURE RESEARCH

In this paper two mean-square stability tests were developed for a jump-linear system driven by a finite-state machine with a first-order Markov input process. The first test was based on conventional Markov jump-linear theory and avoids the use of any higher-order statistics. The second test was developed directly using the higher-order statistics of the machine's output process. The two approaches were illustrated with a simple model for a recoverable computer control system. In this example, it turned out the spectral radii of both stability matrices were always equal. It is conjectured that this is always the case, but it remains to be

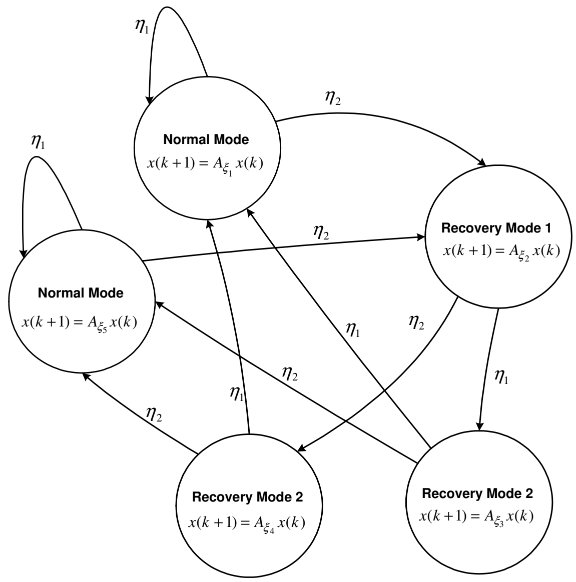

Fig. 6. A unifilar finite-state machine representation of a recoverable computer control system.

proven. In addition, it is possible to extend both approaches to systems with arbitrary r-th order Markov inputs. This is also a topic for further investigation.

VI. ACKNOWLEDGEMENTS

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