On Uniform Convergence in Markov Jump Linear Systems Problems and the Kolmogorov Forward Equation

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Abstract— Uniform convergence of standard transition matrices is a concept which appears in some fundamental results in Markov chain theory and therefore in optimal control, H_∞ control and stability problems of continuous-time Markov Jump Linear Systems (MJLSs) with infinite countable state space of the Markov chain. We identify some classes of standard transition matrices $P = (p_{ij}(t))_{i,j\in\mathbb{N}}$ that exhibits j-uniform convergence of $\frac{o_{ij}(t)}{t} = \frac{p_{ij}(t)-p_{ij}(0)-\dot{p}_{ij}(0)t}{t}$ as $t \to 0$, using tools such as analysis of j-uniform convergence and a version in l_1 of the forward equation.

I. INTRODUCTION

Uniform convergence is a concept which appears in some fundamental results in Markov chain theory and particularly, in optimal control, H_∞ control and stability problems of continuous-time Markov Jump Linear Systems (MJLSs) with infinite countable state space of the Markov chain ([5]-[8]). It is also worth mentioning that i-uniform convergence of $\frac{o_{ij}(t)}{t}$ is naturally required in the semigroup characterization of Markov chains (see, e.g., [4]) and in obtaining a sufficient condition for the Kolmogorov forward equation to hold (see [9]).

MJLSs are modeled as $\dot{x}(t) = A_{\theta(t)}x(t) + B_{\theta(t)}u(t)$, with $t > 0$, $x(0) = x_0$ and $\theta(0) = \theta_0$, where $x(t) \in \mathbb{C}^n$ denotes the state vector and $u(t) \in \mathbb{C}^m$ the input control. $\{\theta(t), t \geq 0\}$ is a standard Markov chain, with right continuous trajectories and a countably infinite state space (see Section II). It introduces randomness in the parameters by means of an arbitrary correspondence $i \mapsto \eta_i$, η_i standing for A_i or B_i and $\theta(t) = i$. A_i and B_i are matrices norm bounded on i and (x_0, θ_0) is a random vector. Sometimes we require a new entry $C_{\theta(t)}w(t)$ in the dynamics, where $\{w(t), t \geq 0\}$ stands either for a random process with finite energy (H_∞ problems) or for a Wiener process. The concern in the jump linear quadratic control problems afore mentioned is to minimize, within a certain class of admissible control policies, the performance

$$\mathcal{J}(\vartheta_0, u) := E[\int_0^\infty \left(\left\|\mathcal{Q}^{1/2}_{\theta(t)}x(t)\right\|^2 + \left\|\mathcal{R}^{1/2}_{\theta(t)}u(t)\right\|^2\right) dt]$$

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where ϑ_0 is the distribution of the initial data $(x(0), \theta(0))$, $\mathcal{Q}_i$ and $\mathcal{R}_i$ are penalty matrices, norm bounded on i and E stands for the expectation of a random variable. In the case of jump H_∞ control problems the concern is to find necessary and sufficient conditions for existence of a stabilizing control such that, for a prescribed value γ,

$$\frac{\|z\|_2}{\|w\|_2} < \gamma$$

for every stochastic processes w with finite energy and any initial random variable $\theta(0)$, where $\|y\|_2 = (\int_0^\infty E[\|y(t)\|^2]dt)^{1/2}$ with $\|\cdot\|$ being the Euclidean norm in $\mathbb{C}^\cdot$ and $z(t)$ is obtained as a linear combination of $x(t)$ and $u(t)$. From an application point of view, MJLSs corresponds to the modeling of physical systems that have their structures subject to abrupt changes (see, e.g., [5], [10] and references therein).

In this paper we identify some classes of standard transition matrices $P = (p_{ij}(t))_{i,j\in\mathbb{N}}$ that exhibits j-uniform convergence of $\frac{o_{ij}(t)}{t} = \frac{p_{ij}(t)-p_{ij}(0)-\dot{p}_{ij}(0)t}{t}$ as $t \to 0$. We do this via an analysis of j-uniform convergence and using a version in l_1 of the forward equation.

II. PRELIMINARIES

We denote $\vartheta := (0, a]$ for arbitrary $a > 0$, sometimes with a subscript, i.e., $\vartheta_i := (0, a_i]$. An $\mathbb{R}$-valued function $r(t)$, defined in some interval ϑ, is said to be of order $o(t)$(or an $o(t)$-function) if $\frac{r(t)}{t} \to 0$ as $t \to 0$. Functions $r_j(t)$, $j \in \mathbb{N}$, defined as above, are said to be of order $o(s)$ uniformly on j if $\frac{r_j(t)}{t} \to 0$ as $t \to 0$ uniformly on j. Similarly, an $\mathbb{R}$-valued function $r(t)$, defined in some interval $(a_0, a]$ is said to be of order $o(a_0 + h)$ if $\frac{r(a_0+h)}{h} \to 0$ as $h \to 0$. The definition "uniformly on j" will apply as above.

Concerning a standard transition matrix $P(t) \equiv (p_{ij}(t))_{i,j\in\mathbb{N}}$ of a continuous time Markov chain, we write:

(i) $\Lambda := (\lambda_{ij} = \dot{p}_{ij}(0))_{i,j\in\mathbb{N}}$ for the associated infinitesimal matrix, where

$$\dot{p}_{ij}(0) = \lim_{t\downarrow 0} \frac{p_{ij}(t) - p_{ij}(0)}{t}, \quad \text{and}$$

(ii) $p_{ij}(t) = p_{ij}(0) + \lambda_{ij}t + o_{ij}(t), \quad t > 0, \quad i,j \in \mathbb{N}$ (1)

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where $p_{ij}(0) = \delta_{ij}$, with $\delta_{ii} = 1$ and $\delta_{ij} = 0$, $i, j \in \mathbb{N}$, $i \neq j$. More generally, since $p_{ij}(t)$ is differentiable,

$$p_{ij}(t+h) = p_{ij}(0) + \dot{p}_{ij}(t)t + o_{t,ij}(t+h), \quad t \geq 0,\ h > 0,\ i,j \in \mathbb{N},$$

where $\dot{p}_{ij}(t+h) = \lim_{h\to 0} \frac{p_{ij}(t+h)-p_{ij}(t)}{h}$. Or else,

$$\frac{o_{t,ij}(t+h)}{h} = \frac{p_{ij}(t+h) - p_{ij}(t)}{h} - \dot{p}_{ij}(t) \to 0 \text{ as } h \to 0$$

and $o_{0,ij}(0+h) \equiv o_{ij}(h)$.

We sometimes write $(p_{ij}(t))$ in lieu of $(p_{ij}(t))_{i,j\in\mathbb{N}}$. We assume that λ_{ii} are finite for all $i \in \mathbb{N}$ and say that an infinitesimal matrix is conservative if $\sum_{j=1}^{\infty} \lambda_{ij} = 0$, $\forall i \in \mathbb{N}$.

Most of the times we shall write STM to denote a standard transition matrix and jSTM a STM whose associated $o_{ij}(t)/t$ functions converge to zero uniformly on j as $t \to 0$, for each $i \in \mathbb{N}$. We denote FEq the Kolmogorov forward equation.

We denote by l_1 the set made up of all infinite sequences of real numbers $x = (x_1, x_2, ...)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ equipped with the usual norm $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$ and by $\mathcal{D}(A)$ the domain of a linear operator A.

III. CONVERGENCE BEHAVIOR OF $\sum_{j=1}^{\infty} f_j(t)$

The lemma and corollaries of this section are classical in the uniform convergence scenario. Nonetheless, since they support the results that appears in Section IV, it is worth stating them here.

Let the $\mathbb{R}$-valued functions $f_j(t)$, $j \in \mathbb{N}$, $t \in \vartheta$, be such that

$$f_j(t) \to 0 \text{ as } t \to 0, \text{uniformly on } j. \tag{2}$$

Lemma 1: Let $\sum_{j=1}^{\infty} f_j(t)$, $t \in \vartheta$, be finite and f_j, $j \in \mathbb{N}$, nonnegative. Then (2) is necessary to having $\sum_{j=1}^{\infty} f_j(t) \to 0$ as $t \to 0$.

Corollary 1: Condition (2) is necessary to having $\sum_{j=1}^{\infty} |f_j(t)| \to 0$ as $t \to 0$, as long as this summation is finite $\forall t \in \vartheta$.

IV. MAIN RESULTS

Despite of the fact that all standard transition matrices have their associated $o_{ij}(t)$ functions converging to zero uniformly on j as $t \to 0$, this may not be necessarily the case of $\frac{o_{ij}(t)}{t}$. Transition matrices that exhibits uniform convergence on j of $\frac{o_{ij}(t)}{t}$ are those conservative matrices associated to Poisson processes (Lemma 6) and, more generally, a subset of transition matrices satisfying the Kolmogorov forward equation (Lemma 8). These classes are not exhaustive in the set of standard transition matrices and we may ask whether there exist conservative or nonconservative matrices that in fact do not converge to zero uniformly on j as $t \to 0$.

We begin (Section IV-A) introducing a version in l_1 of the forward equation, which is the underpinning result used in solving Lemma 8.

A. A version in l_1 of the forward equation

We consider the Banach space l_1 equipped with the usual norm and an arbitrary element $\Lambda = (\lambda_{ij})_{i,j\in\mathbb{N}}$ in the class of the infinitesimal matrices associated with standard transition matrices, and define the operator $\Lambda(\cdot)$ such that, $\forall x \in l_1$, $\Lambda(x) = x\Lambda$, where $x\Lambda$ is defined in the usual way as the matrix product of x, viewed as a row vector, by Λ, i.e., $(x\Lambda)_{k\in\mathbb{N}} = (\sum_{s=1}^{\infty} x_s\lambda_{sk})_{k\in\mathbb{N}}$. To simplify notation, we make no distinction between the matrix and the corresponding operator.

The space l_1 arises naturally in building an analog of the forward equation in Banach space when our concern is studying uniform convergence to zero in the row of $\left(\frac{o_{ij}(t)}{t}\right)_{i,j\in\mathbb{N}}$ or else, to λ_{ij}, $j = 1, 2, ...$, in the row of $\left(\frac{p_{ij}(t)-p_{ij}(0)}{t}\right)_{i,j\in\mathbb{N}}$.

For arbitrary $i \in \mathbb{N}$, equation (3), in Lemma 4, corresponds to the l_1 forward equation associated to the i-th state as in (11). We shall consider, in this lemma, $|\lambda_{kk}|$ uniformly bounded. Note that if this condition is dropped out, then Λ, seen as an operator, is not bounded and generates neither a contraction nor a C_0 semigroup, whenever $\mathcal{D}(\Lambda) = l_1$ (see the lemma below).

Lemma 2: Let A be a linear operator in a Banach space X and $\mathcal{D}(A) = X$. If A is the infinitesimal generator of a semigroup of contractions, say $T_A(t)$, then $T_A(t)$ is a C_0 semigroup, which implies A to be a bounded operator.

Proof: Since $T_A(t)$ is a semigroup of contractions, we have that $\{f \in X :\ \lim_{h\downarrow 0} T_A(h)f = f\} = \overline{\mathcal{D}(A)}$. But $\overline{\mathcal{D}(A)} = X$ (see, e.g., Section 2.2 of [2]). Therefore $T_A(t)$ is a C_0 semigroup. Consequently, A is closed. Reminding that $\mathcal{D}(A) = X$, we have, from the Closed Graph Theorem, that A is bounded. ∎

Remark 1: If we define Λ to be such that $\mathcal{D}(\Lambda) \neq l_1$, then Λ may be allowed to generate a C_0 semigroup, but in this case it may be difficult to assure, in Lemma 4, $\xi^i \in \mathcal{D}(\Lambda)$. The importance of these arguments stems from the fact that, to obtain existence and uniqueness of solution to (3), as well as differentiability, for any initial value in $\mathcal{D}(\Lambda)$, we have to assure Λ to generate a C_0 semigroup (see, e.g., [11]).

Now an auxiliary result.

Lemma 3: $\Lambda \in Blt(l_1)$ iff $|\lambda_{ii}|$ is uniformly bounded.

Proof: First note that the limits $a_j = \sum_{s=1}^{\infty} x_s\lambda_{sj} \in \mathbb{R}$, $j \in \mathbb{N}$ exists for arbitrary $x = (x_1, x_2, ...) \in l_1$. In fact, for every M, N and j, $N > M$, $\left|\sum_{s=M}^{N} x_s\lambda_{sj}\right| \leqslant \sum_{s=M}^{N} |x_s|\,|\lambda_{sj}| \to 0$ as $M, N \to \infty$, so that $\left\{\sum_{s=0}^{N} x_s\lambda_{sj}\right\}_{N\in\mathbb{N}}$ is Cauchy and converges in the complete space $\mathbb{R}$.

(if part): Λ is clearly linear. Now, let $\Lambda(x) = x\Lambda = (a_1, a_2, ...)$, so that

$$\|\Lambda(x)\|_1 = \sum_{j=1}^{\infty} |a_j| = \sum_{j=1}^{\infty} |\sum_{s=1}^{\infty} x_s \lambda_{sj}|$$
$$\leqslant \lim_{N\to\infty} \sum_{j=1}^{N} \lim_{M\to\infty} \sum_{s=1}^{M} |x_s| |\lambda_{sj}|$$
$$= \lim_{N\to\infty} \lim_{M\to\infty} \sum_{s=1}^{M} |x_s| \sum_{j=1}^{N} |\lambda_{sj}|$$
$$\leqslant \lim_{M\to\infty} \sum_{s=1}^{M} 2|x_s| |\lambda_{ss}| \leqslant 2c \sum_{s=1}^{\infty} |x_s| = 2c \|x\|_1 ,$$

where $c \geq |\lambda_{ss}|$ does not depend on s. This shows that Λ takes values in l_1 and is bounded.

(only if part): From the proof above, the class of the infinitesimal matrices Λ that belongs to $Blt(l_1)$ is nonempty. Thus, define $\xi^k = (0, 0, ...0, 1, 0, ...) \in l_1$ with "1" placed arbitrarily in the k-th position. $\Lambda \in Blt(l_1)$ means that there is c_1 such that $\left\|\Lambda(\xi^k)\right\|_1 \leqslant c_1 \left\|\xi^k\right\|_1 = c_1$. But $\left\|\Lambda(\xi^k)\right\|_1 = \sum_{j=1}^{\infty} |\lambda_{kj}| \geq |\lambda_{kk}|$ so that $|\lambda_{kk}| \leqslant c_1\ \forall k \in \mathbb{N}$. ■

Consider the l_1 valued function $x(t) = (x_1(t), x_2(t), ...)$, $t \geq 0$. To avoid confusion with $\dot{x}(t) := (\dot{x}_1(t), \dot{x}_2(t), ...)$, $\dot{x}_j(t)$ being the derivative of each entry $x_j(t)$ in the usual $\mathbb{R}$ norm, we shall denote the derivative of $x(t)$ in the l_1 norm by $\acute{x}(t)$.

Lemma 4: Let $\Lambda \equiv (\lambda_{ij})_{i,j\in\mathbb{N}}$ be an infinitesimal matrix with $|\lambda_{kk}|$ uniformly bounded and let

$$\acute{x}(t) = \Lambda(x(t)) = x(t)\Lambda, \ \ t \geq 0, \qquad (3)$$

be the l_1 differential equation where $\Lambda(x(t))$ is defined above and $x(0) = \xi^i \equiv (0, 0, ..., 0, 1, 0, ...) \in l_1$ is the initial data with "1" placed in the i-th position. Then there exists a unique solution $x(t) \in l_1$ of (3) which is continuous and continuously differentiable in the l_1 norm. Moreover, $x(t)$, continuous and continuously differentiable in $\mathbb{R}$, for each entry $x_j(t)$, is such that $\dot{x}(t) = \acute{x}(t)$ so it satisfies

$$\dot{x}(t) = \Lambda(x(t)) = x(t)\Lambda, \ \ t \geq 0, \qquad (4)$$

with $x_i(0) = \xi^i$. In addition,

$$\frac{x_j(t+h) - x_j(t)}{h} - \dot{x}_j(t) \xrightarrow{\mathbb{R}} 0 \text{ as } h \to 0, \text{ uniformly on } j \in \mathbb{N}, \forall t \geq 0. \qquad (5)$$

Proof: Λ is a linear and bounded operator in Banach space (see Lemma 3), so it generates a C_0 semigroup. Hence, from semigroup theory (see, for instance [11]) and noting that $\xi^i \in l_1 = \mathcal{D}(\Lambda)$, where $\mathcal{D}(\Lambda)$ stands for the domain of Λ, the first assertion follows. Now, this means that, for $t \geq 0$ arbitrarily fixed, there exists $\acute{x}_i(t) \in l_1$ such that the following limit exists.

$$\frac{x(t+h) - x(t)}{h} - \acute{x}(t) \xrightarrow{l_1} 0 \text{ as } h \to 0. \qquad (6)$$

This implies that

$$\left(\frac{x(t+h) - x(t)}{h} - \acute{x}(t)\right)_j \xrightarrow{\mathbb{R}} 0 \text{ as } h \to 0,\ j \in \mathbb{N}$$

or else, $$\frac{x_j(t+h) - x_j(t)}{h} - \acute{x}_j(t) \xrightarrow{\mathbb{R}} 0 \text{ as } h \to 0,\ j \in \mathbb{N} \qquad (7)$$

Now, (7) means that $\dot{x}_j(t) = \acute{x}_j(t)$, $\forall j \in \mathbb{N}$, or else, $\dot{x}(t) = \acute{x}(t)$, for $t \geq 0$ (note that the limits in (6) and (7), $t > 0$, may be considered from the right and from the left, since h assumes positive and negative values). Therefore, $x(t)$ satisfies $\dot{x}(t) = x(t)\Lambda,\ t \geq 0$, with initial data $x(0) = (0, 0, ..., 0, 1, 0, ...)$. Finally, convergence in (7) is in fact uniform on $j \in \mathbb{N}$, so (5) follows. ■

Remark 2: Equation (3) is therefore equivalent to both (4) and (5) in that the set of solutions (the unitary set) that agrees with (3), agrees with (4) and (5), and vice-versa.

B. $j-$uniform convergence of $o_{ij}(t)$

Lemma 5: Let $(p_{ij}(t))_{i,j\in\mathbb{N}}$ be a standard transition matrix. Then, for arbitrary $i \in \mathbb{N}$,

$o_{ij}(t) \to 0$ (and $|o_{ij}(t)|$) $\to 0$ as $t \to 0$ uniformly on $j \in \mathbb{N}$.

Proof: We may appeal to a particular case of Theorem II.2.2 of [1] which says that $0 \leqslant p_{ij}(t) \leqslant 1 - p_{ii}(t)$, $\forall j \in \mathbb{N}$, $j \neq i$ and $t > 0$, reminding that $1 - p_{ii}(t) \to 0$ as $t \to 0$, or else, to [12], Theorem 1 of Section II.2.2, to obtain that $p_{ij}(t) \to 0$ as $t \to 0$ uniformly on j. Now, since $0 \leqslant \lambda_{ij} \leqslant -\lambda_{ii}$, it follows that $\lambda_{ij}t \to 0$ as $t \to 0$ uniformly on j. It is easy to see that $o_{ij}(t) = p_{ij}(t) - \lambda_{ij}t \to 0$ as $t \to 0$ uniformly on $j \neq i$. Clearly, since $\{j \in \mathbb{N}\colon j = i\}$ is finite for fixed i, the latter convergence is uniform on $j \in \mathbb{N}$. Now, using the definition of convergence, it easily follows that uniform convergence of $o_{ij}(t)$ is equivalent to uniform convergence of $|o_{ij}(t)|$. ■

C. Classes of examples of jSTMs

Lemma 6 (Poisson processes): Let $(p_{ij}(t))_{i,j\in\mathbb{N}}$ be the (conservative) transition matrix of a Poisson process with parameter $b > 0$ arbitrarily chosen. Then, $(p_{ij}(t))_{i,j\in\mathbb{N}}$ is a jSTM.

Proof: $o_{ij}(t) := p_{ij}(t) - \lambda_{ij} t = p_{ij}(t)$, where $p_{ij}(t) = e^{-bt}\frac{(bt)^{j-i}}{(j-i)!}$, $j = i+2, i+3, ...$. Moreover, for $t \in (0, 1/b]$, we have that $0 \leqslant bt \leqslant 1$, which implies $(bt)^{j-i} \leqslant (bt)^2$. Hence, $\frac{o_{ij}(t)}{t} \leqslant \frac{o_{ii+2}(t)}{t}$ for $j = i+2, i+3, ...$ and every t in the j-independent interval $(0, 1/b]$. Since $\frac{o_{ii+2}(t)}{t} \to 0$ as $t \to 0$ and $\{1, ... i+1\}$ is a finite set, the result follows. ■

Lemma 7 (processes with positive o_{ij} functions): Let $(p_{ij}(t))_{i,j\in\mathbb{N}}$ be a standard transition matrix with a conservative infinitesimal matrix and such that, for every i, $o_{ij}(t)$ is positive $\forall t \in \vartheta_i$ and $j \in \mathbb{N} - \mathcal{S}_i$, with $\mathcal{S}_i$ being a finite subset of $\mathbb{N}$. Then, $(p_{ij}(t))_{i,j\in\mathbb{N}}$ is a jSTM.

Proof: For a conservative infinitesimal matrix $\sum_{j=1}^{\infty} \frac{o_{ij}(t)}{t} = 0$, $\forall t > 0$, so that $\sum_{j=1}^{\infty} \frac{o_{ij}(t)}{t} \to 0$ as $t \to 0$. Since $\mathcal{S}_i$ is finite, $\sum_{j=1, j\notin\mathcal{S}_i}^{\infty} \frac{|o_{ij}(t)|}{t} = \sum_{j=1, j\notin\mathcal{S}_i}^{\infty} \frac{o_{ij}(t)}{t} \to 0$ as $t \to 0$ so this is the case of $\sum_{j=1}^{\infty} \frac{|o_{ij}(t)|}{t}$. Hence, using the necessity part of Lemma 1, the result follows. ■

The example that follows is supported by Section IV-A.

Lemma 8 (STMs uniquely satisfying the FEq): Let $P(t) = (p_{ij}(t))_{i,j\in\mathbb{N}}$ be a standard transition matrix and $\Lambda = (\lambda_{ij})_{i,j\in\mathbb{N}}$ the associated infinitesimal matrix. Suppose that $|\lambda_{ii}|$ is uniformly bounded and $P(t)$ uniquely satisfies the forward equation

$$\dot{P}(t) = P(t)\Lambda, \; t \geq 0, \tag{8}$$

with $P(0) = \{\delta_{ij}\}_{i,j\in\mathbb{N}}$, in the class of continuous and continuously differentiable functions, where continuity and differentiation are considered in $\mathbb{R}$, for each entry $p_{ij}(t)$, $i, j \in \mathbb{N}$. Then, $P(t)$ is a jSTM, i.e., for each $i \in \mathbb{N}$,

$$\frac{o_{ij}(h)}{h} \to 0 \text{ as } h \to 0, \text{ uniformly on } j \in \mathbb{N} \tag{9}$$

This is true, in fact, $\forall t \geq 0$, i.e.,

$$\frac{o_{ij}(t+h)}{h} = \frac{p_{ij}(t+h) - p_{ij}(t)}{h} - \dot{p}_{ij}(t) \to 0$$
$$\text{as } h \to 0, \text{ uniformly on } j \in \mathbb{N}. \tag{10}$$

Proof: For $i \in \mathbb{N}$ arbitrarily fixed, denote $P_i(t) := (p_{i1}(t), p_{i2}(t), ...)$ and $\dot{P}_i(t) := (\dot{p}_{i1}(t), \dot{p}_{i2}(t), ...)$. $P_i(t)$ clearly belongs to l_1. To see that this is also the case of the latter expression, i.e., that $\sum_{j=1}^{\infty} |\dot{p}_{ij}(t)|$ is finite, refer to equation II.3(4) of [1] or, alternatively, Lemma 3, reminding that $P_i(t)$ satisfies (11). So, from (8),

$$\dot{P}_i(t) = P_i(t)\Lambda, \; t \geq 0, \tag{11}$$

where $P_i(0) = (0, 0, ..., 0, 1, 0, ...)$ with "1" placed in the i-th position. Now, from Lemma 4, there exists $x(t) = (x_1(t), x_2(t), ...)$, continuous and continuously differentiable in $\mathbb{R}$, for each entry $x_j(t)$, that satisfies (11) with the initial data $x(0) = P_i(0)$, and such that (5) holds. Since, by assumption, solutions to (11) are unique, we have that $P_i(t) = x(t)$. In this case (5) reads as (10). In particular, with $t = 0$, we obtain (9). ■

It is worth noticing that, if a jSTM satisfies the forward equation, where limits are considered in $\mathbb{R}$, for each entry $p_{ij}(t)$, $i, j \in \mathbb{N}$, then, regardless of being the unique solution or not, its associated $o_{ij}(t+h)/h$ functions converge to zero uniformly on j as $h \to 0$, $\forall t \geq 0$. To see this note that, by assumption, $\exists\, \delta_j = \delta : \mathbb{R}_{>0} \to \vartheta$, such that, $\forall \varepsilon > 0$,

$$-\varepsilon \leqslant \frac{o_{kj}(h)}{h} = \frac{p_{kj}(h) - p_{kj}(0)}{h} - \dot{p}_{kj}(0) \leqslant \varepsilon,$$
$$0 < h \leqslant \delta(\varepsilon), \; \forall j \in \mathbb{N},$$

for arbitrary $k \in \mathbb{N}$. Multiplying the above expression by $p_{ik}(t)$, we have, for $t \geq 0$, that

$$-\varepsilon p_{ik}(t) \leqslant \frac{p_{ik}(t)p_{kj}(h) - p_{ik}(t)p_{kj}(0)}{h} - p_{ik}(t)\dot{p}_{kj}(0)$$
$$\leqslant \varepsilon p_{ik}(t).$$

Applying $\sum_{k=1}^{M}$on all terms of the above expression, passing to the limit as $M \to \infty$ and reminding the Chapman-Kolmogorov equation (the semigroup property of P), we have that

$$-\varepsilon \leqslant \frac{p_{ij}(t+h) - p_{ij}(t)}{h} - \sum_{k=1}^{\infty} p_{ik}(t)\dot{p}_{kj}(0) \leqslant \varepsilon.$$

Since P satisfies $\dot{P}(t) = P(t)\Lambda$, where limits are in $\mathbb{R}$ for each entry $p_{ij}(t)$, we have that $\dot{p}_{ij}(t) = \sum_{k=1}^{\infty} p_{ik}(t)\dot{p}_{kj}(0)$ and the above expression reads

$$-\varepsilon \leqslant \frac{p_{ij}(t+h) - p_{ij}(t)}{h} - \dot{p}_{ij}(t) =: \frac{o_{ij}(t+h)}{h} \leqslant \varepsilon,$$
$$0 < h \leqslant \delta(\varepsilon), \; \forall j \in \mathbb{N} \text{ and } t \geq 0.$$

Reminding that δ does not depend on j, the above assertion follows.

Remark 3: Suppose that all solutions to (8), in the class of the continuous and continuously differentiable functions for each matrix entry, in the $\mathbb{R}$ norm, are standard transition matrices (clearly, all of them are associated to Λ). Then, they are one, say $P(t)$, so Lemma 8 applies. We justify this, noting that the minimal solution always exists and, being by assumption, a transition matrix, it is unique (see, e.g., [1]).

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