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On the stability of interval matrices

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***Abstract*— In this paper a new method is proposed for computing a lower bound to the stability margin of an interval matrix family, thus providing also a sufficient condition for stability. The method is based on Gershgorin's theorem and on the optimal selection of the eigenvectors of the nominal system: the optimization considerably improves previous bounds reported in literature. The analytical results make the solution extremely simple. Numerical experiments show that several problems involving uncertain systems can be solved efficiently by the proposed method, also in view of the comparisons with other methods proposed in the literature.**

I. Introduction

Much research has been devoted in the past decades to the stability of the family of linear systems characterized by the state space representation: $x \in \mathcal{R}^n$, $A \in \mathcal{R}^{n \times n}$, $q \in \mathcal{R}^p$

$$\sigma x = A(q)x \qquad (1)$$

In equation (1) σ is an operator indicating the time derivation d/dt for continuous-time systems or forward unit time shift for discrete-time systems, and q is a vector of real uncertain parameters, belonging to a predefined set Q, on which the entries of A depend. Several hypotheses can be made on the set Q, as well as on the functional relationship among the entries a_{ij} of A and the parameters. Here we shall consider the special case of a family of interval matrices, as defined in [1]:

$$A_I = \left\{ A \,|\, a_{ij} = q_{ij},\ q_{ij}^- \le q_{ij} \le q_{ij}^+ \right\}. \qquad (2)$$

This definition implies that the system is characterized by a dynamical matrix with entries independent each other, that may take values in given intervals $[q_{ij}^-, q_{ij}^+]$.
Clearly the definition includes matrices with some fixed elements, for which $q_{ij}^- = q_{ij}^+$. An alternative representation of the family A_I is obtained by defining

- the nominal matrix A_0 with entries $a_{ij}^0 = (q_{ij}^- + q_{ij}^+)/2$
- the perturbation matrix δA with entries $\delta a_{ij} \,:\, |\delta a_{ij}| \le \Delta a_{ij} = (q_{ij}^+ - q_{ij}^-)/2$

from which we obtain

$$A_I = \{ A \,|\, A = A_0 + \delta A \}. \qquad (3)$$

The non-negative matrix with entries Δa_{ij} will be denoted by ΔA.

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As usual the set A_I is said Hurwitz stable (Schur stable) if all the eigenvalues of any matrix $A \in A_I$ have negative real part (modulus less than unity). For a long time researchers tried to prove an "extreme point" result which should assert the equivalence between the stability of the family and that of all (or some) of vertex matrices. Unfortunately the claims by Bialas [2] and Jiang [3] that the family A_I is Hurwitz stable (Schur stable respectively) if and only if all the vertex matrices are so, have been shown to be incorrect. However in some special cases stability of vertices does imply stability of the family, as for 2×2 matrices [1], and symmetric matrices [4]. The line of research aiming at a finitely computable necessary and sufficient condition of robust stability is still active: for example in [5] the problem is studied by converting it into the analysis of the robust non-singularity of a larger auxiliary matrix family.
The lack of simple extreme point results is strictly related to the fact that the characteristic polynomial of a system with an interval dynamic matrix has coefficients which are multilinear in the uncertain parameters. A powerful tool for the analysis of such polynomials, and hence of the interval matrix, is provided by the mapping theorem [1],[6]: by its application in [6] it is shown how to solve the problems:

- is an assigned interval family stable?
- given a nominal stable matrix A_0 and possibly weights $\Delta a_{ij} \ge 0$, what is the largest value of ϵ for which the interval family

$$A_I^\epsilon = \left\{ A \,|\, a_{ij}^0 - \epsilon \Delta a_{ij} \le a_{ij} \le a_{ij}^0 + \epsilon \Delta a_{ij} \right\}.$$

  is stable?

The key point is to compute numerically the characteristic polynomials of the vertex matrices and to verify the stability not only of these polynomials, but also of the polynomials whose coefficients belong to the segments connecting the vertices. The stability of segments is verified using the "Segment Lemma" or the "Bounded Phase" condition [6]. Although the stability of segment polynomials is not "finitely computable" in the sense of [5], and is only sufficient for the stability of the family, the check both of vertices and of joining segments seems to give a definite answer to the stability problem of interval matrices. The very drawback is the exponential dependence of the number of vertices and segments: letting some entries of A to be fixed, a number of $p \le n^2$ of free parameters imply $v = 2^p$ vertices and $s = (v-1)v/2 = (2^{2p-1} - 2^{p-1})$ connecting segments, although the actual number of segments to check may be reduced by Karitonov-like expedients. A 3×3 full interval matrix has 512 vertices and 130816 segments, a 4×4 one has 65536 vertices and $2,147,450,880$ segments.

Several approaches have been proposed to avoid the dimensionality curse. Just to give a few examples, in [6] the Lyapunov theorem is used to estimate the parametric stability radius in the 2-norm; in [9], starting from sufficient conditions for the validity of NP-hard semi-infinite systems of LMIs arising from LMIs with uncertain data, it has been proved that given an interval matrix $\mathcal{U}_p = \{A \,|\, |A_{ij} - A_{0_{ij}}| \leq \alpha C_{ij}\}$ one can determine a computable lower bound on the supremum of α for which all instances of $\mathcal{U}_p$ share a common quadratic Lyapunov function; in [7] an explicit expression of the characteristic polynomial of the interval family allows one to use Karitonov's theorem to state a sufficient condition of stability; in [8] a region that contains the eigenvalues of all the matrices in the family is estimated using Gershgorin's theorem.

In this paper we improve the method proposed in [8], to get much less conservative estimates of the stability margin and of the parametric stability radius, using a very simple algorithm, that does not suffer from increasing dimension of the parameter space. Extensive numerical experiments are reported, where our method and the performance of the resulting algorithm are compared to the results achievable by the methods and algorithms proposed in [6] (stability of vertices and segment polynomials, use of Lyapunov theorem).

II. Notation and useful results

- **Notation:** In the complex plane $\mathcal{C}$, the open left plane is denoted by $\mathcal{C}^-$, the open unit disk by $\mathcal{D}_1$. Given a matrix (or vector) X, $|X|$ denotes the matrix (vector) whose elements are the moduli of the elements of X; $X > 0$ ($X \geq 0$) means that all elements of X are positive (nonnegative); if $X \in \mathcal{R}^{n\times n}$, $sp(X)$ is the spectrum of X, $\lambda_k(X)$ denotes the $k-th$ eigenvalue of X.
- **Irreducible matrices [10]:**

  Definition 1 *A matrix $A \in \mathcal{R}^{n\times n}$ is said to be reducible if either*

  a) $n = 1$ *and* $A = 0$; *or*

  b) $n \geq 2$, *there is a permutation matrix* $P \in \mathcal{C}^{n\times n}$, *and there is some integer* $1 \leq r \leq n-1$ *such that*

  $$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

  where $B \in \mathcal{C}^{r\times r}$, $D \in \mathcal{C}^{n-r\times n-r}$, $C \in \mathcal{C}^{r\times n-r}$, *and* $0 \in \mathcal{C}^{n-r\times r}$ *is a zero matrix.*

  Definition 2 *[10, 6.2.21] A matrix $A \in \mathcal{R}^{n\times n}$ is said to be irreducible if it is not reducible.*

- **Nonnegative matrices [10]:**

  Theorem 1 *[10, 8.4.4] Let $A \in \mathcal{R}^{n\times n}$, $\rho(A)$ be the spectral radius of A and suppose that A is irreducible and nonnegative. Then*

  a) $\rho(A) > 0$;

  b) $\rho(A) > 0$ *is an eigenvalue of* A;

  c) *There is a positive vector x such that* $Ax = \rho(A)x > 0$;

  d) $\rho(A) > 0$ *is an algebraically (and hence geometrically) simple eigenvalue of* A.

  Corollary 1 *[10, 8.1.31] Let $A \in \mathcal{R}^{n\times n}$ and suppose that A is nonnegative. If A has a positive eigenvector, then*

  $$\begin{aligned} \rho(A) &= \max_{x>0} \min_{1\leq i\leq n} \frac{1}{x_i}\sum_{j=1}^{n} a_{ij}x_j \\ &= \min_{x>0} \max_{1\leq i\leq n} \frac{1}{x_i}\sum_{j=1}^{n} a_{ij}x_j \end{aligned} \tag{4}$$

III. Problem statement and main result

Let us consider the interval matrix

$$A_I = \{A \,|\, A = A_0 + \delta A\}\,, \text{ with } \Delta a_{ij} \geq |\delta a_{ij}|, \tag{5}$$

and assume that A_0 is diagonalizable. Let $sp(A_0) = \{\lambda_1, \ldots, \lambda_n\}$, $\Lambda = diag([\lambda_1, \ldots, \lambda_n])$, $U = [u_{ij}]$ a matrix whose columns are right eigenvectors of A_0, $V = [v_{ij}] = U^{-1}$ its inverse. Obviously we have:

$$\begin{aligned} sp(A) &= sp(A_0 + \delta A) \\ &= sp(V(A_0 + \delta A)U) \\ &= sp(VA_0U + V\delta AU) \\ &= sp(\Lambda + V\delta AU). \end{aligned}$$

Then for any perturbation matrix the perturbed spectrum is the same as that of the matrix $L = \Lambda + V\delta AU$. By defining $F = |V|\Delta A|U|$, the result proved in [8] is:

Proposition 1 *[8] The eigenvalues of any matrix in the family A_I belong to the union of the disks whose centers are λ_k and whose radii are $\rho_k^r = \sum_{j=1}^{n} f_{kj}, k = 1, \ldots, n$ (sum of the rows of F), and also to the union of the disks whose centers are λ_k and whose radii are $\rho_k^c = \sum_{j=1}^{n} f_{jk}, k = 1, \ldots, n$ (sum of the columns of F).*

For the sake of completeness a detailed proof is in the Appendix. □

We now observe that the matrix U is not uniquely defined: if A_0 is diagonalizable and T is a matrix whose columns are fixed eigenvectors (for instance those with unit 2-norm), then all matrices of the form $U = TH = Tdiag([h_1, \ldots, h_{n-1}, h_n]), h_k \neq 0, k = 1, 2, \ldots, n$, define a similarity transformation that diagonalizes A_0: $sp(A) = sp(\Lambda + V\delta AU) = sp(\Lambda + H^{-1}T^{-1}\delta ATH)$. The matrix F whose row-sums and column-sums define the radii of circles bounding the perturbed spectrum is not uniquely determined by A_0 and ΔA, but depends on n free parameters according to the formula:

$$\begin{aligned} F &= H^{-1}|T^{-1}|\Delta A|T|H \\ &= diag([1/h_1, \ldots, 1/h_{n-1}, 1/h_n])|T^{-1}|\Delta A \\ &\quad |T|diag([h_1, \ldots, h_{n-1}, h_n]), \\ & h_k > 0, k = 1, 2, \ldots, n. \end{aligned}$$

Hence, letting $F_0 = |T^{-1}|\Delta A|T|$, the radii, computed from rows and columns, become respectively:

$$\rho_k^r = \frac{1}{h_k}(h_1 f_{k1}^0 + \ldots + h_n f_{kn}^0),\ k = 1, \ldots, n, \quad (6)$$

$$\rho_k^c = h_k(f_{1k}^0/h_1 + \ldots + f_{nk}^0/h_n),\ k = 1, \ldots, n. \quad (7)$$

The dependence of the radii on the parameters will be emphasized by the notation:

$$\rho_k^r = \rho_k^r(h),\ \rho_k^c = \rho_k^c(h).$$

Since for every choice of $h_1, \ldots, h_n$ Proposition 1 holds, the result of [8] is improved by the following:

Proposition 2 *The eigenvalues of any matrix in the family A_I belong to the set $\mathcal{S}$ defined by: $\mathcal{S} = \mathcal{S}_r \cap \mathcal{S}_c$,*

where

$$\mathcal{S}_x = \bigcap_{h_1>0,\ldots,h_n>0} \Sigma_x(h_1, \ldots, h_n),$$

$$\Sigma_x(h_1, \ldots, h_n) = \bigcup_1^n R_k^x(h)$$

$$R_k^x(h) = \{\mu \in \mathcal{C} : |\mu - \lambda_k| \leq \rho_k^x(h)\},$$

$\rho_k^x(h)$ *given by equations (6-7), with x=r or x=c*

□

Corollary 2 *If $\mathcal{S} \subset \mathcal{C}^-(\mathcal{S} \subset \mathcal{D}_1)$, then A_I is Hurwitz (Schur) stable.*

IV. Stability margin and parametric stability radius

The immediate application of Proposition 2 is to the estimation of two parameters related to the robust stability of an interval matrix family: the stability margin and the parametric stability radius. Since in the literature the definitions are not unique, they are here reported with reference to Hurwitz stability:

- Stability margin of the family A_I:

  $$S_H = -\max_{A \in A_I} (\max_{i=1,\ldots,n}(Re(\lambda_i(A)))$$

  the opposite of the largest real part of the spectra of matrices in A_I. $S_H > 0$ implies stability of the family.
- Parametric stability radius (weighted and unweighted) of a stable matrix A_0 (corresponding to *the parametric stability margin* in l_∞ norm of reference [6]):

  $$r_\infty(A_0) = \sup\{\epsilon | A_0 + \delta A(\epsilon) \text{ is stable}, \ |\delta A(\epsilon)| \leq \epsilon \Delta A\}$$

  where $\Delta a_{ij} = 1$ if the unweighted radius is concerned.

Let

$$\begin{aligned} a_k &= Re[\lambda_k(A_0)], k = 1, \ldots, n, \\ B_H^r(h) &= \max_{k=1,\ldots,n}[a_k + \rho_k^r(h)], \\ B_H^c(h) &= \max_{k=1,\ldots,n}[a_k + \rho_k^c(h)], \end{aligned}$$

then a lower bound to the stability margin is given by

$$B^* = -\min\left\{\min_{h>0} B_H^r(h),\ \min_{h>0} B_H^c(h)\right\}. \quad (8)$$

The quantity B^* will be called the optimal Gershgorin-based lower bound of the stability margin. Its computation is very simple, although it seems to require the solution of two min-max problems.

Let us consider the minimization of the bound based on row-sums $B_H^r(h)$. It is required to find the minimum with respect to $h_1, \ldots, h_n$, of the maximum of the n functions:

$$\begin{aligned} g_i(h) = \ & a_i + \tfrac{1}{h_i}(h_1 f_{i1}^0 + \ldots h_{i-1} f_{i,i-1}^0 + h_i f_{ii}^0 + \\ & h_{i+1} f_{i,i+1}^0 + \ldots + h_n f_{in}^0),\ i = 1, \ldots, n. \end{aligned} \quad (9)$$

Since $F_0 \geq 0$ $h > 0$, the solution is strictly related to the properties of non-negative matrices, recalled in Section II. Indeed define $m = \max|a_i|$ and consider the matrix $M = mI + diag([a_1 \ldots a_n]) + F_0$. The matrix M is clearly non-negative and we assume that it is irreducible. Then by Theorem 1 the eigenvalue of M with largest modulus, say μ, is real, positive and the associated eigenvector, say $\underline{x}$, is positive; moreover by Corollary 1 we have

$$\begin{aligned} \mu &= \max_{i=1,\ldots,n}\left(\frac{1}{x_i}\sum_{j=1}^n x_j M_{ij}\right) \\ &= \min_{h>0} \max_{i=1,\ldots,n}\left(\frac{1}{h_i}\sum_{j=1}^n h_j M_{ij}\right). \end{aligned} \quad (10)$$

The definition of M implies that:

- $\gamma = \mu - m$ is the largest real eigenvalue of $G = diag([a_1 \ldots a_n]) + F_0$;
- $\underline{x}$ is a positive eigenvector of G, associated to γ;
-

$$\begin{aligned} \gamma &= \mu - m \\ &= \min_{h>0} \max_{i=1,\ldots,n}\left(m + a_i + f_{ii}^0 + \frac{1}{h_i}\sum_{j=1, j\neq i}^n h_j f_{ij}^0\right) \\ &\quad -m \\ &= \min_{h>0} \max_{i=1,\ldots,n} g_i(h) = \min_{h>0} B_H^r(h). \end{aligned}$$

The same argument can be applied to the column-sums: since transposition does not change the eigenvalues the result is exactly the same: $\gamma = \min_{h>0} B_H(h)$. Then we conclude:

Proposition 3 *If the non-negative matrix F_0 is irreducible then the optimal Gershgorin-based lower bound of the* *stability margin is $B^* = -\gamma$, where γ is the largest real eigenvalue of $G = diag([Re(\lambda_1), \ldots, Re(\lambda_n)]) + F_0$.*

Remark 1. The assumption of irreducibility of F_0 is not restrictive because this property is generic.

Also a lower bound of the parametric stability radius can be estimated by applying our approach: let $B^*(\epsilon)$ be the lower bound of the stability margin of the interval matrix A_I^{ϵ}. Since we assume A_0 stable, we have $B^*(0) > 0$: then it is obvious to take as estimate of the parametric stability radius the quantity $E^* = \min\{\epsilon > 0 | B^*(\epsilon) = 0\}$. Now $B^*(\epsilon) = 0$ is equivalent to the largest real eigenvalue of $G(\epsilon) = diag([Re(\lambda_1), \ldots, Re(\lambda_n)]) + \epsilon F_0$ to be zero. Then E^* can be characterized as:

$$\begin{aligned} E^* = \quad & \min\{\epsilon > 0| \\ & \det(diag([Re(\lambda_1), \ldots, Re(\lambda_n)]) + \epsilon F_0) = 0\}. \end{aligned} \tag{11}$$

V. NUMERICAL EXPERIMENTS AND APPLICATIONS

4.1 The first set of numerical experiments aims at assessing how conservative is the method we propose in those cases where stability of vertices ensures stability of the interval family. In the examples only Hurwitz stability will be considered.

4.1.1 The 2×2 matrix tested in reference [8]:

$$A_0 = \begin{bmatrix} -3.8 & 1.6 \\ 0.6 & -4.2 \end{bmatrix}, \ \Delta A = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}.$$

The eigenvalues of A_0 are -3,-5; the eigenvector matrix with unit-norm columns is

$$T = \begin{bmatrix} 0.894 & -0.800 \\ 0.447 & 0.600 \end{bmatrix}.$$

The estimate of the stability margin given in reference [8] is $B = 1.78$; the estimate given by the present method is $B^* = 2.19$, obtained by the optimal eigenvector matrix:

$$\begin{aligned} U & = diag([0.2771\,1])T diag([3.614\,1]) \\ & = \begin{bmatrix} 3.2325 & -0.800 \\ 1.6162 & 0.600 \end{bmatrix}. \end{aligned}$$

Using the vertices of the interval family, the margin is computed as $S_H = 2.377$.

4.1.2 Ten thousand random 2×2 interval matrices where generated and their stability was tested either by the exact vertices analysis and by the Gershgorin method: using the exact criterion 3130 where found stable, whereas the Gershgorin method recognized 1737 stable systems. The computational complexity of the exact method was obviously much larger: the *elapsed-time* (computed by the Matlab intrinsic function *tic-toc*) was 7.51 times longer using the vertices than the Gershgorin method.

4.1.3 A symmetric 4×4 interval matrix is defined by 10 independent parameters, so that 1024 vertices must be checked for stability. 100 symmetric matrices A_0 where defined, so to have eigenvalues uniformly distributed in the interval $[-10, 0]$, and each of them was given an uncertainty $\delta A : |\delta a_{ij}| \leq 0.1|a_{ij}^0|$. By the analysis of vertices 81 matrices were found stable: 69 of these were stable also by the optimized Gershgorin method, whereas only 32 were estimated stable by the application of the Gershgorin method without optimization. In this case the difference in CPU time was dramatic: an *elapsed-time* 3331 times longer using the vertices.

4.2 In the following examples the stability margin and the parametric stability radius, as estimated by the optimal Gershgorin method, are compared with those obtained by the analysis both of vertices and of segments joining the vertices along which the bounded phase condition is checked [6].

4.2.1 Let us consider the interval matrix $A_I = A_0 + \delta A$

$$A_0 = \begin{bmatrix} -0.2975 & 0.086333 & 0.0784349 \\ -0.281541 & -1.15707 & -0.313211 \\ 0.0733216 & 0.275475 & -0.37876 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 \\ 0.05 & 0.05 & 0.05 \end{bmatrix}.$$

The eigenvalues of A_0 are $\Lambda = \{-0.3333, -0.5000, -1.0000\}$. The stability margin is estimated as $B^* = 0.04564$; an optimal transformation matrix and the corresponding $F = |(U^*)^{-1}|\Delta A|U^*|$ are

$$U^* = \begin{bmatrix} -1.0789 & 0.0337 & 0.2060 \\ 0.3118 & -0.4530 & 0.3477 \\ 0.1497 & 0.1969 & -0.9147 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.1200 & 0.0533 & 0.1144 \\ 0.3981 & 0.1767 & 0.3795 \\ 0.1895 & 0.0841 & 0.1807 \end{bmatrix}.$$

The estimate of the unweighted parametric stability radius is $E^* = 0.0556$. Extreme points and segment analysis gave the results: stability margin: $S_H = 0.2088$; parametric stability radius: $r_\infty(A_0) = 0.1339$ (note that to verify the bound on the phase along the 130816 segments it took an *elapsed-time*=956.91 seconds, that should be compared to 0.05 seconds of the proposed method).

4.2.2 Let:

$$A_0 = \begin{bmatrix} -1.11121 & 0.45636 & 1.71523 & 0.537766 \\ -1.33367 & -3.42213 & -1.57421 & -0.522113 \\ 0.379664 & 0.853088 & -1.49664 & 1.57577 \\ -1.6426 & -1.08442 & -1.79332 & -4.97002 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0.05 & 0.05 & 0.05 & 0.05 \\ 0.05 & 0.25 & 0.05 & 0.01 \\ 0.03 & 0.02 & 0.05 & 0.05 \\ 0.15 & 0.03 & 0.25 & 0.10 \end{bmatrix}.$$

The nominal spectrum is $sp(A_0) = \{-2 + i, -2 - i, -3, -4\}$. The stability margin is estimated as $B^* = 0.4489$; the weighted parametric stability radius is estimated as $E^* = 1.2410$.

In this example it was possible to compute the eigenvalues of the 65536 vertices, which gave $\max Re[\lambda(A_{vert})] = -1.7527$ with an *elapsed-time* of 954 sec. Segment analysis would have required about 180 days.

4.3 The experiments here reported refer to the linear system studied in example 12.4 of reference [6]:

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & p_1 & -0.7070 & p_2 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ p_3 & -7.5922 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix}, \ C = [0\,1\,0\,0].$$

For the 3 parameters the nominal values and the ranges are given: $p_i = p_i^o + \delta p_i, \ |\delta p_i| \leq \Delta p_i, i = 1, 2, 3,$ $p_1^o = 0.3681, p_2^o = 1.4200, p_3^o = 3.5446; \Delta p_1 = 0.05, \Delta p_2 = 0.01, \Delta p_3 = 0.04$. The nominal open-loop system is unstable: $sp(A(p^o)) = \{-2.0727, -0.2325, 0.2758 - 0.2576i, 0.2758 + 0.2576i\}$. In [6] the system is stabilized by a constant output feedback $K_0 = [-1.63522\, 1.58236]^T$, which produces the spectrum $sp[A(p^o) + B(p^o)K_0C] = \{-19.0122, -0.0629, -0.2441 - 1.4177i, -0.2441 + 1.4177i\}$. Then the authors perform a robustification procedure, based on robustness analysis via Liapunov equation, that provides an estimate of the parametric stability radius in the 2-norm.

The feedback gain thus computed is: $K^* = [-0.99633989\, 1.801833665]^T$ which gives the closed loop spectrum: $sp[A(p_o) + B(p_o)K^*C] = \{-18.3963, -0.0736, -0.2476 - 1.2501i, -0.2476 + 1.2501i\}$. We pose the following problems

P1: Assuming $\Delta p_1 = 0.05, \Delta p_2 = 0.01, \Delta p_3 = 0.04$, estimate the stability margin and the weighted parametric stability radius.

P2: Assuming $\Delta p_1 = \Delta p_2 = \Delta p_3 = \epsilon$, estimate the unweighted parametric stability radius.

P3: Find a feedback gain that improves the robustness of the closed loop system.

Since there are only three parameters, the computation of the stability of vertex polynomials and of the connecting segments can be carried out, and the results are in [6], where also the results from the Lyapunov method are reported. Clearly our approach should be compared with Lyapunov results.

Table 1 summarizes the estimates B^* and E^* of the relevant robustness parameters of the closed-loop system either with feedback matrix K_0 and K^* : they are much less conservative than those achievable by the Lyapunov method of [6]. Moreover, since the evaluation of B^* and E^* is computationally very simple, these parameters can be used within a criterion for optimizing robustness of the closed-loop system: the feedback gain K_1 is found by minimizing the function $J(K) = -B^*(A_0 + B_0KC) - w_1E^*(A_0 + B_0KC) + w_2\|K\|_2$. The norm of K is used to "regularize" the function which otherwise would decrease along a direction. Actually a whole family of optimal feedback matrices may be generated changing weights in the function J, among which the designer may choose. The resulting closed loop nominal spectrum is

$$sp[A(p^o) + B(p^o)K_1C] = \{-50.8157, -0.0830, -0.3010 - 1.1401i, -0.3010 + 1.1401i\}.$$

In this case the method of checking vertices and segments cannot be used for optimization.

TABLE I

Feedback gain	Stability Margin of the family $A_{CL}(p_o + \delta p)$
$K_0 = [-1.63522\ 1.58236]^T$	0.06246
$K^* = [-0.99633989\ 1.801833665]^T$	0.07296
$K_1 = [-1.7759765625\ 5.7232421875]^T$	0.08233
Feedback gain	Parametric stability radius of $A_{CL}(p_o)$
$K_0 = [-1.63522\ 1.58236]^T$	0.2128
$K^* = [-0.99633989\ 1.801833665]^T$	0.2034
$K_1 = [-1.7759765625\ 5.7232421875]^T$	0.2532
Feedback gain	Weighted parametric radius of $A_{CL}(p_o)$
$K_0 = [-1.63522\ 1.58236]^T$	9.63
$K^* = [-0.99633989\ 1.801833665]^T$	10.05
$K_1 = [-1.7759765625\ 5.7232421875]^T$	16.5

VI. CONCLUSIONS

A simple, but very efficient way to checking stability and estimating stability margins of an interval matrix family has been proposed. The method, based on optimized Gershgorin's regions, does not suffer from dimensionality curse and is less conservative than competing approaches, such as those using Lyapunov equation. It is an efficient tool to solve analysis problems in cases where the number of parameters makes impossible the analysis of vertices and edges.

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APPENDIX

Proof Proposition 1: By Gershgorin's theorem the eigenvalues of $\Lambda + V\delta AU$ belong to the set

$$\begin{aligned}\mathcal{S}_{\delta A} = & \bigcup_{k=1,\ldots,n} D_k, \\ D_k = & \left\{\mu \in \mathcal{C} : \left|\mu - (\lambda_k + \sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sk})\right| \leq \right. \\ & \left. \sum_{j=1}^{n,j\neq k}\left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sj}\right|\right\}.\end{aligned}$$

Since

$$\left|\mu - \lambda_k - \sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sk}\right| \geq$$

$$\left| |\mu - \lambda_k| - \left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sk}\right| \right|$$

each disk D_k is contained in the disk (with center in λ_k)

$$\begin{aligned}E_k = & \{\mu \in \mathcal{C} : |\mu - \lambda_k| \leq \\ & \left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sk}\right| + \sum_{j=1}^{n,j\neq k}\left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sj}\right| \\ & = \sum_{j=1}^{n}\left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sj})\right|\right\}.\end{aligned}$$

From the inequality

$$\begin{aligned}&\sum_{j=1}^{n}\left|\sum_{r=1}^{n}\sum_{s=1}^{n} v_{kr}\delta a_{rs}u_{sj}\right| \leq \sum_{j=1}^{n}\left(\sum_{r=1}^{n}\sum_{s=1}^{n} |v_{kr}|\,|\delta a_{rs}|\,|u_{sj}|\right) \\ &\leq \sum_{j=1}^{n}\left(\sum_{r=1}^{n}\sum_{s=1}^{n} |v_{kr}|\,\Delta a_{rs}\,|u_{sj}|\right),\end{aligned}$$

it follows that each disk E_k is contained in a disk R_k with the same center λ_k and a larger radius

$$\rho_k^r = \sum_{j=1}^{n}\left(\sum_{r=1}^{n}\sum_{s=1}^{n} |v_{kr}|\,\Delta a_{rs}\,|u_{sj}|\right) = \sum_{j=1}^{n} f_{kj}.$$

The same argument can be applied to the columns of $\Lambda + V\delta AU$. Then the eigenvalues of $A_0 + \delta A$ belong to the union of disks with centers in the eigenvalues of A_0 and radii given by the sums of the elements of the respective rows (or columns) of $F = |V|\Delta A|U|$. □