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Optimal Sensor and Actuator Location for Descriptor Systems using Generalized Gramians and Balanced Realizations

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Abstract— **This paper presents two methods for optimal sensor and actuator location for linear time-invariant descriptor systems. The objective is the improvement of the state controllability and state observability. Since these two notions are quantified by the corresponding Gramians, the optimal location is based on the maximization of the generalized Gramians. Firstly, a method aims at maximizing the energy provided by the inputs to the system and the energy collected by the outputs. Secondly, state controllability and state observability are jointly improved by considering a balanced realization of descriptor systems. Finally, sensor location is exploited for disturbance decoupling. A numerical example illustrates the efficiency of the proposed methods.**

I. INTRODUCTION

This paper deals with the problem of optimal sensor and actuator location (SAL) for the generic class of linear time-invariant (LTI) descriptor systems ($E\dot{x} = Ax + \dots$). Once the control objectives, and the model had been defined, one has to chose the control structure and design the controller, before verifying the results with simulation and finally implement the solution. While defining the control structure, one of the key points is SAL, in other words, what are the suitable variables to be controlled (actuator location) and what are the suitable variables to be provided to the controller (sensor location) in order to efficiently supervise the plant. Thus, SAL is one of the main issues in control system design. Actuator or sensor selecting is useful, even if all the variables can be manipulated or measured, in order to lower the cost of operation and maintenance.

For usual state-space systems ($\dot{x} = Ax + \dots$), many techniques have arisen to tackle the SAL problem with different objectives such as accessibility, input/output controllability, robust stability face to uncertainties or minimization of the computational cost…

According to the survey made in [8], one of the two most reliable techniques is the improvement of state controllability and observability. In [3], optimal SAL is addressed by maximizing the Gramians of LTI usual systems. In [4] SAL is addressed for flexible structures, and a numerical criterion is proposed to avoid having both very high and very small eigenvalues of the Gramian while maximizing the energetical transfers. In [7] SAL is performed in order to maximize the singular values of the balanced Gramians.

Unfortunatly, to the best of the authors' knowledge, SAL has not been treated in the descriptor case, beside in [5] in which the approach of of [3] is generalized to descriptor systems. In this paper we consider the class of LTI descriptor systems which is known to be more generic than the usual state-space systems. Since it includes static relations, the descriptor formalism can model physical constraints, impulse behaviors or non causality, see [2] for a complete study of the descriptor systems.

Here, SAL for descriptor systems is envisaged from the energetical point of view. A first method consists in selecting the actuators (resp. the sensors) that maximize the energy transmitted from the controller to the system (resp. from the system to the controller or observer). This will be shown to be equivalent to maximize the generalized Gramians. Maximization of the Gramians may lead to efficiently control some state variables and efficiently measure others. In other words, the maximization of the controllability (*resp.* observability) Gramian is an efficient tool to contol (*resp.* estimate) the state variables, but if the goal is to control the outputs with the inputs, another method need to be developped. Input/output performance is improved by maximizing both controllability and observability, thus the SAL is done by maximizing the Gramians of a balanced realization. Moreover, the latter method is exploited for sensor locating with respect to disturbance decoupling. In fact, disturbance decoupling can be considered as minimizing the energetical transfer from unknown input to output while maximizing the one from the command input.

Since the possible location for sensor and/or actuator are finite, these problems can be considered as integer programming problems. Thus, for large scale systems, it is possible to use integer optimization tools in order to avoid to test all the possible locations.

The paper is organized as follows. The second section gives some backgrounds on descriptor systems. The third section is devoted to the definition and the computation of the generalized Gramians. The two proposed solutions to the SAL problem and a method for disturbance attenuation via sensor location are developped in the fourth section. A numerical example illustrates the efficiency of the contribution in the last section.

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II. BACKGROUNDS

In this section, some basics about descriptor systems (taken from [2]) are reminded. We consider the LTI descriptor system, described by a generalized state-space system

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) \\ \quad y(t) = Cx(t) \end{cases} \tag{1}$$

$$\begin{cases} Ex(k+1) = Ax(k) + Bu(k) \\ \quad y(k) = Cx(k) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$ and $y \in \mathbb{R}^m$ are respectively the state variable, the control input and the measured output, and E, A, B and C are real known constant matrix with appropriate dimensions with $r = rank(E) \leq n$. without loss of generality E and A are assumed to be square matrices.

The system (1) (respectively (2)) has a unique solution, for any initial condition, if it is regular *i.e.* $det(sE - A)) \neq 0$. Let $q = deg(det(sE - A))$, a descriptor system has q finite dynamics mode, $(n - r)$ static modes and $(r - q)$ impulsive modes. The finite modes correspond to the finite eigenvalues of the matrix pencil (E, A). The system is called *stable* if and only if the finite modes are stable, *i.e.* if and only if the finite eigenvalues of (E, A) lie in the open left half-plane (respectively in the open unitary disk). The impulsive modes may cause impulse terms or non causality in the response and thus are highly undesirable. A system has no impulsive mode and is called *impulse free* if and only if $deg(det(sE - A)) = r$ holds. A stable and impulse free descriptor system is called *admissible*.

Let introduce the Weierstrass-Kronecker decomposition of the matrix pencil (E, A) [2]. Provided (E, A) is regular, there exist two nonsingular matrices P and Q such that

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \text{ and } PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{3}$$

where N is nilpotent (its nilpotency index is denoted h), and $n_1 + n_2 = n$. The eigenvalues of J are the finite eigenvalues of (E, A). The system (1) (resp. (2)) is equivalent to (4) (resp. (5))

$$\begin{cases} \dot{x}_1(t) = Jx_1(t) + B_1 u(t) \\ N\dot{x}_2(t) = x_2(t) + B_2 u(t) \\ y(t) = C_1 x_1(t) + C_2 x_2(t) \end{cases} \tag{4}$$

$$\begin{cases} x_1(k+1) = Jx_1(k) + B_1 u(k) \\ Nx_2(k+1) = x_2(k) + B_2 u(k) \\ y(k) = C_1 x_1(k) + C_2 x_2(k) \end{cases} \tag{5}$$

with

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; CQ = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \text{ and } x = Q \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6}$$

The subsystems (I_{n_1}, J, B_1, C_1) and (N, I_{n_2}, B_2, C_2) are called *causal* and *noncausal* respectively, the corresponding subspaces are denoted H_c and H_{nc}.

A descriptor system is stable if and only if its causal subsystem is stable. A descriptor system is controllable (resp. observable) if and only if both causal and noncausal subsystems are controllable (resp. observable). Any regular descriptor system can be uniquely defined by its series expansion using the Laurent parameters Φ_k

$$(sE - A)^{-1} = \sum_{k \geq -h} \Phi_k s^{-k-1} \tag{7}$$

$$\text{where } \Phi_k = \begin{cases} Q \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix} P, & for\ k \geq 0 \\ Q \begin{bmatrix} 0 & 0 \\ 0 & (-N)^{-k-1} \end{bmatrix} P, & for\ k < 0 \end{cases} \tag{8}$$

which is valid in some set $0 < |s| \leq R$, for some $R > 0$ [1]. It is interesting to note that $\Phi_0 E$ and $-\Phi_{-1} A$ are the projection on H_c along H_{nc} and on H_{nc} along H_c respectively. Others properties of the Laurent parameters are given in [1] and [10].

III. GENERALIZED GRAMIANS

In the framework of SAL, we need to quantify the controllability and the observability of a descriptor system. For usual state-space system, it is well-known that these notions are related to the Gramians. The generalization of the Gramians to descriptor systems, has been established in [1] and [9]. The generalized Gramians are computed by solving Lyapunov-like equations established in [5] in the continuous-time case, and in [10] in the discrete-time case.

Continuous-time case

The controllability (resp. observability) Gramian of system (1) are decomposed in a causal Gramian (provided that the integral exists) and a non causal Gramian defined by (9) (resp. (10))

$$R_c^c = \int_0^\infty \Phi_0 e^{A\Phi_0 t} BB^T e^{\Phi_0^T A^T t} \Phi_0^T dt, \; R_{nc}^c = \sum_{k=-h}^{k=-1} \Phi_k BB^T \Phi_k^T$$

$$G_c^c = R_c^c + R_{nc}^c \tag{9}$$

$$O_c^c = \int_0^\infty \Phi_0^T e^{A^T \Phi_0^T t} C^T C e^{\Phi_0 A t} \Phi_0 dt, \; O_{nc}^c = \sum_{k=-h}^{k=-1} \Phi_k^T C^T C \Phi_k$$

$$G_o^c = O_c^c + O_{nc}^c \tag{10}$$

The generalized controllability Gramian is determined by solving the Lyapunov-like equations for descriptor systems established by the lemma 1 [5]

Lemma 1: (i) If R_c^c, R_{nc}^c and G_c^c exist they satisfy respectively

$$0 = \Phi_0 A R_c^c + R_c^c A^T \Phi_0^T + \Phi_0 BB^T \Phi_0^T \tag{11}$$

$$0 = \Phi_{-1} E R_{nc}^c E^T \Phi_{-1}^T - R_{nc}^c + \Phi_{-1} BB^T \Phi_{-1}^T \tag{12}$$

$$\begin{aligned} 0 = \; & \Phi_{-1} E G_c^c E^T \Phi_{-1}^T + \Phi_0 BB^T \Phi_0^T + \Phi_{-1} BB^T \Phi_{-1}^T \\ & + \left(\Phi_0 + \frac{\Phi_{-1}}{2}\right) A G_c^c + G_c^c A^T \left(\Phi_0 + \frac{\Phi_{-1}}{2}\right)^T \end{aligned} \tag{13}$$

(ii) If (1) is stable, R_c^c is the unique projection on H_c of the solutions of (11), R_{nc}^c and G_c^c are the unique solutions of (12) and (13) respectively.

(iii) If (1) is stable, (1) is controllable if and only if G_c^c is the unique positive definite solution of (13).

The generalized observability Gramian for continuous-time descriptor system is derived from a dual result, established in [5].

Discrete-time case

The controllability (resp. observability) Gramian of system (2) are decomposed in a causal Gramian provided that the serie converges) and a non causal Gramian defined by (15) (resp. (17)

$$R_c^d = \sum_{k\geq 0} \Phi_k BB^T \Phi_k^T,\ R_{nc}^c = \sum_{k=-h}^{-1} \Phi_k BB^T \Phi_k^T, \quad (14)$$

$$G_c^d = R_c^d + R_{nc}^d \quad (15)$$

$$O_c^d = \sum_{k\geq 0} \Phi_k^T C^T C \Phi_k,\ O_{nc}^c = \sum_{k=-h}^{-1} \Phi_k^T C^T C \Phi_k, \quad (16)$$

$$G_o^d = O_c^d + O_{nc}^d \quad (17)$$

The generalized controllability Gramian is determined by solving the Lyapunov-like equations for descriptor systems established by the lemma 2 [10].

Lemma 2: (i) If R_c^d, R_{nc}^d and G_c^d exist they satisfy respectively

$$\begin{aligned} 0 &= \Phi_0 A R_c^d A^T \Phi_0^T + \Phi_0 BB^T \Phi_0^T - R_c^d & (18)\\ 0 &= \Phi_{-1} E R_{nc}^d E^T \Phi_{-1}^T + \Phi_{-1} BB^T \Phi_{-1}^T - R_{nc}^d & (19)\\ 0 &= (\Phi_0 A - \Phi_{-1} E)\, G_c^d\, (\Phi_0 A - \Phi_{-1} E)^T - G_c^d \\ &\quad + \Phi_{-1} BB^T \Phi_{-1}^T + \Phi_0 BB^T \Phi_0^T & (20) \end{aligned}$$

(ii) If (2) is stable, R_c^d is the unique projection on H_c of the solutions of (18), R_{nc}^d and G_c^d are the unique solutions of (19) and (20) respectively.

(iii) If (2) is stable, (2) is controllable if and only if G_c^d is the unique positive definite solution of (20).

The generalized observability Gramian for discrete-time descriptor system is derived from a dual result, established in [10].

IV. Optimal sensor and actuator location

In this section, the optimal SAL problem is treated. First, the energetic interpretation of the Gramians is extended to descriptor case. In the usual state-space case, it is established that [6]

- the output energy, of an input free system, generated by an arbitrary initial state X_0, is given by $E_y(X) = X_0^T G_o X_0$, where G_o is the observability Gramian;
- the minimal input energy needed to reach a given state X, from null initial condition, is given by $E_u(X) = X^T G_c^{-1} X$, where G_c is the controllability Gramian.

In the case of discrete-time descriptor systems, this is generalized by the following theorem.

Theorem 1: Consider the discrete-time descriptor system (2). The minimal energy input to reach a given state X from $x(0) = 0$ is given by (21). The output energy of the input free system (*i.e.* $u(k) = 0$) generated by a given initial state X is given by (22)

$$\begin{aligned} E_u(X) &= \min_{u, x(0)=0, x(\infty)=X} \sum_{k\geq 0} u(k)^T u(k) \\ &= X^T \left(G_c^d\right)^{-1} X & (21)\\ E_y(X) &= \sum_{k\geq 0} y(k)^T y(k) \\ &= X^T (QP)^{-T} G_o^d (QP)^{-1} X & (22) \end{aligned}$$

moreover, the matrices P and Q are not uniquely defined, but the product QP is unique (see [2]).

Proof: Omitted due to space limitation. ■

In the continuous-time case, such an interpretation is not rigorously feasible since eventual impulse behavior cause infinite energy, but the Gramians still quantify the controllability and the observability. One should note that the causal Gramians are defined like Gramians of usual systems and that the noncausal Gramians are defined like in the discrete-time case. For both usual systems and discrete time descriptor systems the energetical meaning of the Gramians is valid. Based on the previous interpretation of the generalized Gramians, two complementary methodologies of optimal SAL are proposed.

A. Optimization of the state controllability or/and observability

The actuators are chosen in order to minimize the energy that must be provided to the system. The sensors are chosen in order to maximize the energy collected by the measured output. Thus optimal SAL is equivalent to the maximization of the trace of the Gramians. Actuator (resp. sensor) selection sets the matrix B (resp. C) and thus determines the controllability (resp. observability) Gramian.

Optimal SAL methodology (continuous-time case)

Optimal placement n_a actuators is equivalent to finding B that maximizes J_c under the constraint (24).

$$J_c^c = Trace(G_c^c) \quad (23)$$

$$b_{ij} \in \{0, 1\},\ \sum_{i\in\mathcal{C}} b_{ij} = 1,\ j = 1, \ldots, n_a \quad (24)$$

where $\mathcal{C}$ denotes the subset of state variables that can be manipulated. Optimal placement of ns sensors is equivalent to finding C that maximizes J_o under the constraint (26)

$$J_o^c = Trace((QP)^{-T} G_o^c (QP)^{-1}) \quad (25)$$

$$c_{ij} \in \{0, 1\},\ \sum_{j\in\mathcal{O}} c_{ij} = 1\ i = 1, \ldots, n_s \quad (26)$$

where $\mathcal{O}$ denotes the subset of state variables that can be measured. The entries of B (resp. C) are set to 0 or 1 whether the corresponding input (resp. output) is selected or not. It is not restrictive to assume $b_{ij} \in \{0, 1\}$ and $c_{ij} \in \{0, 1\}$ since the system can always be normalized.

The **discrete-time case** is adressed similarly, by selecting B that maximizes $J_c^d = Trace(G_c^d)$ and/or selecting C that maximizes $J_o^d = Trace((QP)^{-T} G_o^d (QP)^{-1})$.

This method improves state controllability and state observability separately. This is of particular interest for the control or the estimation of the state variables (*e.g.* for observer-based diagnosis).

B. Joint optimization of the state controllability and observability

The previous method may lead to efficiently control a subset of the state variables and efficiently observe another subset and then result in a poor controllability of the output by the input. If the objective is to control the output variables, then one should ensure that the input energy is optimally collected by the measurements. Thus, the SAL is based on the maximization of the generalized Gramian of a balanced realization. First we give a realization of a discrete-time descriptor system which is termed balanced in the sense that the minimal input energy (21) and the maximal output energy (22) are linked to a common matrix.

Theorem 2: Let T_1 and T_2 be non singular matrices, such that $(T_1JT_1^{-1}, T_1B_1, C_1T_1^{-1})$ and $(T_2NT_2^{-1}, T_2B_2, C_2T_2^{-1})$ are balanced with diagonal Gramian Σ_1 and Σ_2 respectively (see computation in chap. 3.9 of [11] or [6]). In the discrete-time case, the minimal energy input to reach a given state X from $x(0) = 0$ is given by (27). The output energy of the input free system, generated by a given state X is given by (28)

$$\begin{aligned} E_u(X) &= \min_{u,x(0)=0,x(\infty)=X} \sum_{k\geq 0} u(k)^T u(k) \\ &= X^T \bar{Q}^{-T} \left(\Sigma^d\right)^{-1} \bar{Q}^{-1} X \qquad (27) \\ E_y(X) &= \sum_{k\geq 0} y(k)^T y(k) \\ &= X^T \bar{Q}^{-T} \Sigma^d \bar{Q}^{-1} X. \qquad (28) \end{aligned}$$

where Σ^d is a diagonal positive definite matrix.

Proof: From (20) and by duality, it is easy to derive

$$G_c^d = Q \begin{bmatrix} dLyap(J,B_1) & 0 \\ 0 & dLyap(N,B_2) \end{bmatrix} Q^T$$
$$G_o^d = P^T \begin{bmatrix} dLyap(J^T,C_1^T) & 0 \\ 0 & dLyap(N^T,C_2^T) \end{bmatrix} P \quad (29)$$

where $dLyap(M_1, M_2)$ denotes the positive definite solution of the discrete Lyapunov equation $M_1XM_1^T - X + M_2M_2^T = 0$. Let us define the following Weierstrass-Kronecker decomposition of (2)

$$\bar{Q} = Q \begin{bmatrix} T_1^{-1} & 0 \\ 0 & T_2^{-1} \end{bmatrix},\ \bar{P} = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} P,\ \bar{P}A\bar{Q} = \begin{bmatrix} \bar{J} & 0 \\ 0 & I \end{bmatrix} \quad (30)$$

$$\bar{P}E\bar{Q} = \begin{bmatrix} I & 0 \\ 0 & \bar{N} \end{bmatrix},\ \bar{P}B = \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix},\ C\bar{Q} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \quad (31)$$

where $(\bar{J}, \bar{B}_1, \bar{C}_1)$ and $(\bar{N}, \bar{B}_2, \bar{C}_2)$ are balanced realizations thus $dLyap(\bar{J}, \bar{B}_1) = dLyap(\bar{J}^T, \bar{C}_1^T) = \Sigma_1^d$ and $dLyap(\bar{J}, \bar{B}_2) = dLyap(\bar{J}^T, \bar{C}_2^T) = \Sigma_2^d$ thus (29) becomes

$$G_c^d = \bar{Q}\Sigma\bar{Q}^T,\ G_o^d = \bar{P}^T\Sigma\bar{P},\ with\ \Sigma = \begin{bmatrix} \Sigma_1^d & 0 \\ 0 & \Sigma_2^d \end{bmatrix} \quad (32)$$

Combine (21)-(22) with (32), then (27)-(28) follows. ■

The objective of SAL is to jointly maximize controllability and observability. In other words sensors and/or actuators should be chosen to maximize the ratio of the input energy collected by the outputs, thus a natural criteria is to maximize J^d defined by

$$\begin{aligned} J^d &= Trace(E_y(I_n)\,(E_u(I_n))^{-1}) \qquad (33) \\ &= Trace(\bar{Q}^{-T}\left(\Sigma^d\right)^2 \bar{Q}^T) \qquad (34) \end{aligned}$$

Computation of Σ^d and Σ^c

First determine J, N, B_1, B_2, C_1 and C_2, then Σ_1 and Σ_2 are the Gramians of the balanced realizations of (J, B_1, C_1) and (N, B_2, C_2) respectively.

In the continuous-time case Σ^c is defined by

$$\Sigma^c = \begin{bmatrix} \Sigma_1^c & 0 \\ 0 & \Sigma_2^c \end{bmatrix} \quad (35)$$

$$\begin{aligned} \Sigma_1^c &= cLyap(T_1JT_1^{-1}, T_1B_1) \\ &= cLyap((T_1JT_1^{-1})^T, (C_1T_1^{-1})^T) \end{aligned} \quad (36)$$

$$\begin{aligned} \Sigma_2^c &= dLyap(T_2NT_2^{-1}, T_2B_2) \\ &= dLyap((T_2JT_2^{-1})^T, (C_2T_2^{-1})^T) \end{aligned} \quad (37)$$

where $cLyap(M_1, M_2)$ denotes the positive definite solution of the Lyapunov equation $M_1X + XM_1^T + M_2M_2^T = 0$. In the discrete-time case Σ^d is defined by

$$\Sigma^d = \begin{bmatrix} \Sigma_1^d & 0 \\ 0 & \Sigma_2^d \end{bmatrix} \quad (38)$$

$$\begin{aligned} \Sigma_1^d &= dLyap(T_1JT_1^{-1}, T_1B_1) \\ &= dLyap((T_1JT_1^{-1})^T, (C_1T_1^{-1})^T) \end{aligned} \quad (39)$$

$$\begin{aligned} \Sigma_2^d &= dLyap(T_2NT_2^{-1}, T_2B_2) \\ &= dLyap((T_2JT_2^{-1})^T, (C_2T_2^{-1})^T) \end{aligned} \quad (40)$$

Optimal SAL methodology (continuous-time case)

Optimal placement of n_a actuators and/or n_s sensors is equivalent to finding B and/or C that maximize J^c under the constraint (24) and/or (26).

$$J^c = Trace(\bar{Q}^{-T}\left(\Sigma^c\right)^2 \bar{Q}^T) \quad (41)$$

The **discrete-time case** is adressed similarly, by selecting B and/or C that maximize the criteria $J^d = Trace(\bar{Q}^{-T}\left(\Sigma^d\right)^2 \bar{Q}^T)$ under the constraints (24) and/or (26).

The criteria used for the two proposed methods are based on the $Trace$ of the Gramians to reflect the total energy transmitted from the inputs to the outputs. Nevertheless, one may penalize location where both very high and very low eigenvalues appear, even for the first method (like [4] suggested, for usual systems). One may prefer to avoid poorly controllable (or observable) modes by maximizing the lowest singular value (see [3], for usual systems case). The choice of the criteria closely depends on the control objectives.

Since the possible location for the actuators (resp. sensors) are finite, the possible values of B (resp. C) are finite, thus the two methodologies are integer programming problems. The basic solution is enumeration and numerical checking of all the candidates. For large scale systems integer optimization is a very efficient method to significantly reduce the computational cost. A review of Branch and Bound methods for integer programming in the SAL framework is proposed in [3].

C. Disturbance decoupling via optimal sensor location

Let consider a LTI descriptor system with unknown input $w \in \mathbb{R}^{n_w}$

$$\begin{cases} E\dot{x}(t) = Ax(t) + Bu(t) + B_w w(t) \\ y(t) = Cx(t) \end{cases} \tag{42}$$

$$\begin{cases} Ex(k+1) = Ax(k) + Bu(k) + B_w w(k) \\ y(k) = Cx(k) \end{cases} \tag{43}$$

The previous SAL method can be exploited for disturbance decoupling by selecting the sensors such that the transmitted energy from the unknown to the output is minimized while the energy transmitted from the command input to the output is maximized. According to previous discussion the measurement matrix C should be chosen to minimize Σ_w^c (or Σ_w^d in the discrete-time case) the balanced Gramian of (E, A, B_w, C) while maximizing Σ^c.

Optimal sensor locaion methodology (continuous-time case) Optimal placement of n_s sensors is equivalent to finding C that maximizes J_d^c under the constraint (26).

$$J_d^c = Trace(\bar{Q}^{-T} (\Sigma^c)^2 \bar{Q}^T) \left(Trace(\bar{Q}_w^{-1} (\Sigma_w^c)^2 \bar{Q}_w^T) \right)^{-1} \tag{44}$$

The **discrete-time case** is addressed similarly by chosing C that maximizes J_d^d under the constraint (26).

$$J_d^d = Trace(\bar{Q}^{-T} \left(\Sigma^d\right)^2 \bar{Q}^T) \left(Trace(\bar{Q}_w^{-1} \left(\Sigma_w^d\right)^2 \bar{Q}_w^T) \right)^{-1} \tag{45}$$

V. NUMERICAL EXAMPLE

Let us consider the continuous-time LTI descriptor system defined by the matrix pencil

$$E = \begin{bmatrix} 3 & 3 & 7 & 3 & 0 & 1 \\ 1 & -4 & 5 & 7 & 0 & 1 \\ 4 & 6 & -3 & 0 & 0 & 1 \\ 3 & 3 & 3 & -2 & 0 & 4 \\ 7 & 4 & 7 & 6 & 0 & -2 \\ 6 & 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -21 & 42 & -126.7 & 287 & 1 & 0 \\ 23 & 79 & -284.9 & 323 & 1 & 5 \\ -40 & 14 & 3.4 & -270 & 1 & 3 \\ -21 & -8 & 77.3 & 92 & 4 & 7 \\ -31 & 91 & -246.3 & 264 & -2 & 3 \\ -42 & 24 & -6.4 & 230 & 1 & -1 \end{bmatrix},$$

$$\text{and } B = \begin{bmatrix} 45 & 0 \\ -10 & 5 \\ 70 & 3 \\ 45 & 7 \\ 90 & 3 \\ 90 & -1 \end{bmatrix}$$

Sensor location is performed in order to maximize the collected output energy. Thus, according to section IV.B the sensors are chosen to maximize $J^c = Trace(\bar{Q}^{-T} (\Sigma^c)^2 \bar{Q}^T)$. No positive definite solution to the computation of the Gramian exists for the location of a unique actuator. Positioning 2 sensors, there exists a positive definite solution Σ^c for 4 combinations of sensors. The comparison of the obtained results are displayed in the following table.

Measurements	x_1, x_5	x_2, x_5	x_3, x_5	x_4, x_5
$J_o^c = Trace(G_o^c)$	2.47	2.59	2.53	2.44
J^c	7.69	11.17	1.11	0.41

Applying the first methodology, the 4 solutions are almost equivalent for the state observability since the criterion J_o^c is not significantly different in the four cases. Considering the balanced Gramian an optimal solution for the energetical transfer from the inputs to the ouputs is to measure x_2 and x_5, or in other words to set

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{46}$$

Simulations are runned, for the finite energy input control defined by

$$u_1(t) = \begin{cases} 3, & for\ 1 < t < 5 \\ 0, & else \end{cases} \tag{47}$$

$$u_2(t) = \begin{cases} sin(3t), & for\ t < 2 < \pi \\ 0, & else \end{cases} \tag{48}$$

The following table displays the energy collected by the outputs and the value of the criteria J^c. One should verify that the optimal solution (*i.e.* x_2 and x_5) corresponds to the maximal energy.

Measurements	x_1, x_5	x_2, x_5	x_3, x_5	x_4, x_5
J^c	7.69	11.17	1.11	0.41
Output energy	322	380	261	253

Figure 1 displays the measurements when the sensors are optimally located : x_2 and x_5 are measured. Figure 2 displays the measurements when a non optimal solution is chosen : sensors positioned on x_3 and x_5. The comparison of the collected output energies obtained with the optimal placement (solid line) and a non optimal placement (dashed line) is shown on Figure 3.

VI. CONCLUSION

In this paper, the problem of optimal sensor and actuator location is addressed. Considering an energetical approach the sensors and actuators are located in order to maximize the energy provided to the system via the actuators and the output energy collected by the sensors. Two complementary quantitative methodologies of SAL have been proposed and illustrated. In the first one, state controllability and state observability are considered and optimized separately whereas the second one ensures that the outputs optimally collect the input energy since the balanced realization is used for optimization. The latter method can be extended to perform disturbance decoupling. Both methods are equivalent to integer programming problems, thus the computational cost can be significantly reduced by integer programming.

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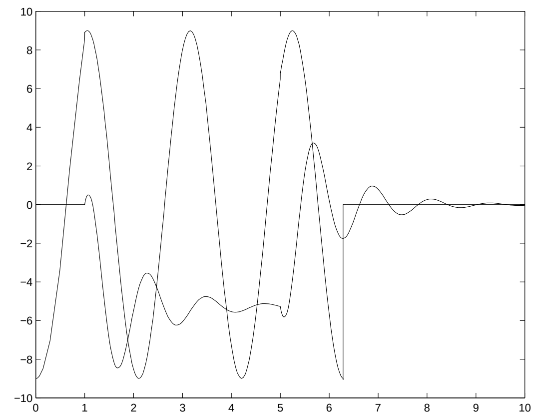

Fig. 1. Measurement of $x_2(t)$ and $x_5(t)$.

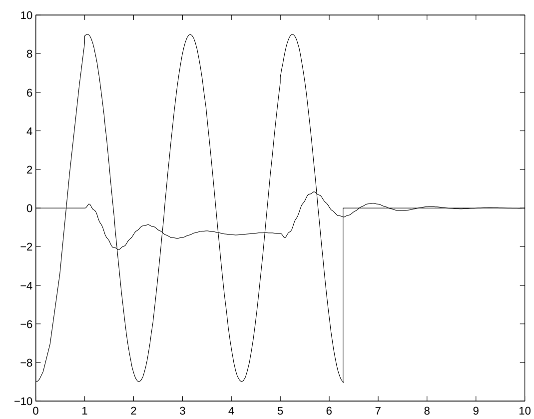

Fig. 2. Measurement of $x_3(t)$ and $x_5(t)$.

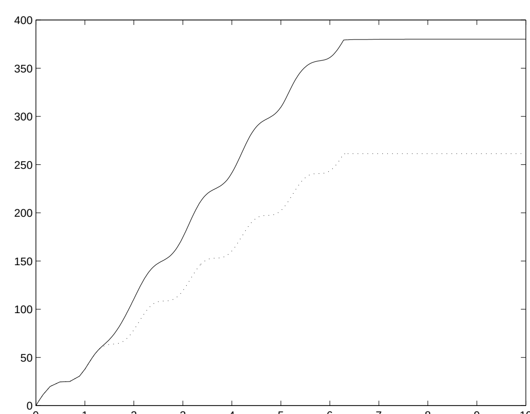

Fig. 3. Comparison of the collected output energy of the optimal placement (solid line) and non optimal placement (dashed line).