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Robust Observer Backstepping Neural Network Control of Nonlinear Systems in Strict Feedback Form

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***Abstract*—An observer for nonlinear systems in strict feedback form is presented. Together with a backstepping neural network controller, the resulting control strategy can be used in systems where only the output is measurable and unmatched additive disturbances with unknown bounds exist. Some assumptions about the system are required in order to apply the separation principle. A stability proof is provided together with simulation results.**

I. Introduction

MOST mathematical functions representing physical systems are nonlinear. Researchers have come up with effective ways to design controllers for some nonlinear systems that can be transformed to specific nonlinear forms. One of the most popular forms is the strict feedback form, which represents many physical systems, e.g., flexible-joint manipulator [1], continuously stirred tank reactor [2], and two-link planar robot [3]. Recently, researchers have begun to use intelligent methods, e.g., neural networks or fuzzy logic, to replace the nonlinear functions representing the systems. Due to their property as universal approximators, they can be used to approximate any continuous nonlinear functions. With these approximators, controller synthesis is less dependent on system modeling.

Practically, it is almost mandatory to involve uncertainties in the controller design process and to handle situations where all required states are not measurable due to unavailable sensor technology or cost.

In this paper, three-layer neural networks (NN) are used as function approximators. All NN weights are tuned online. The separation principle is used to separately design the observer and controller, with key assumptions making this possible. A nonlinear observer is designed, using a NN structure. Additive disturbances with unknown bounds exist in the system and violate the matching condition. These disturbances may depend on the states or vary with time. The controller is designed based on knowledge of estimated states. Backstepping and Lyapunov-based design methods are used in the controller design process.

Section II presents preliminaries. Section III describes the observer design. Section IV describes the controller design. Section V presents simulation results. Conclusions are given in Section VI.

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II. Preliminaries

A. System Description

Consider the system in strict feedback form:

$$\dot{x}_i = f_i(\bar{x}_i) + g_i(\bar{x}_i)(x_{i+1} + d_{ai}(\bar{x}_i, t)), \quad 1 \le i \le m-1,$$
$$\dot{x}_m = f_m(\bar{x}_m) + g_m(\bar{x}_m)(u + d_{am}(\bar{x}_m, t)), \qquad (2.1)$$
$$y = x_1,$$

where $x_i \in \mathbb{R}$, $\bar{x}_i = [x_1, \ldots, x_i]\ i = 1, \ldots, m$, $u, y \in \mathbb{R}$, $f_i(\bullet), g_i(\bullet)$, $i = 1, \ldots, m$ are unknown smooth functions, $d_{ai}(\bar{x}_i, t), i = 1, \ldots, m$ are additive unknown nonlinear disturbances with unknown bounds. Only output y is measurable.

B. Neural Network Basics

We use a 3-layer neural network as in Fig. 1. With its universal approximation property [4], this network can approximate any smooth function to arbitrary accuracy, given enough number of hidden-layer neurons. Each variable in the network can be defined as follows:

$$\bar{Z} = [z_1, z_2, \ldots, z_n, 1]^T \in \mathbb{R}^{n+1},$$
$$V = [v_1, v_2, \ldots, v_l] \in \mathbb{R}^{(n+1)\times l},$$
$$v_i = [v_{i1}, v_{i2}, \ldots, v_{i(n+1)}]^T \in \mathbb{R}^{n+1}, i = 1, 2, \ldots, l,$$
$$g(W, V, Z) = W^T S(V^T \bar{Z}) \in \mathbb{R},$$
$$S(V^T \bar{Z}) = [s(v_1^T \bar{Z}), s(v_2^T \bar{Z}), \ldots, s(v_l^T \bar{Z}), 1]^T \in \mathbb{R}^{l+1},$$
$$W = [w_1, w_2, \ldots, w_l, w_{l+1}]^T \in \mathbb{R}^{l+1}.$$

$s(\bullet)$ can be any appropriate activation function. In this paper we use a sigmoid function $s(z_i) = 1/(1 + e^{-z_i}), \forall z_i \in \mathbb{R}$.

Definition 1: Throughout this paper, we define $(\bullet^*)=(\hat{\bullet})-(\tilde{\bullet})$ where $(\bullet^*)$ is the actual value, $(\tilde{\bullet})$ is estimated error and $(\hat{\bullet})$ is the estimated value.

Assumption 1: Any smooth nonlinear function $h_i^*(\cdot)\in\mathbb{R}$ can be represented by a 3-layer neural network with some constant ideal weights W_i^*, V_i^* as $h_i^*(\bullet)=W_i^{*T}S_i(V_i^{*T}\bar{Z}_i)+\varepsilon_i$, where $\|\varepsilon_i\|<\varepsilon_{iU}$ is the approximation error with unknown $\varepsilon_{iU}>0$.

Assumption 2: On the compact set Ω_z, the ideal neural network weights W_i^*, V_i^* are constant and bounded by $\|W_i^*\|\le W_{iU}$, $\|V_i^*\|_F\le V_{iU}$, $i=1,\ldots,m$, where W_{iU} and V_{iU} are not known.

Assumption 3: Additive disturbances $d_{ai}(\bar{x}_i,t)$ are bounded by $\|d_{ai}(\bar{x}_i,t)\|<d_{aiU}$, $i=1,\ldots,m$, where d_{aiU} are unknown.

Lemma 1: The NN approximation error can be put in a linearly parameterized form in terms of $\tilde{W}$ and $\tilde{V}$ as

$$\hat{W}^T S(\hat{V}^T\bar{Z})-W^{*T}S(V^{*T}\bar{Z}) = \tilde{W}^T(\hat{S}-\hat{S}'\hat{V}^T\bar{Z})+\hat{W}^T\hat{S}'\tilde{V}^T\bar{Z}+d_u$$

where

$$\hat{S}=S(\hat{V}^T\bar{Z}),\ \hat{S}'=diag\left\{\hat{s}_1',\hat{s}_2',\ldots,\hat{s}_l'\right\},$$

$$\hat{s}_i'=s'(\hat{v}_i^T\bar{Z})=\left.\frac{d[s(z_a)]}{dz_a}\right|_{z_a=\hat{v}_i^T\bar{Z}},\ i=1,2,\ldots,l.$$

$s(z_i)=1/(1+e^{-z_i}),\ \forall z_i\in\mathbb{R}$ and can be any suitable activation function. The residual term d_u is bounded by

$$|d_u|\le\|V^*\|_F\|\bar{Z}\hat{W}^T\hat{S}'\|_F+\|W^*\|\|\hat{S}'\hat{V}^T\bar{Z}\|+|W^*|_1.$$

Proof: See [5].

Replacing the unknown functions in (2.1) with neural network estimates, we have

$$\begin{aligned}\dot{\xi}_i &= \hat{f}_i(\bar{\xi}_i)+\hat{g}_i(\bar{\xi}_i)(\xi_{i+1}+d_{ai}(\bar{\xi}_m,t)),\quad 1\le i\le m-1,\\ \dot{\xi}_m &= \hat{f}_m(\bar{\xi}_m)+\hat{g}_m(\bar{\xi}_m)(u+d_{am}(\bar{\xi}_m,t)),\\ \zeta &= \xi_1,\end{aligned}\tag{2.2}$$

where $\hat{f}_i=\hat{W}_{fi}^T S_{fi}(\hat{V}_{fi}^T\bar{Z}_{fi}),\ \hat{g}_i=\hat{W}_{gi}^T S_{gi}(\hat{V}_{gi}^T\bar{Z}_{gi})$, and ξ_i,ζ are states and output of the estimated system, respectively. The partial derivative of $\hat{f}_i$ (or $\hat{g}_i$), with respect to one of its input states, is

$$\frac{\partial\hat{f}_i}{\partial\xi_j}=\hat{W}_{fi}^T\hat{S}_{fi}'v_{jf}\tag{2.3}$$

where $v_j=\left[v_{j1},v_{j2},\ldots,v_{jl}\right]\in\mathbb{R}^l$, $j=1,2,\ldots,n+1$.

III. Observer Design

In this section, we present the design of a nonlinear observer for actual system in (2.1) using estimated system in (2.2). From (2.1), we have the sequence of output derivatives as

$$\begin{aligned}y_e &\triangleq\begin{bmatrix}y_{e_1} & y_{e_2} & \ldots & y_{e_m}\end{bmatrix}^T\triangleq\begin{bmatrix}y & \dot{y} & \ldots & y^{(m-1)}\end{bmatrix}^T,\\ &=H(\bar{x}_m)\triangleq\begin{bmatrix}x_1 & \varphi_1(\bar{x}_2) & \ldots & \varphi_{m-1}(\bar{x}_m)\end{bmatrix}^T,\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}\varphi_1(\bar{x}_2) &= f_1(x_1)+g_1(x_1)(x_2+d_{a1}),\\ \varphi_2(\bar{x}_3) &= \frac{\partial\varphi_1}{\partial x_1}[f_1(x_1)+g_1(x_1)(x_2+d_{a1})]\\ &\quad+\frac{\partial\varphi_1}{\partial x_2}[f_2(\bar{x}_2)+g_2(\bar{x}_2)(x_3+d_{a2})],\\ &\ \vdots\\ \varphi_{m-1}(\bar{x}_m) &= \sum_{j=1}^{m-1}\frac{\partial\varphi_{m-2}}{\partial x_j}\left[f_j(\bar{x}_j)+g_j(\bar{x}_j)(x_{j+1}+d_{aj})\right].\end{aligned}\tag{3.2}$$

The m-derivative of the output is

$$\begin{aligned}y^{(m)} &= \sum_{j=1}^{m-1}\frac{\partial\varphi_{m-1}}{\partial x_j}\left[f_j(\bar{x}_j)+g_j(\bar{x}_j)(x_{j+1}+d_{aj})\right]\\ &\quad+\frac{\partial\varphi_{m-1}}{\partial x_m}[f_m(\bar{x}_m)+g_m(\bar{x}_m)(u+d_{am})],\\ &=\alpha(\bar{x}_m)+\beta(\bar{x}_m)u,\\ &=\alpha(y_e)+\beta(y_e)u.\end{aligned}\tag{3.3}$$

From (3.1) and (3.3), we have

$$\dot{y}_e=Ay_e+B\left[\alpha(y_e)+\beta(y_e)u\right]\tag{3.4}$$

where

$$A=\begin{bmatrix}0 & & \\ \vdots & I & \\ 0 & \cdots & 0\end{bmatrix},\quad B=\begin{bmatrix}0 & \ldots & 0 & 1\end{bmatrix}^T.$$

Similarly we obtain (3.1) and (3.4) for the estimated system (2.2) as follows:

$$\begin{aligned}\zeta_e &\triangleq\begin{bmatrix}\zeta_{e_1} & \zeta_{e_2} & \ldots & \zeta_{e_m}\end{bmatrix}^T=\begin{bmatrix}\zeta & \dot{\zeta} & \ldots & \zeta^{(m-1)}\end{bmatrix}^T,\\ &\triangleq\hat{H}(\bar{\xi}_m)=\begin{bmatrix}\xi_1 & \psi_1(\bar{\xi}_2) & \ldots & \psi_{m-1}(\bar{\xi}_m)\end{bmatrix}^T,\end{aligned}$$

$$\dot{\zeta}_e=A\zeta_e+B\left[\hat{\alpha}(\zeta_e)+\hat{\beta}(\zeta_e)u\right].$$

Replacing ξ_i with its estimated state, $\hat{x}_i$, we have

$$\begin{aligned}\hat{\zeta}_e &\triangleq\begin{bmatrix}\hat{\zeta}_{e_1} & \hat{\zeta}_{e_2} & \ldots & \hat{\zeta}_{e_m}\end{bmatrix}^T,\\ &\triangleq\hat{H}(\bar{\hat{x}}_m)=\begin{bmatrix}\hat{x}_1 & \psi_1(\bar{\hat{x}}_2) & \ldots & \psi_{m-1}(\bar{\hat{x}}_m)\end{bmatrix}^T,\end{aligned}$$

$$\dot{\hat{\zeta}}_e=A\hat{\zeta}_e+B\left[\hat{\alpha}(\hat{\zeta}_e)+\hat{\beta}(\hat{\zeta}_e)u\right].$$

Assumption 4: System (2.1) and estimated system (2.2) are uniformly completely observable, i.e., the mappings $H(\bar{x}_m)$ and $\hat{H}\left(\bar{\xi}_m\right)$ are invertible with respect to $\bar{x}_m$ and $\bar{\xi}_m$ and their inverses, $\bar{x}_m=H^{-1}(y_e)$ and $\bar{\xi}_m=\hat{H}^{-1}\left(\zeta_e\right)$, are smooth.

Assumption 5: The differences between functions f_i, g_i in (2.1) and their estimates $\hat{f}_i, \hat{g}_i$ in (2.2) are bounded.

Consider an observer for the estimated system (2.2) as

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \\ \vdots \\ \dot{\hat{x}}_m \end{bmatrix} = \begin{bmatrix} \hat{f}_1(\hat{x}_1) + \hat{g}_1(\hat{x}_1)\hat{x}_2 \\ \hat{f}_2(\hat{\bar{x}}_2) + \hat{g}_2(\hat{\bar{x}}_2)\hat{x}_3 \\ \vdots \\ \hat{f}_m(\hat{\bar{x}}_m) + \hat{g}_m(\hat{\bar{x}}_m)u \end{bmatrix} + \left[\frac{\partial H(\hat{\bar{x}}_m)}{\partial \hat{\bar{x}}_m}\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right], \quad (3.5)$$

$$\hat{\zeta} = \hat{x}_1,$$

where $\varepsilon = diag\left[\eta, \eta^2, \ldots, \eta^m\right], 0 < \eta \le 1$, and $\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]$ is jacobian. η is a design parameter and $L = [l_1, l_2, \ldots, l_m]^T$ is such that $s^m + l_1 s^{m-1} + \ldots + l_m$ is a Hurwitz polynomial.

To show stability properties of the observer estimation error dynamics, $\tilde{x} = \hat{x} - x$, we proceed as follows.
For $i = 1$,

$$\dot{\hat{\zeta}}_{e_1} = \hat{f}_1(\hat{x}_1) + \hat{g}_1(\hat{x}_1)\hat{x}_2 + I_1\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right],$$
$$= \hat{\zeta}_{e_2} + I_1\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right].$$

For $2 \le i \le m-1$,

$$\dot{\hat{\zeta}}_{e_i} = \sum_{j=1}^{i}\Big\{\left(\partial \psi_{i-1}/\partial \hat{x}_j\right)\left(\hat{f}_j(\hat{\bar{x}}_j) + \hat{g}_j(\hat{\bar{x}}_j)\hat{x}_{j+1}\right.$$
$$\left. + I_j\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right]\right)\Big\},$$
$$= \hat{\zeta}_{e_{i+1}} + \sum_{j=1}^{i}\left(\partial \psi_{i-1}/\partial \hat{x}_j\right)\left[I_j\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right]\right].$$

For $i = m$,

$$\dot{\hat{\zeta}}_{e_m} = \hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u$$
$$+ \sum_{j=1}^{m} \partial \psi_{m-1}/\partial \hat{x}_j\left[I_j\left[\partial \hat{H}(\hat{\bar{x}}_m)/\partial \hat{\bar{x}}_m\right]^{-1} \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right]\right],$$

where $I_j \in \mathbb{R}^{1\times m}$ is a row vector whose jth element is 1 and 0 otherwise. We then can write the above in matrix form as

$$\dot{\hat{\zeta}}_e = A\hat{\zeta}_e + B\left[\hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u\right] + \varepsilon^{-1} L\left[\zeta - \hat{\zeta}\right] \quad (3.6)$$

Define the observer error, $\tilde{\zeta}_e = \hat{\zeta}_e - y_e$. Let $C \in \mathbb{R}^{1\times m} = [1, 0, \ldots, 0]$, then the observer error dynamics are given by

$$\dot{\tilde{\zeta}}_e = (A - \varepsilon^{-1} LC)\tilde{\zeta}_e + B[\hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u - \alpha(y_e) - \beta(y_e)u].$$

Define another coordinate transformation:

$$\tilde{v} = \varepsilon^{'}\tilde{\zeta}_e, \ \ \varepsilon^{'} \triangleq diag\left[\frac{1}{\eta^{m-1}}, \frac{1}{\eta^{m-2}}, \ldots, 1\right].$$

In the new coordinates, the error dynamics become

$$\dot{\tilde{v}} = \frac{1}{\eta}(A - LC)\tilde{v} + B[\hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u - \alpha(y_e) - \beta(y_e)u].$$

By proper choice of $L, A - LC$ is Hurwitz. Let P be the solution to the Lyapunov equation

$$P(A - LC) + (A - LC)^T P = -I$$

and consider the Lyapunov candidate $V = \tilde{v}^T P \tilde{v} > 0$. Its time derivative along the $\tilde{v}$ trajectories is
$\dot{V} = -\tilde{v}^T\tilde{v}/\eta + 2\tilde{v}^T PB[\hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u - \alpha(y_e) - \beta(y_e)u].$

Assumption 6: α, β are globally Lipschitz functions and input u is bounded.

With the above assumption, we have

$$\left|\hat{\alpha}(\hat{\zeta}_e) + \hat{\beta}(\hat{\zeta}_e)u - \alpha(y_e) - \beta(y_e)u\right| \le k\left|\tilde{\zeta}_e\right|, \quad k > 0. \quad (3.7)$$

Thus, we have

$$\dot{V} \le -\frac{\|\tilde{v}\|^2}{\eta} + 2k\|P\|\|\tilde{v}\|^2. \quad (3.8)$$

Therefore, if we choose $\eta < \bar{\eta}, \ \ \bar{\eta} = \min\{1/2k\|P\|, 1\}$, from the Lyapunov theorem, we can conclude that $\tilde{v}$ and hence $\tilde{\zeta}_e$ will converge to zero asymptotically. Therefore, from Assumption 4, $\tilde{x}$ will converge to zero asymptotically.

To prove that $\tilde{\zeta}_e$ is bounded at all time, we proceed as follows.

From $\lambda_{\min}(P)\|\tilde{\zeta}_e\|^2 \le V(t) \le 1/(\eta^{2(m-1)})\lambda_{\max}(P)\|\tilde{\zeta}_e\|^2$, we have

$$\dot{V}(t) \le -\left(1/\eta - 2k\|P\|\right)V(t)/\lambda_{\max}(P),$$
$$V(t) \le 1/(\eta^{2(m-1)})\lambda_{\max}(P)\|\tilde{\zeta}_e(0)\|^2$$
$$\exp\left\{-\left(1/\eta - 2k\|P\|\right)t/\lambda_{\max}(P)\right\},$$
$$\|\tilde{\zeta}_e\| \le 1/(\eta^{(m-1)})\sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}\|\tilde{\zeta}_e(0)\|,$$
$$\exp\left\{-\left(1/\eta - 2k\|P\|\right)t/\left(2\lambda_{\max}(P)\right)\right\}.$$

IV. Controller Design

Assumption 7: g_i has known signs. There exist known constants $g_{iU} > 0$ such that $\|g_i(\cdot)\| \le g_{iU} \ \ \forall i = 1, \ldots, m-1$.

The control objective is to make the output y follow the desired trajectory y_d as closely as possible, while all the signals in the closed-loop system are bounded. We proceed using the following steps.

Step 1:

Let $z_1 = \hat{x}_1 - x_{1d} = x_1 + \tilde{x}_1 - x_{1d}$ be the error between the estimate of x_1 and the desired output in the first subsystem in the backstepping design. Let x_{2d} be a virtual control input and introduce the error variable $z_2 = \hat{x}_2 - x_{2d}$. Choose the virtual control as

$$x_{2d} = -\hat{g}_1^{-1}[c_1 z_1 + \hat{f}_1 + \dot{\tilde{x}}_1 - \dot{x}_{1d} - u_{2dvsc}],$$
$$= -\left[\hat{W}_{g1}^T S_{g1}(\hat{V}_{g1}^T \bar{Z}_{g1})\right]^{-1}[c_1 z_1 + \hat{W}_{f1}^T S_{f1}(\hat{V}_{f1}^T \bar{Z}_{f1}) \quad (4.1)$$
$$- \dot{x}_{1d} - u_{2dvsc}].$$

From Assumptions 1, 3 and Lemma 1, we have
$\left|d_{uf1}\right| + \left|\varepsilon_{f1}\right| + \left|d_{ug1}x_{2d}\right| + \left|\varepsilon_{g1}x_{2d}\right| + \left|g_1 d_{a1}\right| \le K_1^{*T}\varphi_1$
where

$$K_1^* = [\|V_{f1}^*\|_F,\ \|W_{f1}^*\|,\ \|W_{f1}^*\|_1 + \varepsilon_{f1U} + g_{1U}d_{a1U},\ \|V_{g1}^*\|_F,$$
$$\|W_{g1}^*\|,\ \|W_{g1}^*\|_1 + \varepsilon_{g1U}]^T,$$
$$\varphi_1 = [\|\bar{Z}_{f1}\hat{W}_{f1}^T\hat{S}'_{f1}\|_F,\ \|\hat{S}'_{f1}\hat{V}_{f1}^T\bar{Z}_{f1}\|,\ 1,\ \|\bar{Z}_{g1}\hat{W}_{g1}^T\hat{S}'_{g1}x_{2d}\|_F,$$
$$\|\hat{S}'_{g1}\hat{V}_{g1}^T\bar{Z}_{g1}x_{2d}\|,\ \|x_{2d}\|]^T.$$

We let the variable structure control term be in the form

$$u_{(i+1)dvsc} = -\hat{K}_i^T\bar{\varphi}_i,$$

$$\bar{\varphi}_i = \begin{bmatrix} \|\bar{Z}_{fi}\hat{W}_{fi}^T\hat{S}'_{fi}\|_F \frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\|\bar{Z}_{fi}\hat{W}_{fi}^T\hat{S}'_{fi}\|_F\right) \\ \|\hat{S}'_{fi}\hat{V}_{fi}^T\bar{Z}_{fi}\|\frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\|\hat{S}'_{fi}\hat{V}_{fi}^T\bar{Z}_{fi}\|\right) \\ \frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\right) \\ \|\bar{Z}_{gi}\hat{W}_{gi}^T\hat{S}'_{gi}x_{(i+1)d}\|_F \frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\|\bar{Z}_{gi}\hat{W}_{gi}^T\hat{S}'_{gi}x_{(i+1)d}\|_F\right) \\ \|\hat{S}'_{gi}\hat{V}_{gi}^T\bar{Z}_{gi}x_{(i+1)d}\|\frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\|\hat{S}'_{gi}\hat{V}_{gi}^T\bar{Z}_{gi}x_{(i+1)d}\|\right) \\ \|x_{(i+1)d}\|\frac{2}{\pi}\arctan\left(\frac{z_i}{\mu_i}\|x_{(i+1)d}\|\right) \end{bmatrix}. \quad (4.2)$$

where μ_i is a small positive number. $\hat{K}_i$ approximates K_i^*.

The $\dot{z}_1$ equation becomes

$$\dot{z}_1 = \dot{\hat{x}}_1 - \dot{x}_{1d},$$
$$= \dot{x}_1 - \dot{x}_{1d} + \dot{\tilde{x}}_1,$$
$$= \left[\varepsilon_{f1} - \tilde{W}_{f1}^T(\hat{S}_{f1} - \hat{S}'_{f1}\hat{V}_{f1}^T\bar{Z}_{f1}) - \hat{W}_{f1}^T\hat{S}'_{f1}\tilde{V}_{f1}^T\bar{Z}_{f1} - d_{uf1}\right]$$
$$- c_1z_1 - \hat{K}_1^T\bar{\varphi}_1 + g_1d_{a1} + [\varepsilon_{g1} - \tilde{W}_{g1}^T(\hat{S}_{g1} - \hat{S}'_{g1}\hat{V}_{g1}^T\bar{Z}_{g1})$$
$$- \hat{W}_{g1}^T\hat{S}'_{g1}\tilde{V}_{g1}^T\bar{Z}_{g1} - d_{ug1}]x_{2d} + g_1(x_2 - x_{2d}).$$

Choose the Lyapunov function as

$$V_1 = \frac{1}{2}z_1^2 + \frac{1}{2}\tilde{W}_{f1}^T\Gamma_{wf1}^{-1}\tilde{W}_{f1} + \frac{1}{2}tr\left\{\tilde{V}_{f1}^T\Gamma_{vf1}^{-1}\tilde{V}_{f1}\right\}$$
$$+ \frac{1}{2}\tilde{W}_{g1}^T\Gamma_{wg1}^{-1}\tilde{W}_{g1} + \frac{1}{2}tr\left\{\tilde{V}_{g1}^T\Gamma_{vg1}^{-1}\tilde{V}_{g1}\right\} + \frac{1}{2}\tilde{K}_1^T\Gamma_{k1}^{-1}\tilde{K}_1, \quad (4.3)$$

where $\Gamma_{wf1}, \Gamma_{vf1}, \Gamma_{wg1}, \Gamma_{vg1}, \Gamma_{k1} > 0$. We use the following σ-modification weight updating laws:

$$\dot{\hat{W}}_{fi} = \dot{\tilde{W}}_{fi} = \Gamma_{wfi}[(\hat{S}_{fi} - \hat{S}'_{fi}\hat{V}_{fi}^T\bar{Z}_{fi})z_i - \sigma_{wfi}\hat{W}_{fi}], \quad (4.4)$$

$$\dot{\hat{V}}_{fi} = \dot{\tilde{V}}_{fi} = \Gamma_{vfi}[\bar{Z}_{fi}\hat{W}_{fi}^T\hat{S}'_{fi}z_i - \sigma_{vfi}\hat{V}_{fi}], \quad (4.5)$$

$$\dot{\hat{W}}_{gi} = \dot{\tilde{W}}_{gi} = \Gamma_{wgi}[(\hat{S}_{gi} - \hat{S}'_{gi}\hat{V}_{gi}^T\bar{Z}_{gi})x_{(i+1)d}z_i - \sigma_{wgi}\hat{W}_{gi}], \quad (4.6)$$

$$\dot{\hat{V}}_{gi} = \dot{\tilde{V}}_{gi} = \Gamma_{vgi}[\bar{Z}_{gi}\hat{W}_{gi}^T\hat{S}'_{gi}x_{(i+1)d}z_i - \sigma_{vgi}\hat{V}_{gi}], \quad (4.7)$$

$$\dot{\hat{K}}_i = \dot{\tilde{K}}_i = \Gamma_{ki}[\bar{\varphi}_iz_i - \sigma_{ki}\hat{K}_i]. \quad (4.8)$$

The σ terms in the update laws are used to prevent $\hat{W}_i, \hat{V}_i, \hat{K}_i$ from growing unboundedly by maintaining $\hat{W}_i, \hat{V}_i, \hat{K}_i$ values around their initial values. Also, we use the fact that

$$2\tilde{W}^T\hat{W} = \|\tilde{W}\|^2 + \|\hat{W}\|^2 - \|W^*\|^2 \geq \|\tilde{W}\|^2 - \|W^*\|^2,$$
$$2tr\left\{\tilde{V}^T\hat{V}\right\} = \|\tilde{V}\|_F^2 + \|\hat{V}\|_F^2 - \|V^*\|_F^2 \geq \|\tilde{V}\|_F^2 - \|V^*\|_F^2.$$

We can find $\dot{V}_1$ as follows:

$$\dot{V}_1 = z_1\varepsilon_{f1} - z_1d_{uf1} - c_1z_1^2 - \hat{K}_1^T\bar{\varphi}_1z_1 + z_1g_1d_{a1} + z_1\varepsilon_{g1}x_{2d}$$
$$- z_1d_{ug1}x_{2d} + z_1g_1z_2 + \tilde{K}_1^T\bar{\varphi}_1z_1 - \tilde{W}_{f1}^T\sigma_{wf1}\hat{W}_{f1} - tr\left\{\tilde{V}_{f1}^T\sigma_{vf1}\hat{V}_{f1}\right\}$$
$$- \tilde{W}_{g1}^T\sigma_{wg1}\hat{W}_{g1} - tr\left\{\tilde{V}_{g1}^T\sigma_{vg1}\hat{V}_{g1}\right\} - \tilde{K}_1^T\sigma_{k1}\hat{K}_1,$$
$$\leq -c_1z_1^2 + K_1^{*T}\varphi_1|z_1| - \hat{K}_1^T\bar{\varphi}_1z_1 + z_1g_1z_2 + \tilde{K}_1^T\bar{\varphi}_1z_1 - \tilde{W}_{f1}^T\sigma_{wf1}\hat{W}_{f1}$$
$$- tr\left\{\tilde{V}_{f1}^T\sigma_{vf1}\hat{V}_{f1}\right\} - \tilde{W}_{g1}^T\sigma_{wg1}\hat{W}_{g1} - tr\left\{\tilde{V}_{g1}^T\sigma_{vg1}\hat{V}_{g1}\right\} - \tilde{K}_1^T\sigma_{k1}\hat{K}_1.$$

From [6], we have the property

$$0 \leq |\alpha| - \alpha\frac{2}{\pi}\arctan\left(\frac{\alpha}{\eta}\right) \leq 0.2785\eta, \quad \forall\alpha \in \mathbb{R}.$$

Therefore, we have

$$\dot{V}_1 \leq -c_1z_1^2 + z_1g_{1U}z_2 - \frac{\sigma_{wf1}}{2}\|\tilde{W}_{f1}\|^2 - \frac{\sigma_{vf1}}{2}\|\tilde{V}_{f1}\|_F^2 - \frac{\sigma_{wg1}}{2}\|\tilde{W}_{g1}\|^2$$
$$- \frac{\sigma_{vg1}}{2}\|\tilde{V}_{g1}\|_F^2 - \frac{\sigma_{k1}}{2}\|\tilde{K}_1\|^2 + \xi_1,$$

where

$$\xi_1 = 0.2785\eta_1(\|V_{f1}^*\|_F + \|W_{f1}^*\| + \|W_{f1}^*\|_1 + \varepsilon_{f1U} + g_{1U}d_{a1U}$$
$$+ \|V_{g1}^*\|_F + \|W_{g1}^*\| + \|W_{g1}^*\|_1 + \varepsilon_{g1U}) + \frac{\sigma_{wf1}}{2}\|W_{f1}^*\|^2$$
$$+ \frac{\sigma_{vf1}}{2}\|V_{f1}^*\|_F^2 + \frac{\sigma_{wg1}}{2}\|W_{g1}^*\|^2 + \frac{\sigma_{vg1}}{2}\|V_{g1}^*\|_F^2 + \frac{\sigma_{k1}}{2}\|K_1^*\|^2.$$

The term $z_1g_{1U}z_2$ will be cancelled in the next step.

Step i: $(2 \leq i \leq m-1)$

Let $z_i = \hat{x}_i - x_{id}$ be the error of this subsystem. Let $x_{(i+1)d}$ be a virtual control input. We introduce the error variable $z_{i+1} = \hat{x}_{i+1} - x_{(i+1)d}$ and choose the virtual control as

$$x_{(i+1)d} = -\hat{g}_i^{-1}[g_{(i-1)U}z_{i-1} + c_iz_i + \hat{f}_i + \dot{\tilde{x}}_i - \dot{x}_{id} - u_{(i+1)dvsc}],$$
$$= -\left[\hat{W}_{gi}^TS_{gi}(\hat{V}_{gi}^T\bar{Z}_{gi})\right]^{-1}[g_{(i-1)U}z_{i-1} + c_iz_i \quad (4.9)$$
$$+ \hat{W}_{fi}^TS_{fi}(\hat{V}_{fi}^T\bar{Z}_{fi}) - \dot{x}_{id} - u_{(i+1)dvsc}],$$

where $u_{(i+1)dvsc}$ is as (4.2). The extra term $g_{(i-1)U}z_{i-1}$ is used to cancel the effect of the term $z_{i-1}g_{i-1}z_i$ from the previous step.

Similar to the derivation of step 1, the $\dot{z}_i$ equation becomes

$$\dot{z}_i = \left[\varepsilon_{fi} - \tilde{W}_{fi}^T(\hat{S}_{fi} - \hat{S}'_{fi}\hat{V}_{fi}^T\bar{Z}_{fi}) - \hat{W}_{fi}^T\hat{S}'_{fi}\tilde{V}_{fi}^T\bar{Z}_{fi} - d_{ufi}\right]$$
$$- g_{(i-1)U}z_{i-1} - c_iz_i - \hat{K}_i^T\bar{\varphi}_i + g_id_{ai}$$
$$+ \left[\varepsilon_{gi} - \tilde{W}_{gi}^T(\hat{S}_{gi} - \hat{S}'_{gi}\hat{V}_{gi}^T\bar{Z}_{gi}) - \hat{W}_{gi}^T\hat{S}'_{gi}\tilde{V}_{gi}^T\bar{Z}_{gi} - d_{ugi}\right]x_{(i+1)d}$$
$$+ g_i(x_{i+1} - x_{(i+1)d}).$$

Choose the Lyapunov function

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 + \frac{1}{2}\tilde{W}_{fi}^T\Gamma_{wfi}^{-1}\tilde{W}_{fi} + \frac{1}{2}tr\left\{\tilde{V}_{fi}^T\Gamma_{vfi}^{-1}\tilde{V}_{fi}\right\} + \frac{1}{2}\tilde{W}_{gi}^T\Gamma_{wgi}^{-1}\tilde{W}_{gi} + \frac{1}{2}tr\left\{\tilde{V}_{gi}^T\Gamma_{vgi}^{-1}\tilde{V}_{gi}\right\} + \frac{1}{2}\tilde{K}_i^T\Gamma_{ki}^{-1}\tilde{K}_i,$$

where $\Gamma_{wfi}, \Gamma_{vfi}, \Gamma_{wgi}, \Gamma_{vgi}, \Gamma_{ki} > 0$ and use the weight update laws (4.4) to (4.8). The derivative $\dot{V}_i$ can be found as in the previous step to be

$$\dot{V}_i \le \sum_{k=1}^{i}\left\{-c_k z_k^2 - \frac{\sigma_{wfk}}{2}\left\|\tilde{W}_{fk}\right\|^2 - \frac{\sigma_{vfk}}{2}\left\|\tilde{V}_{fk}\right\|_F^2 - \frac{\sigma_{wgk}}{2}\left\|\tilde{W}_{gk}\right\|^2 - \frac{\sigma_{vgk}}{2}\left\|\tilde{V}_{gk}\right\|_F^2 - \frac{\sigma_{kk}}{2}\left\|\tilde{K}_k\right\|^2 + \xi_k\right\} + z_i g_{iU} z_{i+1}.$$

Step m:

This is the last step. Let $z_m = \hat{x}_m - x_{md}$ be the error of the last subsystem. We select the control input as

$$\begin{aligned} u &= -\hat{g}_m^{-1}[g_{(m-1)U} z_{m-1} + c_m z_m + \hat{f}_m + \dot{\tilde{x}}_m - \dot{x}_{md} - u_{vsc}], \\ &= -\left[\hat{W}_{gm}^T S_{gm}(\hat{V}_{gm}^T \bar{Z}_{gm})\right]^{-1}[g_{(m-1)U} z_{m-1} + c_m z_m \\ &\quad + \hat{W}_{fm}^T S_{fm}(\hat{V}_{fm}^T \bar{Z}_{fm}) - \dot{x}_{md} - u_{(m+1)dvsc}], \end{aligned} \tag{4.10}$$

where $u_{(m+1)dvsc}$ is similar to (4.2) but replacing $x_{(i+1)d}$ with u.

The $\dot{z}_m$ equation becomes

$$\begin{aligned} \dot{z}_m &= \left[\varepsilon_{fm} - \tilde{W}_{fm}^T(\hat{S}_{fm} - \hat{S}'_{fm}\hat{V}_{fm}^T\bar{Z}_{fm}) - \hat{W}_{fm}^T\hat{S}'_{fm}\tilde{V}_{fm}^T\bar{Z}_{fm} - d_{ufm}\right] \\ &\quad - g_{(m-1)U} z_{m-1} - c_m z_m - \hat{K}_m^T\bar{\varphi}_m + g_m d_{am} \\ &\quad + \left[\varepsilon_{gm} - \tilde{W}_{gm}^T(\hat{S}_{gm} - \hat{S}'_{gm}\hat{V}_{gm}^T\bar{Z}_{gm}) - \hat{W}_{gm}^T\hat{S}'_{gm}\tilde{V}_{gm}^T\bar{Z}_{gm} - d_{ugm}\right]u. \end{aligned}$$

Choose the Lyapunov function

$$\begin{aligned} V_m = \sum_{k=1}^{m}\Big\{ & \frac{1}{2}z_k^2 + \frac{1}{2}\tilde{W}_{fk}^T\Gamma_{wfk}^{-1}\tilde{W}_{fk} + \frac{1}{2}tr\left\{\tilde{V}_{fk}^T\Gamma_{vfk}^{-1}\tilde{V}_{fk}\right\} \\ & + \frac{1}{2}\tilde{W}_{gk}^T\Gamma_{wgk}^{-1}\tilde{W}_{gk} + \frac{1}{2}tr\left\{\tilde{V}_{gk}^T\Gamma_{vgk}^{-1}\tilde{V}_{gk}\right\} + \frac{1}{2}\tilde{K}_k^T\Gamma_{kk}^{-1}\tilde{K}_k\Big\}, \end{aligned} \tag{4.11}$$

where $\Gamma_{wfm}, \Gamma_{vfm}, \Gamma_{wgm}, \Gamma_{vgm}, \Gamma_{km} > 0$ and use the weight update laws (4.4) to (4.8). By following the derivation as in the previous step, the derivative of the Lyapunov function of the whole system is given by

$$\dot{V}_m \le \sum_{k=1}^{m}\left\{-c_k z_k^2 - \frac{\sigma_{wfk}}{2}\left\|\tilde{W}_{fk}\right\|^2 - \frac{\sigma_{vfk}}{2}\left\|\tilde{V}_{fk}\right\|_F^2 - \frac{\sigma_{wgk}}{2}\left\|\tilde{W}_{gk}\right\|^2 - \frac{\sigma_{vgk}}{2}\left\|\tilde{V}_{gk}\right\|_F^2 - \frac{\sigma_{kk}}{2}\left\|\tilde{K}_k\right\|^2 + \xi_k\right\}.$$

Let $\varsigma = \min_{1\le k\le m}\{2c_k\} > 0$, $\delta = \sum_{k=1}^{m}\xi_k \ge 0$ and choose

$$\sigma_{wfk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{wfk}^{-1}\right\},\ \sigma_{vfk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{vfk}^{-1}\right\},$$
$$\sigma_{wgk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{wgk}^{-1}\right\},\ \sigma_{vgk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{vgk}^{-1}\right\},$$
$$\sigma_{kk} \ge \varsigma\lambda_{\max}\left\{\Gamma_{kk}^{-1}\right\}, \qquad k = 1,\ldots,m,$$

using the Rayleigh-Ritz inequality, we have

$$\begin{aligned} \dot{V}_m &\le \sum_{k=1}^{m}\left\{-c_k z_k^2 - \frac{\sigma_{wfk}}{2}\left\|\tilde{W}_{fk}\right\|^2 - \frac{\sigma_{vfk}}{2}\left\|\tilde{V}_{fk}\right\|_F^2 - \frac{\sigma_{wgk}}{2}\left\|\tilde{W}_{gk}\right\|^2 - \frac{\sigma_{vgk}}{2}\left\|\tilde{V}_{gk}\right\|_F^2 - \frac{\sigma_{kk}}{2}\left\|\tilde{K}_k\right\|^2 + \xi_k\right\}, \\ &\le \sum_{k=1}^{m}\Big\{-\varsigma 0.5 z_k^2 - \varsigma 0.5\tilde{W}_{fk}^T\Gamma_{wfk}^{-1}\tilde{W}_{fk} - \varsigma 0.5 tr\left\{\tilde{V}_{fk}^T\Gamma_{vfk}^{-1}\tilde{V}_{fk}\right\} \\ &\quad - \varsigma 0.5\tilde{W}_{gk}^T\Gamma_{wgk}^{-1}\tilde{W}_{gk} - \varsigma 0.5 tr\left\{\tilde{V}_{gk}^T\Gamma_{vgk}^{-1}\tilde{V}_{gk}\right\} \\ &\quad - \varsigma 0.5\tilde{K}_k^T\Gamma_{kk}^{-1}\tilde{K}_k + \xi_k\Big\}, \\ &\le -\varsigma V_m + \delta. \end{aligned}$$

From the uniform ultimate boundedness theorem [7], we conclude that all errors $z_i, \tilde{W}_i, \tilde{V}_i, \tilde{K}_i,\ i = 1,\ldots,m$ are uniformly ultimately bounded and therefore all closed loop signals are uniformly ultimately bounded. Since $V_m(t,\bullet)$ is positive definite and $\dot{V}_m \le -\varsigma V_m + \delta$, where $\varsigma > 0$ and $\delta \ge 0$ are bounded constants, then from Lemma 2.1 in [8], we have

$$V_m(t,\bullet) \le \frac{\delta}{\varsigma} + \left(V_m(t_0,\bullet) - \frac{\delta}{\varsigma}\right)e^{-\varsigma t}, \quad \forall t \ge t_0.$$

Therefore, we have that as $t \to \infty$, $V_m(t,\bullet) \le \delta/\varsigma$ and since we have

$$V_m(t,\bullet) \le \frac{\delta}{\varsigma} + \left(V_m(t_0,\bullet) - \frac{\delta}{\varsigma}\right)e^{-\varsigma t}, \quad \forall t \ge t_0,$$

$$\frac{1}{2}\sum_{k=1}^{m} z_k^2 \le \frac{\delta}{\varsigma} + \left(V_m(t_0,\bullet)\right)e^{-\varsigma t},$$

$$|z_1| \le \sqrt{2\left(\frac{\delta}{\varsigma} + \left(V_m(t_0,\bullet)\right)e^{-\varsigma t}\right)},$$

then when $t \to \infty$, $|z_1| = |y - y_d| \le \sqrt{2\delta/\varsigma}$.

Remark 1: The $\dot{\tilde{x}}_i$ terms in (4.1), (4.9), and (4.10) are bounded and converge to zero asymptotically based on the proof in Section III. Their transient values are treated as NN estimation errors and are handled by variable structure control terms.

V. Simulation Results

The proposed observer-controller is applied to the system:

$$\begin{aligned} \dot{x}_1 &= 0.5x_1 + (1 + 0.1x_1^2)(x_2 - (2 + \sin x_1)), \\ \dot{x}_2 &= x_1 x_2 + (2 + \cos x_1)(u - 0.3(e^{x_1} + e^{-x_1})), \\ y &= x_1. \end{aligned} \tag{5.1}$$

The system has $-(2 + \sin x_1)$ and $-0.3(e^{x_1} + e^{-x_1})$ as additive disturbances and is similar to an example used in [5].It can be verified that this system satisfies all required assumptions of the proposed control scheme. Only output y is measurable and all nonlinear functions are unknown. The control objective is to guarantee that (i) all closed-loop signals remain bounded, (ii) the output y follows the desired trajectory generated by passing a square wave of amplitude 10, zero mean, and 20-s period into the filter $1/(s+2)^3$. The observer (3.5), the controller (4.10) with variable structure terms (4.2), and weight update laws (4.4)-(4.8) are used. Number of hidden-layer nodes is 10. The design parameters are given by

$\Gamma_{wfi} = \Gamma_{vfi} = \Gamma_{wgi} = \Gamma_{vgi} = 10,\ \Gamma_{ki} = 1,\ c_i = 15,\quad \forall i = 1,2$,
$\sigma_{wfi} = \sigma_{vfi} = \sigma_{wgi} = \sigma_{vgi} = \sigma_{ki} = 0.1,\quad \forall i = 1,2$,
$\eta = 0.1,\ L = [10\ \ 20]^T,\ \mu_i = 0.1$.

Sampling period is 1 ms. All initial values are set to 0.1. Simulation results are shown in Figures 2-3. Overall tracking performance is good, as shown in Fig. 1(a) and (b). This is the result of good observer performance, as shown in Fig. 1(c) and (d), and controller performance, as shown in Fig. 1(e) and (f). In Fig. 2, the estimated functions $\hat{f}_1, \hat{g}_1$ are different from the actual functions f_1, g_1 but the differences are bounded. The control input does not chatter since a smooth version of the variable structure control is used.

VI. Conclusion

We have presented a model-free output feedback control system design for a nonlinear system in strict feedback form. Three-layer neural networks are used as identifiers of the unknown plant functions. The observer objective is to estimate the actual states using actual plant output, plant input, and estimated plant functions from the identifier. The controller is used to reduce error between estimated states and their desired values. Simulation results show good overall tracking performance for an example system.

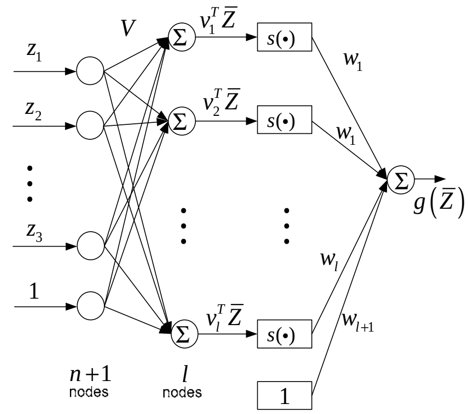

Fig. 1. A 3-layer neural network.

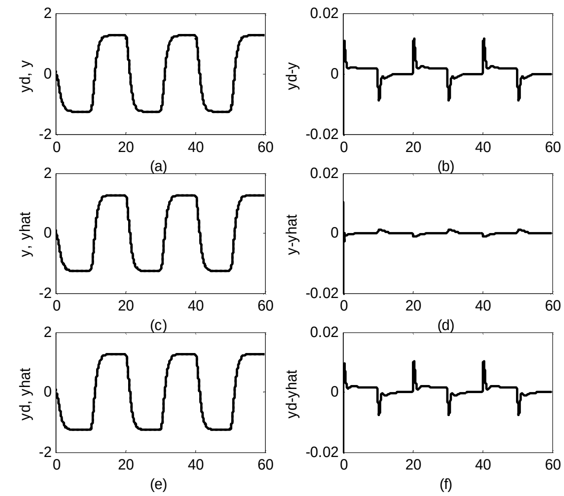

Fig. 2. Control system performance during 60 s: (a) actual output y versus desired output y_d, (b) overall tracking error $y_d - y$, (c) actual output y versus estimated output $\hat{y}$, (d) observer error, (e) desired output y_d versus estimated output $\hat{y}$, (f) controller error.

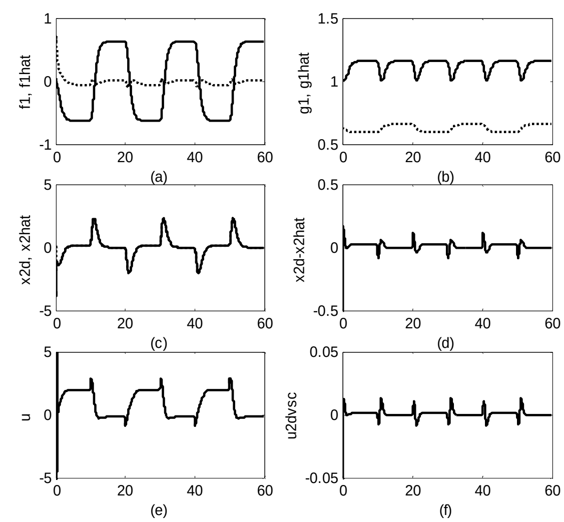

Fig. 3. Closed-loop signals during 60 s: (a) actual f_1 versus estimated $\hat{f}_1$, (b) actual g_1 versus estimated $\hat{g}_1$, (c) controller performance, desired state x_{2d} versus estimated state $\hat{x}_2$, (d) error $x_{2d} - \hat{x}_2$, (e) actual control input u, (f) variable structure control input u_{2dvsc}.

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