Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Parameter Governors For Discrete-Time Nonlinear Systems With Pointwise-in-Time State and Control Constraints

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***Abstract*— Parameter Governors are add-on control schemes that adjust parameters (such as gains or offsets) in the nominal control laws so that to avoid violation of pointwise-in-time state and control constraints and to improve the overall system performance. As compared to more general Model Predictive Controllers, parameter governors tend to be more conservative but the computational effort needed to implement them on-line can be relatively low. In this paper we study the properties of two particular classes of parameter governors, namely, the feedforward governors and the gain governors.**

I. Introduction

This paper is motivated by the observation that reference governor-like schemes [1], [2], [4], [5], [6] can adjust parameters in nominal control laws other than the true reference commands. In this way, generalized Parameter Governors can be constructed that enforce pointwise-in-time constraints and improve system performance with a relatively low on-line computational effort. Figure 1 illustrates the application of a parameter governor to a discrete-time nonlinear system which is controlled by a nominal feedback law dependent on a parameter:

$$x(t+1) = f(x(t), u(t), r(t), \theta(t)),\quad u(t) = u_c(x(t), r(t), \theta(t)). \tag{1}$$

Here x is the state, $r(t)$ is the reference command, $u(t)$ is the control input and $\theta(t)$ is a parameter in the control function u_c. The state vector $x(t)$ may include both plant states and controller states and depending on the form of the control law and parameter governor, f may explicitly depend on $r(t)$ and $\theta(t)$. As Figure 1 suggests the parameter governor is mainly intended to modify parameters, $\theta(t)$, of the closed-loop system and not the reference command.

The basic mechanism for the adjustment of $\theta(t)$ is similar to the reference governor: The $\theta(t)$ is selected so that with $\theta(t+k) \equiv \theta(t)$, the constraints are satisfied for $k \in \mathbf{Z}^+$ and a cost functional is minimized. The assumption $\theta(t+k) = \theta(t)$ for $k \in \mathbf{Z}^+$ is conservative as a basis for selecting the value of $\theta(t)$, but it helps to keep the on-line computational effort low. Examples of parameter governors that we subsequently study in more detail in this paper include *Feedforward Governors* and *Gain Governors*.

A constant reference command assumption, $r(t) \equiv r$ for all $t \in \mathbf{Z}^+$, will be made in this paper for the analysis of the properties of the feedforward governor and of the gain governor. These schemes, however, also function when $r(t)$ changes with time. In fact, as we discuss later, both for the feedforward governor and for the gain governor the adjustment of $\theta(t)$ can provide an effective mechanism for dealing with large changes in $r(t)$ and the augmentation of a separate reference governor to assure feasibility may be unnecessary.

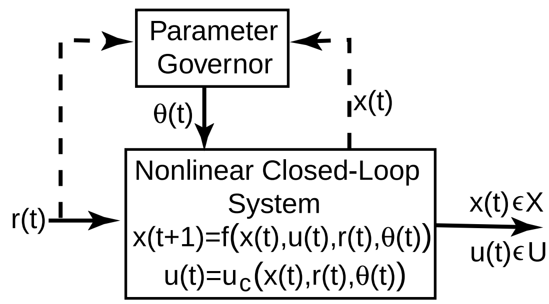

Fig. 1. The parameter governor.

It is important to the development of both feedforward governor and gain governor that with $\theta(t+k) \equiv \theta(t)$ and $r(t+k) = r(t)$ for $k > 0$ the outputs of the system converge to the same steady-state value, $y_e(r(t))$, which is independent of $\theta(t)$. This property enables the treatment of stage-additive cost functionals that unlike the cost functionals in [2] penalize the deviation of the system output from the desired steady-state value, $y_e(r(t))$, corresponding to the actual reference command, $r(t)$. Explicit terminal set conditions can be avoided even though the horizon over which the cost is computed is finite, provided this horizon satisfies appropriate assumptions.

II. The Feedforward Governor

In the feedforward governor case, $x = (x_p, x_i)$ in (1), $r(t) \equiv r \in \mathbf{R}^m$ for all $t \in \mathbf{Z}^+$ and

$$x_p(t+1) = f_p(x_p(t), u(t)), \tag{2}$$

$$x_i(t+1) = x_i(t) + y(t) - r, \quad y(t) = h(x_p(t)), \tag{3}$$

so that f in (1) depends on $x(t)$, $u(t)$ and r:

$$x(t+1) = f(x(t), u(t), r). \tag{4}$$

Here $x_p(t) \in \mathbf{R}^p$ is referred to as the state of the plant (although in reality x_p may also include controller states), $u(t) \in \mathbf{R}^m$ is the control input, $y(t) \in \mathbf{R}^m$ is an output of the system of the same dimension as the control input, and

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$x_i(t) \in \mathbf{R}^m$ is the state of a discrete-time integrator. The functions f_p and h are assumed to be continuous in their arguments. We assume that corresponding to the reference command signal $r(t) \equiv r$ there is a unique steady-state equilibrium for the plant state, denoted by $x_{pe}(r)$, and for the control input, denoted by $u_e(r)$, such that

$$y_e(r) = h(x_{pe}(r)) = r, f_p(x_{pe}(r), u_e(r)) = x_{pe}(r).$$

The control signal $u(t)$ is generated as a sum of the feedforward, $u_e(r)$, a feedback function u_{fb}, dependent continuously on the states $x_p(t)$ and $x_i(t)$, and an adjustable offset $\theta(t)$:

$$u(t) = u_e(r) + u_{fb}(x_p(t), x_i(t), r) + \theta(t). \qquad (5)$$

The integrator (3) is essential to the feedforward governor operation and it is intended to eliminate the influence of the constant offset term, $\theta(t)$, on the steady-state values of x_p and u.

The pointwise-in-time constraints are imposed on the state of the system and on $\theta(t)$ that have the following form

$$(\theta(t), x(t)) \in C, \quad \forall t \in \mathbf{Z}^+, \qquad (6)$$

where $C \subset \mathbf{R}^{m+p+m}$. Because of (5), the control constraints can always be recast as equivalent constraints on $x(t)$ and $\theta(t)$ and transformed into the form (6). Additionally, we assume that the offset is selected from a given set Θ, $\theta(t) \in \Theta$.

The on-line selection of $\theta(t) \in \Theta$ for each $t \in \mathbf{Z}^+$ is based on the minimization of a cost function subject to the constraints imposed by (6). The cost function has the following form,

$$J(x(t), \theta(t), r, T) = ||\theta(t)||^2_{\Psi_\theta} + \sum_{k=t}^{k=t+T} Q\left(\begin{array}{c} x_p\big(k-t|x(t), r, \theta(t)\big) - x_{pe}(r), \\ u\big(k-t|x(t), r, \theta(t)\big) - u_e(r) \end{array} \right), \qquad (7)$$

where $Q(a,b) \geq 0$ is the incremental cost; $\Psi_\theta = (\Psi_\theta)^{\mathrm{T}} \geq 0$, $||\theta(t)||^2_{\Psi_\theta} = \theta(t)^{\mathrm{T}}\Psi_\theta\theta(t)$; $x_p(k-t|x(t), r, \theta(t))$ denotes the plant state and $u(k-t|x(t), r, \theta(t))$ denotes the control input predicted $k-t$ steps ahead given initial state $x(t)$, reference command r and assuming $\theta(t+s) = \theta(t)$ for $s \in \mathbf{Z}^+$. The $T > 0$ is the finite horizon that needs to be selected in agreement with our subsequent assumptions. The cost function (7) must be minimized with respect to $\theta(t) \in \Theta$ subject to meeting constraints (6) restricted to the same finite horizon:

$$\Big(\theta(t), x(k-t|x(t), r, \theta(t))\Big) \in C, \quad k = t, t+1, \cdots, t+T, \qquad (8)$$

where $x(k-t|x(t), r, \theta(t))$ denotes the state of (4) predicted $k-t$ steps ahead given initial state $x(t)$, reference command r and assuming $\theta(t+s) = \theta(t)$ for $s \in \mathbf{Z}^+$. Note that (7) and (8) can be evaluated on-line using predictions of $x(k-t|x(t), r, \theta(t))$, $u(k-t|x(t), r, \theta(t))$ based on the model (2)-(5). If C has an inequality characterization

$$C = \{(\theta, x) \in \mathbf{R}^{m+p+m} : \ g_j(\theta, x, r) \leq 0, \ j = 1, \cdots, q\},$$

then the constraint (8) becomes

$$\max_{j=1,\cdots,q; k=t,\cdots,t+T} g_j(\theta(t), x(k-t|x(t), \theta(t), r), r) \leq 0.$$

The mechanism by which changing $\theta(t)$ may prevent constraint violation and enhance the performance can now be made intuitively clear. To simplify the exposition of the basic idea, suppose that u_{fb} has the form $u_{fb}(x_p, x_i, r) = \bar{u}_{fb}(x_p, r) - \epsilon \cdot x_i + \theta$ and let $\bar{x}_{pe}(r;\theta)$ denote the equilibrium of (2), (5) with $x_i(t+k) = 0$, $\theta(t+k) = \theta$, $k \in \mathbf{Z}^+$. If $\epsilon > 0$ is sufficiently small and the steady-state output monotonicity condition [3] holds then the closed loop system is singularly perturbed, the variable x_i is a slow variable, the variable x_p is a fast variable, and $x_p(t) \to \bar{x}_{pe}(r)$ [3]. Suppose now that the constraints are imposed on plant states only, i.e., we require that $x_p(t) \in C_p$ for all $t \in \mathbf{Z}^+$. With $\theta(t+k) \equiv \theta(t)$ for all $k \in \mathbf{Z}^+$ in (5), $x_p(t+k)$ for $k \in \mathbf{Z}^+$ first rapidly converges to a neighborhood of $\bar{x}_{pe}(r; \theta(t) - \epsilon x_i(t))$, and then progresses along the manifold $\mathcal{M} = \{\zeta \in \mathbf{R}^p : \ \exists\theta \text{ s.t. } \zeta = \bar{x}_{pe}(r;\theta)\}$ towards $\bar{x}_{pe}(r; 0) = x_{pe}(r)$. By a proper selection of $\theta(t)$, the fast part of the trajectory of $x_p(t+k)$ convergent to a neighborhood of $\bar{x}_{pe}(r; \theta(t) - \epsilon x_i(t))$ can be made to avoid the constraints while the slow part of this trajectory satisfies the constraints if the constraints are satisfied near $\mathcal{M}$. Through the on-line adjustment of $\theta(t)$ for each $t \in \mathbf{Z}^+$ the advantage can repeatedly be taken of the fast dynamics of the system (as opposed to slower drift along $\mathcal{M}$), and therefore the performance and convergence speed can be significantly enhanced. Furthermore, since changes in θ shift $\bar{x}_{pe}(r;\theta)$ in a similar manner as r does, it may be possible to effectively accommodate even large changes in r with this approach, without a need for a separate reference governor.

Figure 2 illustrates this intuitive picture with simulation results for an example second order system. Here $x_p = (x_{p1}, x_{p2})$, $y = x_{p1}$, where x_{p1} is a position variable and x_{p2} is a velocity variable. The trajectory of $x_p(t+k)$ with $x_p(t) = (x_{p1}(t), x_{p2}(t)) = (0, 0)$, $x_i(t) = 0$, $\theta(t+k) \equiv \theta(t) = -200$, $k \in \mathbf{Z}^+$, $r = 3.5 \times 10^{-3}$, and marked as (a) in Figure 2, exhibits clearly visible fast and slow parts. It avoids violation of the two constraints also indicated in the figure. The first constraint is on the overshoot by x_{p1}: $x_p(t+k) \in C_p = \{(x_{p1}, x_{p2}) : \ x_{p1} \leq 4 \times 10^{-3}\}$, $k \in \mathbf{Z}^+$, see the dashed line (c) in Figure 2. The second constraint restricts the velocity x_{p2} when x_{p1} crosses a pre-defined position: $x_p(t+k) \in C_p = \mathbf{R}^2 \setminus \{(x_{p1}, x_{p2}) : \ x_{p1} = 3.33 \times 10^{-3}, |x_{p2}| > 0.1\}$, $k \in \mathbf{Z}^+$, see the dashed lines (d) and (e) in Figure 2. For comparison, the trajectory (b) with $x_i(t+k) \equiv 0$ and $\theta(t+k) \equiv 0$ for $k \in \mathbf{Z}^+$ violates both constraints. Note that for both trajectories (a) and (b), $x_p(t+k) \to x_{pe}(r) = (3.5 \times 10^{-3}, 0)$ as $k \to \infty$ and, in fact, trajectories of the plant states corresponding to any $\theta(t)$ converge to the same equilibrium $x_{pe}(r) = (3.5 \times 10^{-3}, 0)$.

The rigorous theoretical results are based on the following assumptions. These assumptions are somewhat stronger

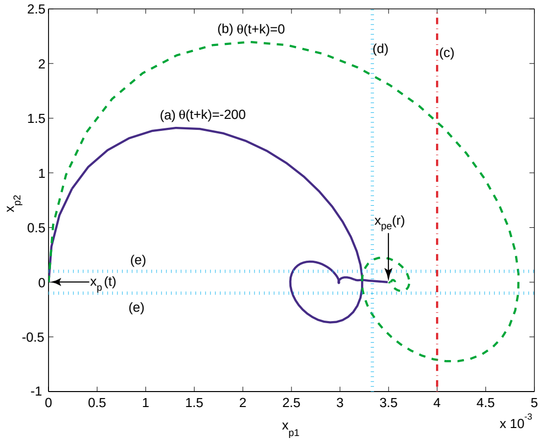

Fig. 2. Trajectories of plant states with $r = 3.5 \times 10^{-3}$: (a) With $\theta(t+k) \equiv \theta(t) = -200$ and $x_i(t+k)$ evolving according to (3); (b) With $\theta(t+k) \equiv \theta(t) = 0, x_i(t+k) \equiv x_i(t) = 0$. Also shown constraints: (c) $x_{p1}(t+k) \leq 4 \times 10^{-3}$ for $k \in \mathbf{Z}^+$; (d),(e): $|x_{p2}(t+k)| \leq 0.1$ if $x_{p1}(t+k) = 3.33 \times 10^{-3}$ for $k \in \mathbf{Z}^+$.

than really needed but they simplify the exposition of the main ideas.

(A1) The set $C \subset \mathbf{R}^{m+p+m}$ in (6) is compact and $\theta(t) \in \Theta$, where $\Theta \subset \mathbf{R}^m$ is a compact set.

(A2) There exists $\delta > 0$ such that for all $\theta \in \Theta$, $(\theta, x_{pe}(r), x_{ie}(r,\theta)) + \delta \cdot \mathcal{B}_{m+p+m} \subset C$, where $x_{pe}(r)$, $x_{ie}(r,\theta)$ denote the unique equilibrium values of x_p and x_i for given constant r and θ and where $\mathcal{B}_{m+p+m}$ is the unit ball in $\mathbf{R}^{m+p+m}$.

(A3) $x(k|\bar{x}, r, \theta) \to (x_{pe}(r), x_{ie}(r,\theta))$ as $k \to \infty$ for all $\theta \in \Theta$ and $(\theta, \bar{x}) \in C$.

(A4) There exist $k_1^* \in \mathbf{Z}^+$ and $0 \leq q < 1$ such that for all $(\theta, \bar{x}) \in C$, $\theta \in \Theta$ and $k \geq k_1^*$,

$$Q\Big(x_p(k|\bar{x}, r, \theta) - x_{pe}(r), u(k|\bar{x}, r, \theta) - u_e(r)\Big) \leq q \cdot Q\Big(\bar{x}_p - x_{pe}(r), u(0|\bar{x}, r, \theta) - u_e(r)\Big). \tag{9}$$

(A5) The function Q in (7) is continuous and satisfies $Q(0,0) = 0$, $Q(a,b) > 0$ if $(a,b) \neq 0$.

The assumption (A1) may require that artificial constraints be added in for the state variables that remain unconstrained by the virtue of the problem formulation. The assumption (A2) is reasonable and can be interpreted as a strict steady-state feasibility condition. The assumptions (A3) and (A4) characterize the needed stability properties of the system when $\theta(t)$ is maintained at a constant value; they generically hold if the state convergence is exponential. The assumption (A5) is imposed on the cost function and not on the original plant itself; it restricts the incremental cost to be strictly positive-definite, which is essential for our subsequent convergence results.

The assumptions (A2) and (A3) and the compactness of C and Θ imply the following

Proposition 1: There exists $k_2^* \in \mathbf{Z}^+$ such that for all $\theta \in \Theta$ and $(\theta, \bar{x}) \in C$, if $(\theta, x(k|\bar{x}, r, \theta)) \in C$ for $k = 0, \cdots, k_2^*$, then $(\theta, x(k|\bar{x}, r, \theta)) \in C$ for all $k \in \mathbf{Z}^+$.

The main result characterizing the response properties of the feedforward governor is given by the following

Theorem 2: Suppose (A1)-(A5) hold and the initial state $x(0)$ is feasible in a sense that there exists $\theta(0) \in \Theta$ such that $(\theta(0), x(k|x(0), r, \theta(0)) \in C$ for all $k \geq 0$. Let $\theta(t) = \theta^*(t),\ t \geq 0$ be generated by minimizing (7) subject to (8) with $T > max\{k_1^*, k_2^*\}$ where k_1^* is defined in (A4) and k_2^* is defined in Proposition 1. Suppose that $x^*(t), u^*(t)$ are the resulting state and control trajectories. Then, $x^*(t)$ remains feasible for all $t \geq 0$ (i.e., constraints $(\theta^*(t), x^*(t)) \in C$ are satisfied for all $t \geq 0$) and $x_p^*(t) \to x_{pe}(r)$, $u^*(t) \to u_e(r)$ as $t \to \infty$. Furthermore, $||\theta^*(t)||^2_{\Psi_\theta}$ converges to a limit.

Proof: Using (A4), (7) and that $\theta^*(t)$ is a feasible choice (guarantees constraint satisfaction) at time $t+1$ but not necessarily optimal, we obtain

$$J(x^*(t+1), \theta^*(t+1), r, T) \leq J(x^*(t+1), \theta^*(t), r, T) \leq J(x^*(t), \theta^*(t), r, T) - (1-q) Q \begin{pmatrix} x_p^*(t) - x_{pe}(r), \\ u^*(t) - u_e(r) \end{pmatrix}. \tag{10}$$

Since $0 \leq q < 1$, $Q(a,b) \geq 0\ \forall a, b$, the sequence $\{J(x^*(t), \theta^*(t), r, T)\}$ is bounded and non-increasing with t. Therefore, it has a limit as $t \to \infty$ and

$$Q\Big(x_p^*(t) - x_{pe}(r), u^*(t) - u_e(r)\Big) \to 0 \text{ as } t \to \infty. \tag{11}$$

By (A5), $x_p^*(t) \to x_{pe}(r)$, $u^*(t) \to u_e(r)$. Finally, the convergence of $J(x^*(t), \theta^*(t), r, T)$ to a limit, (11) and (7) imply that $||\theta^*(t)||^2_{\Psi_\theta}$ converges to a limit. The proof is complete.

Remark 1: The feasibility of the initial state $x(0) = (x_p(0), x_i(0))$ can be affected by the value of $x_i(0)$. Since x_i is typically a part of the nominal control law or of the feedforward governor, it can often be assigned appropriately at $t = 0$ to help ensure that $x(0)$ is feasible.

Remark 2: Suppose the value of $x_i(t)$ can be reset during the on-line operation. Then this reset provides a similar mechanism for avoiding constraint violation and improving performance as does the adjustment of $\theta(t)$. In particular, if $u_{fb}(x_p, x_i, r) = \bar{u}_{fb}(x_p, r) - \epsilon \cdot x_i + \theta$, and if $\theta(t) = \bar{\theta}$ has been chosen, then the predicted trajectories of x_p and u with $\theta(t+k) = \bar{\theta}$ for $k \in \mathbf{Z}^+$ are the same as if $\theta(t+k) \equiv 0$ and $x_i(t)$ was reset to $x_i(t) - \bar{\theta}/\epsilon$.

Remark 3: Theorem 2 applies when Θ consists only of a finite number of elements. In this case the minimization of (7) subject to (8) can be accomplished via a finite number of on-line model simulations for each value of $\theta(t) \in \Theta$, where each simulation is run for the finite horizon T or until the first time instant when constraints become violated.

Remark 4: The result in Proposition 1 enables to relax the conditions $\big(\theta, x(k-t|x(t), r, \theta)\big) \in C$ *for all* $k \in \mathbf{Z}^+$ to (8), provided that the time horizon T in (8) is sufficiently large. This property is expected from similar developments in the reference governor case [1]; it is also related to the finite-determination properties of maximum constraint admissible sets [8]. Theorem 2 demonstrates that if T is sufficiently large so that (A4) applies then the *explicit* terminal set conditions, usually required with the receding horizon control approach, are not needed. Note that larger values of the horizon T do not increase the dimensionality of the optimization problem that needs to be solved to determine $\theta(t)$ - it remains equal to m. Larger Ts do, however, increase the computational effort required to simulate the model on-line over longer time horizons. A practical procedure for selecting T can be based on estimating, via multiple off-line simulations for different $(\theta,\bar{x}) \in C$, $\theta \in \Theta$, the guaranteed time, $k^*(\theta,\bar{x}) \in \mathbf{Z}^+$, after which the constraints are not violated and the expected decay in the incremental cost is at least a factor of $q < 1$. More precisely, $(\theta, x(k|\bar{x},r,\theta)) \in C$ and (9) applies for $k > k^*(\theta,\bar{x})$. Then an acceptable horizon can be specified based on the condition $T \geq \max_{(\theta,\bar{x})\in C, \theta\in\Theta} k^*(\theta,\bar{x})$.

Remark 5: It is not necessary to update $\theta(t)$ every $t \in \mathbf{Z}^+$. The updates can be less frequent, for example, every $t \in I_u = \{0, n, 2\cdot n, 3\cdot n, \cdots; n \in \mathbf{Z}^+, n > 1\}$. Whenever $t \notin I_u$, $\theta(t)$ can be kept constant, $\theta(t) = \theta(t-1)$. As long as for $t \in I_u$, $\theta(t) \in \Theta$ is selected to minimize the cost (7) subject to the constraints (8), the properties in Theorem 2 are guaranteed to hold. The time interval, $k\cdot n \leq t < (k+1)\cdot n$, allows for a larger time (n sampling periods) to calculate the optimal value of $\theta((k+1)n)$ in case these computations cannot be completed within a single sampling period. Note that the value of $x((k+1)n)$ can be predicted via on-line simulations assuming that $\theta(t) = \theta(kn)$ for $kn \leq t \leq (k+1)n$. Note also that suboptimal values of $\theta(t) = \bar{\theta}(t) \neq \theta^*(t)$ may be acceptable as long as the cost non-increase is guaranteed, i.e., $J(x(t+1), \bar{\theta}(t+1), r, T) \leq J(x(t+1), \theta(t), r, T)$.

From Theorem 2, $||\theta^*(t)||^2_{\Psi_\theta}$ converges to a limit. Suppose that

$$\lim_{t\to\infty} ||\theta^*(t)||_{\Psi_\theta} = \lim_{t\to\infty} \sqrt{(\theta^*(t))^{\mathrm{T}}\Psi_\theta\theta^*(t)} = v_{lim} \geq 0. \quad (12)$$

It turns out that under appropriate, additional assumptions, $v_{lim} = 0$ and $x_i^*(t) \to 0$ so that *asymptotically* the feedforward governor becomes inactive. The additional assumptions are:

(A6) f_p, h, u_{fb} in (2), (3), (5) are Lipshitz in all arguments.

(A7) $0 \in int\Theta$, Θ is convex.

(A8) Q is twice continuously differentiable in all arguments.

(A9) The matrix Ψ_θ in (7) is positive definite, $\Psi_\theta > 0$.

We also need an additional assumption to ensure the convergence $x_i^*(t) \to x_{ie}(r, \theta^*(t))$ when $x_p^*(t) \to x_{pe}(r)$, $u^*(t) \to u_e(r)$ which is slightly stronger than the convergence $u_{fb}(x_p^*(t), x_i^*(t), r) \to u_{fb}(x_{pe}(r), x_{ie}(r,\theta^*(t), r)$ already guaranteed by Theorem 2:

(A10) For all $\theta \in \Theta$, if x_p is sufficiently close to $x_{pe}(r)$ and $u_{fb}(x_p, x_i, r) + \theta$ is sufficiently close to 0, then u_{fb} is invertible with respect to x_i and the inverse is a continuous function.

This assumptions holds, for example, if u_{fb} has the form $u_{fb}(x_p, x_i, r) = \bar{u}_{fb}(x_p, r) - \epsilon \cdot x_i$, $\epsilon > 0$.

Theorem 3: Suppose (A6)-(A10) hold in addition to the assumptions of Theorem 2. Then, all conclusions of Theorem 2 remain valid, and $v_{lim} = \lim_{t\to\infty} ||\theta^*(t)||_{\Psi_\theta} = 0$.

Proof: The idea of the proof is to compare the optimal decision at time t, $\theta^*(t)$, with an alternative decision,

$$\hat{\theta}(t) \triangleq \theta^*(t) \cdot \frac{v_{lim}}{v_{lim} + \epsilon_2},$$

where $\epsilon_2 > 0$, and to demonstrate that if $v_{lim} > 0$, t is sufficiently large and ϵ_2 is sufficiently small, then $\hat{\theta}(t)$ is a feasible choice at time t and actually results in a smaller value of the cost (7) than $\theta^*(t)$. In particular, our assumptions can be shown to imply for an appropriate constant $L_Q > 0$ that

$$\begin{aligned}
&J(x(t), \hat{\theta}(t), r, T) - J(x(t), \theta^*(t), r, T) \\
&\leq -\epsilon_2 \frac{||\theta^*(t)||_{\Psi_\theta}}{(v_{lim} + \epsilon_2)^2}(2v_{lim} + \epsilon_2) \\
&+ \Bigg(T \cdot L_Q \frac{\epsilon_2 ||\theta^*(t)||_{\Psi_\theta}}{v_{lim} + \epsilon_2} \times \\
&\times \sup_{k=0,\cdots,T} \left\| DQ(x_p^*(t+k) - x_{pe}(r), u^*(t+k) - u_e(r)) \right\| \Bigg) \\
&+ O(\epsilon_2^2).
\end{aligned}$$

If $v_{lim} > 0$, $\epsilon_2 > 0$ is sufficiently small and $t \in \mathbf{Z}^+$ is sufficiently large, the first term can be made to strictly dominate in absolute value the third term and, in view of $x_p^*(t+k) \to x_{pe}(r)$, $u^*(t+k) \to u_e(r)$ (the result of Theorem 2) and DQ being continuous with $DQ(0,0) = 0$ (which follows from (A5), (A8) since $(0,0)$ is the minimizer of Q) the first term can also dominate the second term so that

$$J(x(t), \hat{\theta}(t), r, T) < J(x(t), \theta^*(t), r, T).$$

The proof is complete.

Remark 6: Assumption (A7) is essential for the result in Theorem 3. For example, appropriate counter-examples can be constructed if the set Θ is discrete-valued.

Remark 7: Assumption (A6) can be relaxed to any condition guaranteeing locally Lipshitz continuity of the state and control trajectories with respect to θ.

III. Gain Governor

For the gain governor case, $x = x_p$ in (1), $r(t) \equiv r$ for all $t \in \mathbf{Z}^+$ and

$$x_p(t+1) = f_p(x_p(t), u(t)), \quad (13)$$

where $x_p(t) \in \mathbf{R}^p$ is referred to as the state of the plant (although in reality x_p may also include controller states) and $u(t) \in \mathbf{R}^m$ is the control input. The function f_p is assumed to be continuous in its arguments. We assume that corresponding to the reference command signal $r(t) \equiv r$ there is a unique steady-state equilibrium for the plant state, denoted by $x_{pe}(r)$, and for the control input, denoted by $u_e(r)$, such that

$$f_p(x_{pe}(r), u_e(r)) = x_{pe}(r).$$

The control signal $u(t)$ is generated as a sum of the nominal feedforward, $u_e(r)$, a feedback function u_{fb}, dependent continuously on $x_p(t)$ and the parameters in the control law $\theta(t) \in \mathbf{R}^s$:

$$u(t) = u_e(r) + u_{fb}(x_p(t), r, \theta(t)), \tag{14}$$

where $u_{fb}(x_{pe}(r), r, \theta) = 0$ for all $\theta \in \Theta$. For example, $\theta(t)$ may represent some or all of the gains in the feedback law.

The rationale for the gain governor is easy to understand in the case of systems with control constraints. Specifically, the gain governor can lower the gains when it becomes necessary to avoid violating the control constraints; the gain governor can increase the gains when there is no danger of constraint violation and doing so improves the performance. The gain governor generalizes the approach of multi-mode control for systems with state and control constraints [7].

The pointwise-in-time constraints are imposed on $x_p(t)$ and on $\theta(t)$:

$$(\theta(t), x_p(t)) \in C, \quad \forall t \in \mathbf{Z}^+, \tag{15}$$

where $C \subset \mathbf{R}^{s+p}$. Because of (14), the control constraints can always be recast as equivalent constraints on $x(t)$ and $\theta(t)$.

The on-line selection of $\theta(t)$ for each $t \in \mathbf{Z}^+$ is based on the minimization of a cost function

$$J(x(t), \theta(t), r, T) = ||\theta(t)||^2_{\Psi_\theta} + \sum_{k=t}^{k=t+T} Q\bigg(x_p(k-t|x(t), r, \theta(t)) - x_{pe}(r)\bigg) \tag{16}$$

which is similar to (7) subject to the same constraints (8) as in the feedforward governor case:

$$\bigg(\theta(t), x_p(k-t|x(t), r, \theta(t))\bigg) \in C, \quad k = t, t+1, \cdots, t+T. \tag{17}$$

Slight modifications to our basic assumptions (A1)-(A9) need to be made for our results in Theorems 2 and 3 to hold. We list the modified assumptions here.

These assumptions are somewhat stronger than really needed but they simplify the exposition of the main ideas.

(A1') The set $C \subset \mathbf{R}^{s+p}$ in (15) is compact and $\theta(t) \in \Theta$, where $\Theta \subset \mathbf{R}^s$ is a compact set.

(A2') There exists $\delta > 0$ such that for all $\theta \in \Theta$, $(\theta, x_{pe}(r)) + \delta\mathcal{B}_{s+p} \subset C$, where $x_{pe}(r)$ denotes the equilibrium value of x_p for a given constant r and θ and $\mathcal{B}_{s+p}$ is the unit ball in $\mathbf{R}^{s+p}$.

(A3'): $x(k|\bar{x}, r, \theta) \to x_{pe}(r)$ as $k \to \infty$ for all $\theta \in \Theta$ and $(\theta, \bar{x}) \in C$.

(A4') There exists $k_1^* \in \mathbf{Z}^+$ and $0 \le q < 1$ such that for all $(\theta, \bar{x}) \in C$, $\theta \in \Theta$ and $k \ge k_1^*$,

$$Q\bigg(x_p(k|\bar{x}, r, \theta) - x_{pe}(r)\bigg) \le q \cdot Q\bigg(\bar{x} - x_{pe}(r)\bigg).$$

(A5') The function Q in (16) is continuous and satisfies $Q(0) = 0$, $Q(a) > 0$ if $a \neq 0$.

(A6') The functions

$$\Phi_{k,\bar{x},r}(\theta) = \begin{bmatrix} x_p(k|\bar{x}, r, \theta) \\ u(k|\bar{x}, r, \theta) \end{bmatrix},$$

are locally Lipshitz as functions of θ for all $(\theta, \bar{x}) \in C$, $\theta \in \Theta$ and $k = 0, \cdots, T$.

(A7') $0 \in int\Theta$, Θ is convex.

(A8') Q is twice continuously differentiable.

(A9') The matrix Ψ_θ in (7) is positive definite, $\Psi_\theta > 0$.

The assumption (A6') can be replaced by one of the usual conditions on smooth dependence of the solution of a difference equation on parameters; it can be satisfied with relative ease. The assumption similar to (A10), that concerns the convergence behavior of x_i, is not needed.

The response properties of the gain governor are derived analogously to the feedforward governor. They are summarized in the following two theorems:

Theorem 4: Suppose assumptions (A1')-(A5') hold and the initial state $x(0)$ is feasible in a sense that there exists $\theta(0) \in \Theta$ such that $(\theta(0), x_p(k|x(0), r, \theta(0)) \in C$ for all $k \ge 0$. Let $\theta(t) = \theta^*(t),\ t \ge 0$ be generated according to minimizing (16) subject to (17) with $T > max\{k_1^*, k_2^*\}$ where k_1^* is defined in (A4') and k_2^* is defined in Proposition 1. Suppose that $x^*(t), u^*(t)$ are the resulting state and control trajectories. Then, $x^*(t)$ remains feasible for all $t \ge 0$ (in particular, constraints $(\theta^*(t), x^*(t)) \in C$ are satisfied for all $t \ge 0$) and $x_p^*(t) \to x_{pe}(r)$, $u^*(t) \to u_e(r)$ as $t \to \infty$. Furthermore, $||\theta^*(t)||^2_{\Psi_\theta}$ converges to a limit.

Theorem 5: Suppose assumptions (A6')-(A9') hold in addition to the assumptions of Theorem 4. Then, all conclusions of Theorem 4 remain valid, and $||\theta^*(t)||_{\Psi_\theta} \to 0$.

Remark 8: The same off-line numerical procedure can be used for specifying an acceptable horizon T in the gain governor case as was used in the feedforward governor case (see Remark 4).

IV. Example

Our first example is an application of a feedforward governor to a system consisting of a mass, m, attached to a spring with the spring constant k_s and a damper with a damping constant, c_d, acted on by an electromagnetic force from a coil. The equations describing the motion of the mass in continuous-time are of the form [5]

$$\begin{aligned} \dot{x}_{p1} &= x_{p2}, \\ \dot{x}_{p2} &= \frac{1}{m}\bigg(\frac{k_a v}{(k_b + L - x_{p1})^2} - k_s x_{p1} - c_d x_{p2}\bigg), \end{aligned} \tag{18}$$

where x_{p1} is the position of the mass (m), x_{p2} is the velocity of the mass (m/s), and $v = i^2$, where i is the current through the coil.

A feedback transformation, $u = (k_a v)/(k_b + L - x_{p1})^2$, converts (18) to a linear system,

$$\begin{aligned} \dot{x}_{p1} &= x_{p2}, \\ \dot{x}_{p2} &= \frac{1}{m}\Big(u - k_s x_{p1} - c_d x_{p2}\Big). \end{aligned} \tag{19}$$

Then (19) is converted to discrete-time form (2) using the fourth order Runge-Kutta approximation.

To assure that $v \geq 0$ (so that it is realizable with a physical current), the following pointwise-in-time constraint needs to be satisfied:

$$v(t) = \frac{u(t)(k_b + L - x_{p1}(t))^2}{k_a} \geq 0. \tag{20}$$

The nominal system (18) does not contain an integral action. Thus to enable the application of the feedforward governor, first the integral action is added-in as in (3),

$$x_i(t+1) = x_i(t) + T_s \cdot (x_{p1}(t) - x_{pe1}(r)),\ \ x_{pe1}(r) = r. \tag{21}$$

We now introduce the feedback law modified by the feedforward governor as

$$u(t) = k_s r - c_a x_{p2}(t) + \theta(t) + k_i x_i(t), \tag{22}$$

where $c_a = 100$, $k_i = -100$. The feed-forward offset θ is selected by the feedforward governor from the set $\Theta = \{-200, -175, -150, -100, -50, -25, -10, 0, 10, 25, 50, 100\}$.

Consider first the situation when such a system is intended for positioning the mass to a desired position, $r = 0.875 \cdot L$, $x_{pe}(r) = (r, 0)$. To avoid collision of the mass with the coil, the following constraint is imposed,

$$x_{p1}(t) \leq L. \tag{23}$$

The set C in (6) reflects the constraints (20), (23) and additional, artificial constraints imposed to make it compact, namely, $-x_{p1} \leq \frac{5}{4}L$, $|x_{p2}| \leq 3$, $|x_i| \leq 1$. The cost function has the form (7) with

$$Q = q_1(x_{p1} - r)^2 + q_2(x_{p2})^2 + r_1(u - u_e(r))^2,$$

where $q_1 = 10^4$, $q_2 = 1$, $r_1 = 10^{-5}$, and with $\Psi_\theta = 5 \times 10^{-4}$.

The off-line numerical procedure described in Remark 4 was used to select T. It yielded $T = 111$.

The simulated responses to the initial condition $x_p(0) = (0, 0)$, $x_i(0) = 0$ are plotted in Figure 3. Figure 3 shows that without the feedforward governor ($k_i = 0, \theta(t) \equiv 0$) the trajectory violates the overshoot constraint, $x_{p1}(t) \leq L$, but it satisfies this constraint with the feedforward governor. While the overshoot constraint can also be met by maintaining $\theta(t)$ at a constant value (equal to 150 which is the value that the feedforward governor selects at $t = 0$), the on-line, dynamic adjustment of $\theta(t)$ (see Figure 4) provides a faster convergence to the set-point. The control constraint, $v = i^2 \geq 0$, is also satisfied.

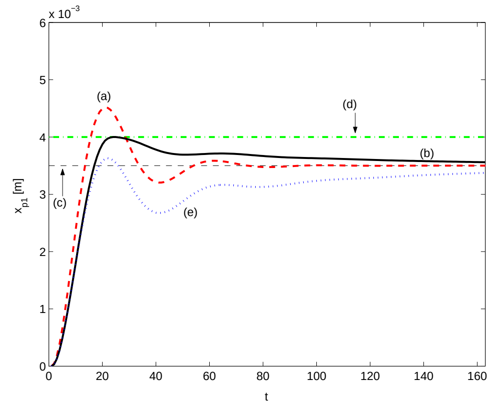

Fig. 3. Time response of x_{p1} without feedforward governor (a) and with feedforward governor (b). The position overshoot constraint is (d) and the set-point r is indicated by line (c). The response (e) corresponds to $\theta(t) \equiv -150$.

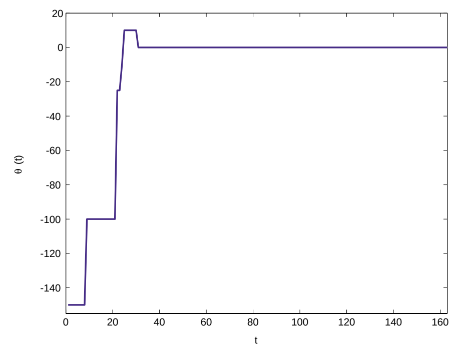

Fig. 4. Time history of $\theta(t)$ adjusted by the feedforward governor.

Consider next a different control objective which is to bring the mass to a desired position, $x_d = 0.8325 \cdot L$, in finite-time where a locking mechanism locks and holds the mass in place. To avoid the locking mechanism damage and provide sufficient time for it to catch the mass, the velocity of the mass, once it reaches the locking position, must be less than 0.1. This results in pointwise-in-time constraints of the form $|x_{p2}(t)| \leq 0.1$ if $x_{p1}(t) = x_d = 0.8325 \cdot L$. In this situation we can apply the control (22) configured to drive the system to a set-point past the locking position, but recognizing that the actual motion will cease once the locking position is crossed. Specifically, we select $r = 0.8327 \cdot L$, $x_{pe}(r) = (r, 0)$. Figure 5 shows that without the feedforward governor ($k_i = 0, \theta(t) \equiv 0$) the system exceeds the velocity specification at the locking time instant 15 times. The feedforward governor with the dynamic adjustment of $\theta(t)$ results in the mass reaching the locking mechanism at $t = 23$ with the acceptable velocity. As this figure also shows the mass can be made to reach the locking mechanism with acceptable velocity by maintaining $\theta(t) \equiv -175$ (which is the value that the feedforward governor selects at $t = 0$) but the locking position is reached much slower, only at $t = 400$.

Our second example is an application of a gain governor

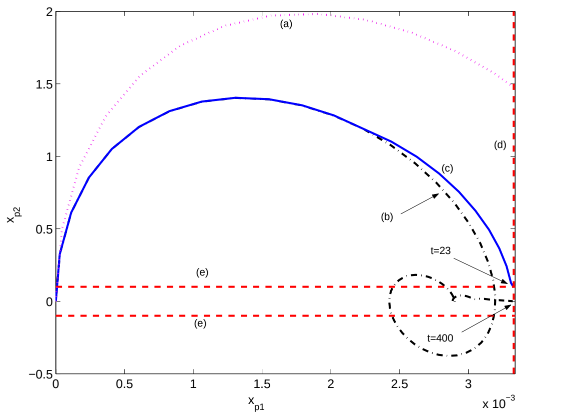

Fig. 5. Phase plane trajectories of x_{p1} and x_{p2} without feedforward governor (a), with $\theta(t) \equiv -175$ (b), and with the feedforward governor (c). The position x_d is indicated by (d) and the velocity constraints are indicated by (e).

to the double integrator,

$$\dot{x}_{p1} = x_{p2}, \quad \dot{x}_{p2} = u,$$

under an input saturation constraint, $|u| \leq 1$. The nominal control law has the form, $u = -\omega_n^2(x_{p1} - r) - 2\zeta\omega_n x_{p2}$, where $\zeta = 0.5$, $r = 0$ so that $x_{pe}(r) = x_{pe}(0) = (0, 0)$, $u_e(r) = u_e(0) = 0$, and $\omega_n = \omega_{n,0} + \theta$, $\omega_{n,0} = 10$. The continuous-time system is discretized assuming that the sampling period is $T_s = 0.01$. The parameter $\theta(t)$ is selected by the gain governor from the set $\theta(t) \in \Theta = \{-9.9, -9.8, \cdots, -0.1, 0, 0.01, 0.05, 0.1, 0.2, \cdots, 0.9, 1\}$, to minimize a cost of the form (16) with $\Psi_\theta = 10^{-4}$, $Q = 10 \cdot (x_{p1} - r)^2 + 0.1 \cdot x_{p2}^2$. The negative values in Θ provide a mechanism to slow down the system while the positive values speed up the response. The horizon was selected according to Remark 8 as $T = \frac{3.5}{T_s}$. The set C in (15) reflects the control constraint, and additional constraints, $-1 \leq x_{p1} \leq 1$, $-2 \leq x_{p2} \leq 2$, were added to make it compact.

Figure 6 shows the time response of x_{p1} with and without the gain governor. The response (a) of the nominal controller with $\omega_n(t) \equiv \omega_{n,0} = 10$ is very fast if there is no saturation, but it behaves poorly with the saturation, see the response (b). The use of a controller with a fixed lower gain $\omega_n(t) \equiv 0.1$ avoids control input saturation (see the response (c)) but it significantly slows down the system thereby sacrificing the performance. Finally, with the gain governor the system avoids control constraint violation, and the speed of response is much better than (c), see response (d) in Figure 6. Figure 7 shows the behavior of $\theta^*(t)$ for the response (d). The response of $\theta^*(t)$ is non-monotonic so that the system is first slowed down to prevent violation of the control constraint, and then made faster once close to the desired equilibrium; ultimately, $\theta^*(t)$ settles to zero in finite time.

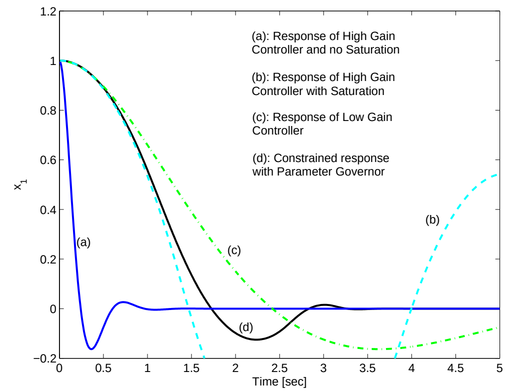

Fig. 6. Time response of $x_1 = x_{p1}$ without the gain governor and with the gain governor: (a) With high gain controller and no input saturation; (b) With high gain controller and input saturation; (c) With low gain controller and input saturation; (d) With the gain governor.

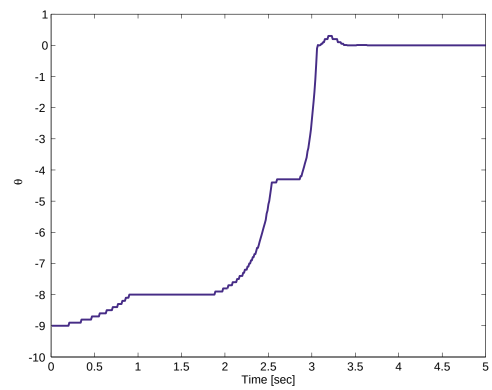

Fig. 7. Time response of parameter θ in the control law.

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