Robust Inventory Control Systems

Jin Wang
Department of Chemical Engineering
West Virginia University Institute of Technology
405 Fayette Pike, Montgomery, WV 25136
jin.wang@mail.wvu.edu

Erik B. Ydstie
Department of Chemical Engineering
Carnegie Mellon University
5000 Forbes Ave.,Pittsburgh,PA
ydstie@andrew.cmu.edu

***Abstract*— Two further developments of the inventory control strategy are studied in this paper. First we use high gain (sliding mode) adaptive control to handle the system uncertainties caused by modelling errors and unmeasured disturbances. It is proved that this control law makes the uncertain system globally stable. The parameters of the resulting controller are easy to tune. Second, inventory control was primarily developed for systems with relative degree equal to one. In this paper we develop an objective-based control strategy which allows application of the inventory control idea to systems with higher relative degree. Several simulation studies illustrate the application of the novel method.**

I. INTRODUCTION

Passivity theory [1], [2], [3] provides an effective means to design control systems for wide range of electro-mechanical systems. We used passivity theory to develop control systems for chemical processes [4]. The central feature of passivity based process control is that the so-called "inventory" variables decay with time. Applications of this method have been reported in recent papers [5], [6], [7].

The inventory of a process is defined as any extensive variable such as total mass, amount of moles of a chemical species or the total energy. The reason for choosing inventory species for control is that many large scale chemical processes can be modelled as networks of unit operations whose dynamics are described by balances of inventories, interconnected by material and energy flow [4]. In such systems, a quadratic error function between the inventories and their ideal objective values, is a suitable storage function for passivity design. This choice ensures that the process is passive and the process inventories will converge to their setpoints when we use strictly passive feedback. For example, the inventory vector may be defined so that

$$v^T = [U, M_1, M_2...M_n] \tag{1}$$

The inventory dynamics are than given by the conservation law:

$$\frac{dv}{dt} = \psi = J_{in} - J_{out} + p \tag{2}$$

where U and $M_i, i = 1, 2..n$ denotes the energy and component masses; J_{in} and J_{out} are flows into and out of the system; and p the rate of production. The control task is to derive the control input u from

$$\psi = -k(v - v^*) + \dot{v}^* \tag{3}$$

Many chemical engineering systems take the J_{in} as u, we can easily calculate it from (3). The essence of the solvability of the control law from (3) is that, the relative degree of this kind of system is equal to 1, and ψ is invertible with respect to the control variables.

The inventory control approach may not function well if there are model uncertainties, errors in the measurements and noise acting on the system. It is also limited to systems that have relative degree equal to one. The purpose of the current paper is to design a strategy to overcome these limitations. To make the method more robust, we develop a sliding mode approach with an adaptation feature to estimate the size of the unmodeled errors. The inventory control approach with sliding mode control is then extended to systems with relative degree larger than one by using an objective based method.

Based on the brief discussion above we have decided to address two issues, which will make the inventory control more robust and applicable to a wider class of systems.

- First, we want to consider uncertainty and develop the nonlinear robust control approaches which provide system dynamics with an invariance property to uncertainties. We achieve this using sliding mode control, to satisfy the Lyapunov stability conditions for the inventory control systems.
- Second, we extend the inventory control method to systems that have higher relative degree than one. This may make inventory control applicable to a wide-range of physical systems than just chemical processes.

II. ROBUST INVENTORY CONTROLLER DESIGN

Let the inventories be represented by the vector v. The dynamics of many chemical processes like reactors, distillation columns, bio-reactors and distributed processes with fluid flow, can then be represented by the inventory balance,

$$\frac{dv}{dt} = p(x) + \phi(d, x, u) \tag{4}$$

where, $p(x)$ is called the production rate, and $\phi(d, x, u)$ is called the inventory supply function which depends on the state variable x, the manipulated variables u and the disturbance variables d. The storage function $\frac{1}{2}(v-v^*)^T(v-v^*)$ can be used to show that the mapping $\phi \rightarrow v - v^*$ is passive and the system is stabilized by any strictly passive feedback [4].

Considering now the inventory system (4) with uncertainties,

$$\dot{v} = p(x) + \phi(d, y, u) + \triangle \tag{5}$$

$\triangle$ denotes a lumped uncertainty which is possibly nonlinear and time-varying. The uncertainty is not known but is assumed to be bounded so that:

$$|\triangle| \leq \delta \tag{6}$$

The objective of this paper is to show that sliding mode control provides a method to control the system dynamics so that they have an invariance property with respect to uncertainties once the system dynamics are controlled in the sliding surface [8], [9].

To develop the sliding mode control strategy, we define the inventory error

$$e(t) = v(t) - v^*(t) \tag{7}$$

where $v^*(t)$ is the desired inventory trajectory. The inventory error dynamics are then described by,

$$\begin{aligned} \dot{e}(t) &= \dot{v} - \dot{v}^* \\ &= p + \phi - \dot{v}^* + \triangle \end{aligned} \tag{8}$$

Following the sliding mode approach [8], we now define a switching surface.

$$S(t) = (\frac{d}{dt} + k_0)\int_0^t e(\tau)d\tau = 0,\ S(0) = 0 \tag{9}$$

The dynamics while in sliding mode can be writen as

$$\dot{S}(t) = 0$$

which leads the inventory error dynamics,

$$\dot{e}(t) = -k_0 e(t) \tag{10}$$

By solving this equation formally for the control input, we obtain an expression for control law called the equivalent control by solving,

$$p + \phi = -k_0 e + \dot{v}^* - \hat{\delta} sgn(S(t)) \tag{11}$$

where $S(t)$ is given by equation (9), the estimate $\hat{\delta}$ satisfies the condition

$$|\triangle| \leq \hat{\delta} \tag{12}$$

and

$$sgn(z) = \left\{ \begin{array}{ll} +1, & z > 0 \\ -1, & z < 0 \end{array} \right\}$$

Remark 1: With no uncertainty, the control law (11) becomes

$$p + \phi = -k_0 e + \dot{v}^* \tag{13}$$

This is the same as proposed in [4]. The control design (13) corresponds to the design of the first order sliding control (9).

We will use the following adaptation algorithm for the bound of δ

$$\dot{\hat{\delta}} = \alpha|S(t)| \tag{14}$$

where α is a positive constant, The closed-loop system is then derived from equations (8) and (11), so that,

$$\dot{e}(t) = -k_0 e(t) - \hat{\delta} sgn(S(t)) + \triangle \tag{15}$$

Lemma 1: Let $\eta > 0$ be a real constant, If the switching surface $S(t)$ of the controlled system satisfies,

$$\frac{1}{2}\frac{d}{dt}S^2 \leq -\eta|S| \tag{16}$$

then $S(t)$ converge to zero and the sliding surface, $S(t) = 0$ exists [8].

To achieve perfect tracking, all system trajectories have to converge to S and stay on the S afterwards. We have to determine a control law such that the above condition (16) is satisfied.

Theorem 1: The control law designed by (11) stabilizes the inventory system with uncertainties (8) and the error converges to the sliding mode.

Proof: According to our control design,

$$\begin{aligned} S(t)\dot{S}(t) &= S(t)(\dot{e}(t) + k_0 e(t)) \\ &= S(t)(-\hat{\delta} sgn(S(t)) + \triangle) \\ &= -\hat{\delta}|S(t)| + \triangle S(t) \\ &\leq -\hat{\delta}|S(t)| + |\triangle||S(t)| \\ &= -(\hat{\delta} - |\triangle|)|S(t)| \\ &= -\eta|S(t)| \end{aligned} \tag{17}$$

where $\eta = \hat{\delta} - |\triangle|$. Therefore, our controller designed by (11) satisfies the existence condition of the sliding mode described by Lemma 1. That implies, when the inventory error $e(t)$ is trapped into the sliding surface, the dynamics of the system is governed by (10), which is always stable, so the inventory error $e(t)$ will converge to zero.

Theorem 2: The sliding mode controller with the adaptation algorithm (14) makes the controlled system (8) asymptotically convergent to the switching surface $S(t) = 0$, and further guarantees that the system is stable.

Proof: Define a Lyapunov function candidate

$$V(S(t), \tilde{\delta}) = \frac{1}{2}S^2(t) + \frac{1}{2\alpha}\tilde{\delta}^2 \tag{18}$$

where

$$\tilde{\delta} = \hat{\delta} - \delta \tag{19}$$

Then

$$\begin{aligned} \dot{V}(S(t), \tilde{\delta}) &= S(t)\dot{S}(t) + \frac{1}{\alpha}(\hat{\delta} - \delta)\dot{\hat{\delta}} \\ &= S(t)(\dot{e}(t) + k_0 e(t)) + \frac{1}{\alpha}(\hat{\delta} - \delta)\alpha|S(t)| \\ &= S(t)(-\hat{\delta} sgn(S(t)) + \triangle) + |S(t)|(\hat{\delta} - \delta) \\ &\leq |S(t)|(|\triangle| - \delta) \leq 0 \end{aligned} \tag{20}$$

This implies,

$$P(t) \equiv |S(t)|(|\triangle| - \delta) \leq -\dot{V}(S(t), \tilde{\delta}) \quad (21)$$

then

$$\int_0^t P(\tau)d\tau \leq V(S(0), \tilde{\delta}(0)) - V(S(t), \tilde{\delta}(t)) \quad (22)$$

As $V(S(0), \tilde{\delta}(0))$ is bounded and $V(S(t), \tilde{\delta}(t))$ is bounded and non-increasing, we conclude that,

$$\lim_{t \to \infty} \int_0^t P(\tau)d\tau < \infty \quad (23)$$

And clearly, $\dot{P}(t)$ is bounded $\iff$ $P(t)$ is uniformly continuous, by Barbalat's Lemma [8]

$$\lim_{t \to \infty} P(t) = 0 \quad (24)$$

Hence, $S(t) \to 0$ as $t \to \infty$. Furthermore, the stability is obtained and the inventory error $e(t)$ converges to the switching surface.

The above sliding mode design is equivalent to the passivity-based control. The mapping

$$p + \phi \to e(t) \quad (25)$$

is passive with the storage function

$$V(S(t), \tilde{\delta}) = \frac{1}{2}S^2(t) + \frac{1}{2\alpha}\tilde{\delta}^2 \quad (26)$$

and the supply rate

$$w = |S(t)|(|\triangle| - \delta) \quad (27)$$

with the control law

$$p + \phi = -k_0 e + \dot{v}^* - \hat{\delta} sgn(S(t)) \quad (28)$$

III. Inventory Control for Relative Degree Greater Than One

For some systems, for example mechanical system, we find that it is hard to solve u from (3) because the relative degree is greater than 1. In such cases, it is not possible to converge to a sliding surface that has relative degree equal to one. We need to extend the order of the error equation accordingly. For example, for a system with relative degree two, we use the second order error equation,

$$\ddot{e} + k_1\dot{e} + k_0 e = 0 \quad (29)$$

where k_0 and k_1 are chosen so that the error dynamics are stable($e(t) \to 0$). Of course, we can extent this to nth order error equation, where the coefficient vector $k = [k_0, k_1 ... k_{n-1}]^T$ is appropriately chosen so that the polynomial $s^{n-1} + k_{n-1}s^{n-2} + ... + k_0$ is Hurwitz. We call this approach Objective Based Control Design.

The detailed procedure for Objective-Based Inventory Control Design is as follows:

Step 1. Define inventory v and its equilibrium v^* based on the physical model.

Step 2. Write the inventory error balance equations, and solve the control law from these equations. The feedback gains $(k_0, k_1, ...)$ satisfy the Hurwitz.

Step 3. Divide the control law into two parts, a feed-forward and a feedback term. The feedback term (u_{fb}) is synthesized by the feedback gains. The feedforward term (u_{ff}) is the controller output at steady state, such that u_{ff} can be solved from $u_{ff} = u|_{v=v^*}$. And eventually the control law is re-written as $u = u_{ff} + u_{fb}$.

Remark 2: The system stability is guaranteed by the Hurwitz condition. For first order inventory error equation, there is only one feedback gain, stability follows as long as $k_0 > 0$. For higher order inventory error equations, the feedback gain constants can be easily chosen by pole-placement same as in [10].

Remark 3: It is not necessary to use the feedforward term in *Step 3* to achieve stability since the system stability is guaranteed via the Hurwitz condition, and the instant performance can be adjusted by the parameters(k_i). But solving the feedforward term helps improve disturbance rejection and setpoint tracking performance.

IV. Examples

To demonstrate the effectiveness of the design developed in this paper, we will apply our methods to a few benchmark control problems. The first two examples demonstrate the effectiveness of sliding mode inventory control for handling system uncertainties. The last one example demonstrate the objective based design method for higher order systems.

Example 1: (Nonlinear tank problem) Consider the liquid surge tank shown in Fig.1 with one inlet (flowing from the upstream process) and one outlet stream (flowing to the downstream process). Based on the material balance, the overall mass balance equation is written as:

$$\frac{dV\rho}{dt} = F_{in}\rho - F_{out}\rho \quad (30)$$

where V is the volume of liquid in the tank; ρ is the liquid density; F_{in} and F_{out} are inlet and outlet volumetric flowrates respectively. This mass balance describes how the volume of the liquid hold-up changes with time. It is often desirable to use tank height h, rather than volume as the state variable. If we assume a constant tank cross-sectional area A, we can express the tank volume as $V = Ah$. We also know that the flowrate out of the tank can be approximated so that it is proportional to the square root of the height of the liquid in the tank,

$$F_{out} = \beta\sqrt{h}$$

where β is a flow coefficient. This gives

$$\frac{dh}{dt} = -\frac{\beta\sqrt{h}}{A} + \frac{F_{in}}{A} \quad (31)$$

For this model we refer to h as the state variable, inlet flowrate (F_{in}) as the input variable and β and A as parameters. Let's define the inventory (objective function),

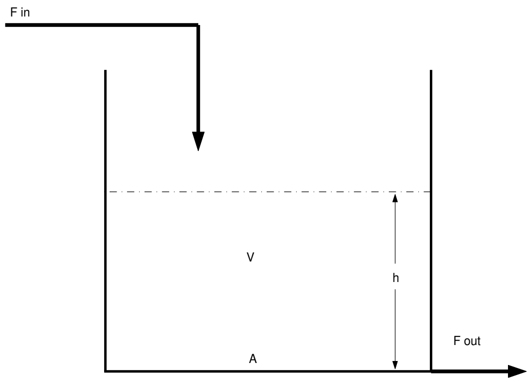

Fig. 1. Liquid Surge Tank

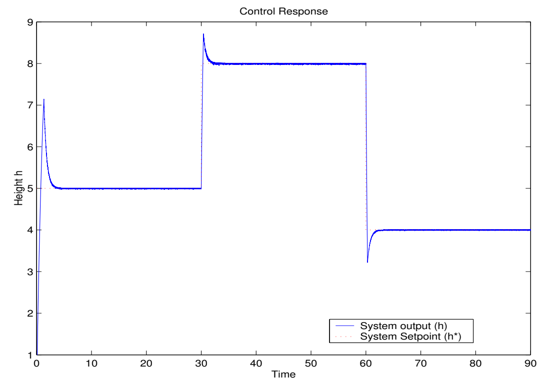

Fig. 2. Control Response of tracking a square setpoint (h^*)

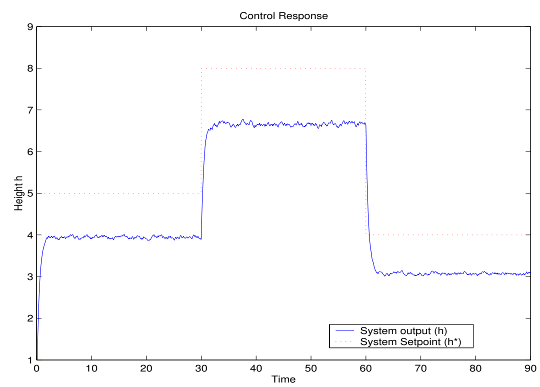

Fig. 3. Control Response of tracking a square setpoint (h^*) without sliding mode

$v = h$, and its equilibrium $v^* = h^*$. The inventory error equation is written as,

$$(\dot{v} - \dot{v}^*) + k_0(v - v^*) = 0 \tag{32}$$

It leads to

$$F_{in} = \beta\sqrt{h} + \dot{h}^*A - k_0(h - h^*)A \tag{33}$$

In steady state, $h = h^*$, the feedforward term $u = u_{ff} = \beta\sqrt{h^*} + \dot{h}^*A$. Therefore, the finial control law becomes,

$$F_{in} = \beta\sqrt{h^*} + \dot{h}^*A - k_0(h - h^*)A \tag{34}$$

This is normal inventory control. In this study, $A = 1$, $\beta = 0.5$, a parameter uncertainty is considered, the flow coefficient becomes $\beta + \Delta\beta$. In this case, the inventory equation becomes

$$\begin{aligned} \frac{dh}{dt} &= \frac{F_{in} - (\beta + \Delta\beta)\sqrt{h}}{A} \\ &= \frac{F_{in} - \beta\sqrt{h}}{A} - \frac{\Delta\beta\sqrt{h}}{A} = \frac{F_{in} - \beta\sqrt{h}}{A} + \Delta \end{aligned} \tag{35}$$

where $\Delta = -\frac{\Delta\beta\sqrt{h}}{A}$. We derive the control law for the inventory control system with this uncertainty from (11):

$$F_{in} = \beta\sqrt{h} - k_0(h - h^*)A + \dot{h}^*A - \hat{\delta}sgn(S(t)) \tag{36}$$

In the same way, considering the feedforward term can be obtained from steady state, the finial control law for the parameter uncertainty becomes,

$$F_{in} = \beta\sqrt{h^*} - k_0(h - h^*)A + \dot{h}^*A - \hat{\delta}sgn(S(t)) \tag{37}$$

The control objective is to control the flowrate inlet F_{in} so that h tracks the square setpoints shown in Fig.2. To investigate the effectiveness of the proposed control system, the following case with parameter variations and time-varying disturbance are considered here: The uncertainty term in this simulation, $\Delta\beta \in [-0.2 \;\; 0.2]$ is a random variable which is unknown for our control design. In addition, the parameters of the controller with adaptive learning are given as follows:

$$k_0 = 2, \; \alpha = 2$$

The control response is plotted in Fig.2. To show the function of sliding mode in the control system, we use the normal inventory control (33) without sliding mode for the uncertain system under the same conditions, the result is shown in Fig.3. It is clear that the sliding mode controller plays an important role in handling this parameter uncertainty.

Example 2: (Relative degree is equal to 1) In this example, we use the nonlinear control problem developed by Byrnes and Isidori [11]:

$$\begin{aligned} \dot{x} &= -x + \mu x^2 + u \\ y &= x \end{aligned} \tag{38}$$

Assume the exosystem (reference inputs or disturbance [11]) satisfies:

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1 \end{aligned}$$

This control problem is to have the system output y track the reference w_1. Let the inventory be defined $v = x$, we get the error equation

$$\dot{e} = \dot{v} - \dot{v}^* = p + \phi - \dot{v}^* = -k_0(v - v^*) \tag{39}$$

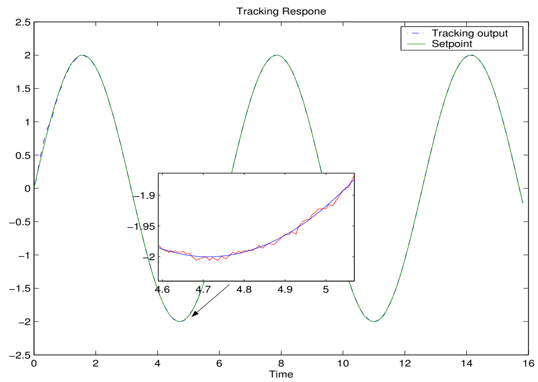

Fig. 4. Tracking Response

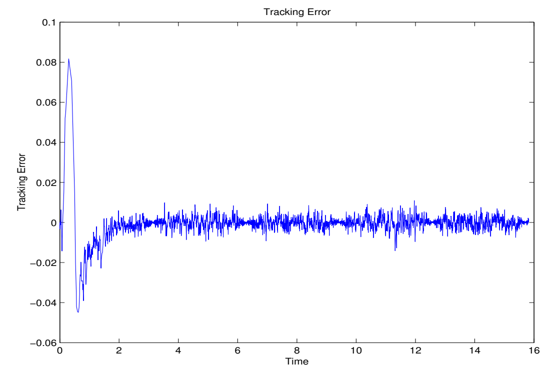

Fig. 5. Tracking Error

Hence, the control law,

$$u = x + \dot{v}^* - \mu x^2 - k_0(x - v^*) \tag{40}$$

Clearly, the the control law consists of feedforward term u_{ff} and feedback term u_{fb}. The feedforward reflects system input in the steady state, i.e. $u_{ff} = v^* + \dot{v}^* - \mu {v^*}^2 = w_1 + w_2 - w w_1^2$, The control law is the same as Byrnes and Isidori [11] approach obtained by Output Regulation Theory:

$$u = w_1 + w_2 - \mu w_1^2 - k_0(x - w_1) \tag{41}$$

Consider an unknown random disturbance $| d | \leq 1$ acting on the system, which is shown in Fig.6.

$$\begin{aligned} \dot{x} &= -x + \mu x^2 + u + d \\ y &= x \end{aligned} \tag{42}$$

The sliding mode control law based on (11) then becomes

$$u = w_1 + w_2 - \mu w_1^2 - k_0(x - w_1) - \hat{\delta} sgn(S(t)) \tag{43}$$

In this example, $\mu = 2$, choose $k_0 = 2$, $\alpha = 30$. Our trajectory function is given by $w_1 = 2\sin t$. Fig.5 and Fig.4 shows the tracking error and tracking response. If we don't use the sliding mode to overcome the disturbance, system output diverges as shown in Fig.7.

Remark 4: We applied our sliding mode inventory control strategy to a time-varying tracking problem with system uncertainty, which can not be solved by output regulation theory [11] due to the unknown disturbance. In addition, the output regulation theory requires that the exosystem is central stable [11], [12]. The approach proposed here does not have this limitation. In another words, sliding mode inventory control strategy can be used to track more general references. Another advantage of the proposed approach over the approach of output regulation theory is that, there is no need to solve the regulator equations which is the most difficulty for output regulation theory [11], [12], [10].

Example 3: (Relative degree is equal to 2) We illustrate here the application of our control strategy to a simple pendulum (Fig.8). This problem which has been extensively

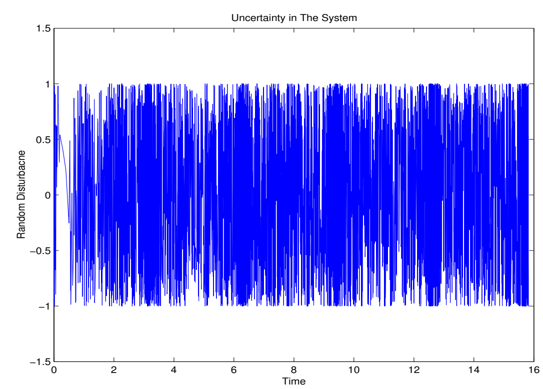

Fig. 6. System Disturbance During the whole Process

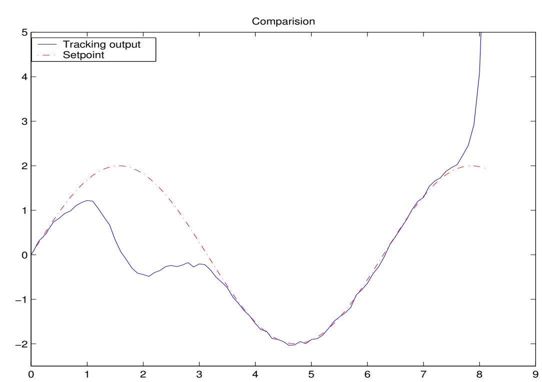

Fig. 7. Without Sliding Mode term($-\hat{\delta} sgn(S(t))$)

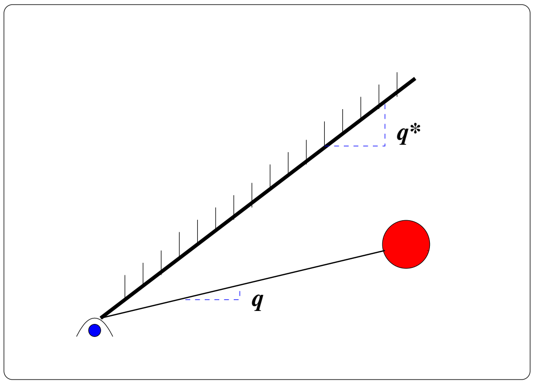

Fig. 8. Simple Pendulum

studied in nonlinear control literature [2], can be described by the equation:

$$ml^2\ddot{q} + mgl\sin(q) = u \tag{44}$$

where q, m, l and g are the angle, the ball mass, pendulum length and gravity acceleration respectively. The torque is used as a control input u. Our objective is to control the pendulum so that its angle q reaches a given setpoint q^*. We choose the inventory variable $v = q$, and error $e = v - v^*$. We then get

$$\ddot{v} = \ddot{q} = \frac{u - mgl\sin v}{ml^2} \tag{45}$$

According to the second order error surface equation (29), we have,

$$\frac{u - mgl\sin v}{ml^2} + k_1\dot{q} + k_0(q - q^*) = 0 \tag{46}$$

which leads to,

$$u = mgl\sin(q) - k_1ml^2\dot{q} - k_0ml^2(q - q^*) = u_{ff} + u_{bf} \tag{47}$$

which is the same as (3.3) in [2]. In steady state, $q = q^*$ and $\dot{q} = 0$,

$$u = u_{ff} = mgl\sin(q^*) \tag{48}$$

it derives the control law,

$$u = mgl\sin(q^*) - k_1ml^2\dot{q} - k_0ml^2(q - q^*) \tag{49}$$

which is the same as the Passivity-Based Control (3.6) in [2].

V. Conclusions

Two issues about inventory control are discussed in this paper. First we developed an improved, sliding mode-based, robust inventory control strategy was proposed for controlling process systems with bounded uncertain parameters and disturbances. The approach is based on an adaptive parameter law for estimating the magnitude of the disturbance. A stability analysis based on Lyapunov theory was given to motivate the theory. Secondly, we developed an extension of the original inventory control theory which makes this method applicable to systems with relative degree larger than one.

Several benchmark example results show that the control strategies proposed here can effectively control nonlinear plants with uncertainties. Good performance, simple structure and adjustment of parameters make the proposed strategies attractive to industrial application.

VI. Acknowledgments

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