

Passivation Designs for CDMA Uplink Power Control

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Abstract—This paper develops new classes of CDMA power control designs by exploiting passivity properties of a gradient-type algorithm proposed in the literature. The new control algorithms offer further design-flexibility, which can be exploited for improved performance and robustness. In our first design, we extend the base station algorithm with Zames-Falb multipliers which preserve its passivity properties. In our second design, we broaden the mobile power update laws with more general, dynamic, passive controllers.

I. INTRODUCTION

In wireless communication networks, power must be regulated to maintain a satisfactory quality of service for users. Increased power ensures longer transmission distance and higher data transfer rate, but it also consumes battery and produces greater amount of interference to neighboring users. In code division multiple access (CDMA) systems, this problem has been studied as an optimization problem, where the i^{th} user minimizes its power p_i , while maximizing its signal-to-interference ratio (SIR)

$$\gamma_i(p) := \frac{Lh_i p_i}{\sum_{k \neq i} h_k p_k + \sigma^2}, \quad (1)$$

where L is the spreading gain of the CDMA system, h_i is the channel gain between the i^{th} mobile and the base station, and σ^2 is the noise variance containing the contribution of the secondary background interference. To regulate the power of each user, Deb *et al.* [1], Zander [2], and Yates [3], pose the constrained optimization problem,

$$\min_i p_i \quad \text{subject to } \gamma_i(p) \geq \gamma_i^{\text{tar}}, \quad (2)$$

where γ_i^{tar} is a threshold chosen to ensure adequate quality of service. An alternative noncooperative game-theoretic formulation is given by Alpcan *et al.* [4], [5], where each user tries to maximize

$$\max_i J_i = U_i(\gamma_i(p)) - P_i(p_i). \quad (3)$$

In this formulation, U_i is a utility function for the i^{th} user, which represents the demand for bandwidth, and P represents the cost of power. The authors then propose the gradient update law

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$$\dot{p}_i = -\lambda_i \frac{\partial J_i}{\partial p_i} = \frac{dU_i}{d\gamma_i} \frac{L\lambda_i h_i}{\sum_{k \neq i} h_k p_k + \sigma^2} - \lambda_i \frac{dP_i(p_i)}{dp_i}, \quad \lambda_i > 0, \quad (4)$$

and prove asymptotic stability of the Nash equilibrium under several assumptions on the functions $U_i(\cdot)$ and $P_i(\cdot)$, and on the number of users.

In this paper, we present new classes of power controllers which include the design of (4) above as a special case. These classes are obtained by exploiting passivity properties of the feedback interconnection of the p_i -subsystems, in which the mobile i updates its own power based upon a feedback from the base station generated by a static algorithm. In our first design, we generalize this base station static algorithm with Zames-Falb multipliers which preserve its passivity properties and, thus, the system stability. In our second design, we broaden the mobile power update laws with more general, dynamic, passive controllers. The additional design flexibility, offered by the new controllers, can be exploited for improved performance, robustness, etc. When applied to the first-order controller (4), our passivity-based stability analysis eliminates the restriction on the number of users, employed in [4]. In this paper, we consider a single base station. For multiple stations, the same switching-based analysis presented in Alpcan *et al.* [4] is applicable. The passivity-based methodology in this paper is similar to our earlier work for wired networks [6]. However, the physical structure and the design set-up are fundamentally different.

The paper is organized as follows. In Section 2 we present our passivity-based stability analysis, and design an extended class of price algorithms for the base station. In Section 3, we present our generalized power update laws for the mobiles. Conclusions are given in Section 4. Throughout the paper, we will use *projection functions* to ensure nonnegative values for physical quantities, such as power. Given a function $f(x)$, its positive projection is defined as

$$(f(x))_x^+ := \begin{cases} f(x) & \text{if } x > 0, \text{ or } x = 0 \text{ and } f(x) \geq 0 \\ 0 & \text{if } x = 0 \text{ and } f(x) < 0. \end{cases}$$

If x and $f(x)$ are vectors, then $(f(x))_x^+$ is interpreted in the component-wise sense. When $(f(x))_x^+ = 0$, we say that the projection is *active*. When $(f(x))_x^+ = f(x)$, we say that the projection is *inactive*.

II. PASSIVITY-BASED STABILITY ANALYSIS

In this section, we first present a passivity-based stability proof for the algorithm (4), and next exploit this passivity property to derive broader classes of controllers. The following assumption is used throughout the paper:

Standing Assumption: *The function $P_i(\cdot)$ in (3) is twice continuously differentiable, nondecreasing, and strictly convex in p_i , i.e.,*

$$\frac{\partial P_i(p_i)}{\partial p_i} \geq 0, \quad \frac{\partial^2 P_i(p_i)}{\partial p_i^2} > 0, \quad \forall p_i, \quad (5)$$

and

$$U_i(\gamma_i) = u_i \log(\gamma_i + L), \quad (6)$$

where u_i is a constant, and γ_i and L are as in (1).

As shown in Alpcan *et al.* [4], [5], this assumption ensures that a unique Nash equilibrium exists for the game (3). The choice of the logarithmic utility function in (6) is meaningful because it represents the maximum achievable bandwidth as in Shannon's Theorem [7]. Noting from (1) and (6) that

$$\frac{dU_i(\gamma_i)}{d\gamma_i} = \frac{u_i}{\gamma_i + L} = \frac{u_i \left(\sum_{k \neq i} h_k p_k + \sigma^2 \right)}{L \left(\sum_i h_i p_k + \sigma^2 \right)}, \quad (7)$$

and substituting (7) in (4), we rewrite the controller (4) as

$$\dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \frac{u_i \lambda_i h_i}{\sum_k h_k p_k + \sigma^2} \right)_{p_i}^+, \quad (8)$$

where the projection $(\cdot)_{p_i}^+$ is added to ensure positivity of p_i . To prepare for our passivity analysis, we let M be the number of the mobiles,

$$h := [h_1 \quad h_2 \quad \cdots \quad h_M]^T \quad (9)$$

$$q := \varphi(y) = -\frac{1}{y + \sigma^2} \quad (10)$$

$$y := h^T p \quad (11)$$

$$w := -h \cdot q \quad (12)$$

and represent (8) as in Figure 1, where the diagonal entries Σ_i of the forward block are given by

$$\Sigma_i : \quad \dot{p}_i = \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i \right)_{p_i}^+. \quad (13)$$

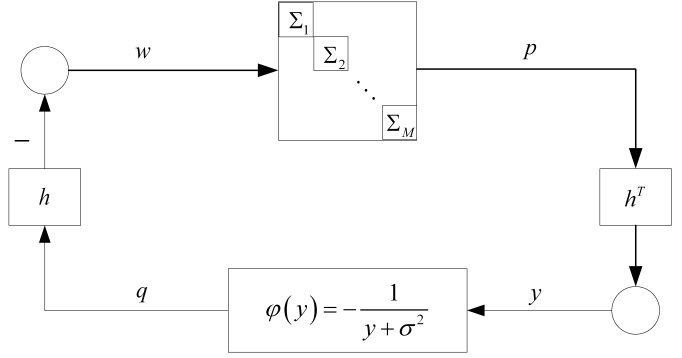


Fig. 1. First-order gradient algorithm of CDMA power control.

In this representation the forward block corresponds to the mobiles and the feedback path corresponds to the base station.

Denoting by p^* the unique Nash equilibrium, and by y^* , q^* and w^* , the corresponding values in (10), (11) and (12), we prove in Proposition 1 below that the forward block is passive from $(w - w^*)$ to $(p - p^*)$. Next, because the feedback block is a nondecreasing function of y , it satisfies the *sector* property

$$(q - q^*)(y - y^*) \geq 0. \quad (14)$$

Because pre-multiplication by h and post-multiplication by its transpose, h^T , preserve passivity, stability of the equilibrium follows from the Circle Criterion:

Proposition 1: Consider the feedback system (9)-(13), represented as in Figure 1. The forward system from $(w - w^*)$ to $(p - p^*)$ is passive, and the equilibrium $p = p^*$ is globally asymptotically stable.

Proof: The derivative of the storage function

$$V(p - p^*) = \frac{1}{2} \sum_i \frac{1}{u_i \lambda_i} (p_i - p_i^*)^2 \quad (15)$$

along the solution of (13) is

$$\begin{aligned} \dot{V} &= \sum_i \frac{1}{u_i \lambda_i} (p_i - p_i^*) \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i \right)_{p_i}^+ \\ &\leq \frac{1}{u_i \lambda_i} (p_i - p_i^*) \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i \right). \end{aligned}$$

This follows because, if the projection is active, $p_i = 0$ and $-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i < 0$, which means that the left hand side of the inequality is zero, and the right hand side is non-negative. Next, by adding and subtracting $u_i \lambda_i w_i^*$, we get

$$\begin{aligned} \dot{V} &\leq \sum_i \frac{1}{u_i \lambda_i} (p_i - p_i^*) \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + \underbrace{u_i \lambda_i w_i^*}_{\frac{dP_i(p_i^*)}{dp_i}} - u_i \lambda_i w_i + u_i \lambda_i w_i \right) \\ &= \sum_i \frac{1}{u_i} (p_i - p_i^*) \left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i(p_i^*)}{dp_i} \right) + (p - p^*)^T (w - w^*), \end{aligned} \quad (16)$$

where the first term satisfies

$$\sum_i \frac{1}{u_i} (p_i - p_i^*) \left(-\frac{dP_i(p_i)}{dp_i} + \frac{dP_i(p_i^*)}{dp_i} \right) < 0, \quad \forall p_i \neq p_i^*$$

since each P_i is strictly concave. Thus, we conclude

$$\dot{V} < (p - p^*)^T (w - w^*), \quad \forall p \neq p^*, \quad (17)$$

which proves passivity from $(w - w^*)$ to $(p - p^*)$. Finally, substituting $w - w^* = -h \cdot (q - q^*)$ and $y - y^* = h^T (p - p^*)$, we conclude from (14) that

$$\begin{aligned} (p - p^*)^T (w - w^*) &= (p - p^*)^T [-h \cdot (q - q^*)] \\ &= -[h^T (p - p^*)]^T (q - q^*) \\ &= -(y - y^*) (q - q^*) \leq 0 \end{aligned} \quad (18)$$

which, combined with (17), proves GAS. \square

Note that, unlike the proof in [4], we have made no assumptions on the number of users. A further advantage of our passivity approach is that it gives us further design flexibility. We now use this flexibility to present a broader class of algorithms for the base station. To this end, we exploit the monotone increasing property of the feedback nonlinearity in Figure 1 and augment it with the following class of multipliers which, as proved in Zames and Falb [8], preserves passivity of the feedback block:

Definition 1: The class of proper transfer functions $Z(s)$ is called inverse-Zames-Falb if $Z(s)$ is strictly stable and

$$1/Z(s) = (m_0 - F(s) + \eta s), \quad (19)$$

where m_0, η are positive constants and the impulse response of $F(s)$, $f(t)$, satisfies

$$f(t) > 0, \quad \int_0^\infty f(t) < m_0. \quad (20)$$

Our new base station algorithm is the cascade of $Z(s)$ and the nonlinearity $\varphi(\cdot)$ in (10), as depicted in Figure 2 with the dashed feedback block. Its stability follows from the same arguments as in Proposition 1, because a monotone first-third quadrant nonlinearity (10) cascaded with inverse-Zames-Falb multiplier (19) is passive [8], [9]:

Theorem 1: Consider the feedback interconnection shown in Figure 2, where the mobile power control law is given by (13) and the base station price update (10) is replaced by

$$q = -Z(s) \left[\frac{1}{y + \sigma^2} \right]. \quad (21)$$

If $Z(s)$ is designed to be an inverse-Zames-Falb multiplier as in (19) with $Z(0) = 1$, then the equilibrium is the same as the one in (9)-(13), and is GAS.

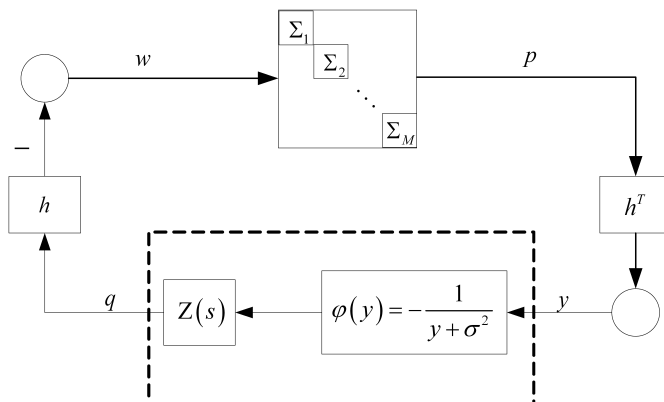


Fig. 2. The modified base station price update.

Proof: At the equilibrium,

$$-Z(s) \left[\frac{1}{y^* + \sigma^2} \right] = -Z(0) \frac{1}{y^* + \sigma^2} = -\frac{1}{y^* + \sigma^2} = q^*,$$

which implies the equilibrium is unchanged. Due to the linearity of $Z(s)$, we can represent the return system as the cascade of the inverse-Zames-Falb multiplier $Z(s)$ and the nonlinearity

$$q - q^* = \varphi(y) - \varphi(y^*) = -\frac{1}{y + \sigma^2} + \frac{1}{y^* + \sigma^2},$$

driven by $(y - y^*)$. Since $\bar{\varphi}(y - y^*) := \varphi(y) - \varphi(y^*)$ is first-third quadrant and monotone, it follows from the property of inverse-Zames-Falb multipliers that the return system is passive, i.e.,

$$\int_0^T (q - q^*) (y - y^*) dt \geq -\mu(x_Z(0)), \quad (22)$$

where x_Z is the internal state of $Z(s)$ and $\mu(\cdot)$ is a nonnegative function (see [8], [9] for proofs of this property). We now show the closed loop stability by using V in Proposition 1. The derivative of V along the solution again satisfies (16). Integrating both sides and denoting $\tilde{p} := p - p^*$, we get

$$\begin{aligned} V(\tilde{p}(T)) - V(\tilde{p}(0)) &\leq -\int_0^T W(\tilde{p}(t)) dt + \int_0^T (p - p^*)^T (w - w^*) dt \\ &= -\int_0^T W(\tilde{p}(t)) dt - \int_0^T (y - y^*) (q - q^*) dt, \end{aligned} \quad (23)$$

where

$$W(\tilde{p}(t)) := \sum_i \frac{1}{u_i} (p_i - p_i^*) \left(\frac{dP_i(p_i)}{dp_i} - \frac{dP_i(p_i^*)}{dp_i} \right)$$

is positive definite as in (16) and the equality in (23) follows from (18). Substituting (22) in (23), we obtain

$$V(\tilde{p}(T)) \leq V(\tilde{p}(0)) + \mu(x_Z(0)) - \int_0^T W(\tilde{p}(t)) dt \leq V(\tilde{p}(0)) + \mu(x_Z(0)), \quad (24)$$

which proves boundedness of \tilde{p} . To prove $\tilde{p} \rightarrow 0$, we note from (24) that

$$\int_0^T W(\tilde{p}(t)) dt \leq V(\tilde{p}(0)) + \mu(x_Z(0)).$$

Then, using boundedness of \dot{p} , we can apply Barbalat's Lemma [11] to conclude that $\tilde{p} \rightarrow 0$ asymptotically. Since $Z(s)$ is stable, its internal states also converge to the corresponding equilibrium values.

III. AN EXTENDED CLASS OF POWER CONTROL ALGORITHMS

In Section 2, we extended the price update law for the base station using the passivity property of the forward block from $(w - w^*)$ to $(p - p^*)$, and the monotone non-decreasing property of the feedback block in Figure 1. In this section, we first prove another passivity property in Figure 1, this time from $(w - w^*)$ to \dot{p} , and next use it to generalize the power update law for the mobiles.

Proposition 2: Consider the feedback system (9)-(13), represented as in Figure 1. The forward system from $(w - w^*)$ to \dot{p} , and the return system from \dot{y} to $-(q - q^*)$ are both passive.

Proof: For the forward system, we let

$$V_1(p - p^*) = \sum_i \frac{1}{u_i \lambda_i} (\lambda_i (P_i(p_i) - P_i(p_i^*)) - u_i \lambda_i w_i^* (p_i - p_i^*))$$

where $V_1(0) = 0$. The derivative of each component of V_1 with respect to p_i is

$$\frac{\partial V_1}{\partial p_i} = \frac{1}{u_i \lambda_i} \left(\lambda_i \frac{dP_i(p_i)}{dp_i} - u_i \lambda_i w_i^* \right)$$

which, when set to zero, has the unique solution at $p = p^*$. Because the second derivative is

$$\frac{\partial^2 V_1}{\partial p_i^2} = \frac{1}{u_i} P_1''(p_i) > 0,$$

we conclude that V_1 is a positive definite function. Next, we note that the derivative of V_1 is

$$\begin{aligned} \dot{V}_1 &= \sum_i \frac{1}{u_i \lambda_i} \left(\lambda_i \frac{dP_i(p_i)}{dp_i} - u_i \lambda_i w_i^* \right) \dot{p}_i \\ &= \sum_i \frac{1}{u_i \lambda_i} \left(\lambda_i \frac{dP_i(p_i)}{dp_i} - u_i \lambda_i w_i \right) \dot{p}_i + (w_i - w_i^*) \dot{p}_i \\ &= \sum_i \frac{1}{u_i \lambda_i} \left(\lambda_i \frac{dP_i(p_i)}{dp_i} - u_i \lambda_i w_i \right) \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} + u_i \lambda_i w_i \right)_{p_i}^+ \\ &\quad + \sum_i (w_i - w_i^*) \dot{p}_i. \end{aligned} \quad (25)$$

Because the first term is negative definite, as can be shown from the uniqueness of equilibrium p^* and the discussion

in Appendix C in [10], the forward system from $(w_i - w_i^*)$ to \dot{p} is passive. Now consider the return system, and let

$$\begin{aligned} V_2(y - y^*) &= \int_{y^*}^y (\varphi(\xi) - \varphi(y^*)) d\xi \\ &= \int_{y^*}^y \left(-\frac{1}{\xi + \sigma^2} + \frac{1}{y^* + \sigma^2} \right) d\xi \end{aligned} \quad (26)$$

where $V_2(0) = 0$, $\nabla V_2(0) = \left(\frac{1}{y^* + \sigma^2} - \frac{1}{y + \sigma^2} \right) \Big|_{y=y^*} = 0$, and $\nabla^2 V_2 = \frac{1}{(y + \sigma^2)^2} > 0$, so V_2 is a non-negative definite function. This return system from \dot{y} to $(q - q^*)$ is passive since

$$\dot{V}_2 = - \left(\frac{1}{y + \sigma^2} - \frac{1}{y^* + \sigma^2} \right) \dot{y} = (q - q^*) \dot{y}. \quad \square$$

We note that stability of the equilibrium p^* , proved in Proposition 1, can also be established from Proposition 2, because the negative feedback interconnection of two passive systems is stable. The advantage of this passivity perspective is that it allows us to design a broader class of power control schemes for the mobiles. A natural generalization is to replace the first-order control law (8) by a more general class of passive systems:

Theorem 2: Consider the feedback interconnection shown in Figure 2, where the base station price update is given by (10) and the mobile power control law (13) is replaced by

$$\begin{aligned} \dot{\xi}_i &= \left(A_i \xi_i + B_i \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} - q_i \right) \right)_{\xi_i}^+, \quad \xi_i \in R^{n_i} \\ \dot{p}_i &= \left(C_i \xi_i + D_i \left(-\lambda_i \frac{dP_i(p_i)}{dp_i} - q_i \right) \right)_{p_i}^+. \end{aligned} \quad (27)$$

If the i^{th} subsystem (A_i, B_i, C_i, D_i) has the structure

$$\begin{aligned} A_i &= \begin{bmatrix} -a_{i1} & & & 0 \\ & -a_{i2} & & \\ & & \ddots & \\ 0 & & & -a_{i,n_i} \end{bmatrix}, \quad B_i = \begin{bmatrix} b_{i1} \\ b_{i2} \\ \vdots \\ b_{i,n_i} \end{bmatrix}, \\ C_i &= [c_{i1} \quad c_{i2} \quad \cdots \quad c_{i,n_i}], \quad D_i = \delta_i, \end{aligned} \quad (28)$$

with $a_{ij} > 0$, $b_{ij} > 0$, $c_{ij} > 0$, $\delta_i > 0$, $\forall i, j$, then the equilibrium $(\xi, p) = (0, p^*)$ of the interconnected system is GAS.

Proof: We first show that the modified system (27) is passive from $(w - w^*)$ to \dot{p} . Consider the following positive definite function for the i^{th} mobile:

$$\begin{aligned} V_{1_i}(\xi_i, p_i - p_i^*) &= \sum_{j=1}^{n_i} \frac{c_{ij}}{2b_{ij}} \xi_{ij}^2 \\ &\quad + \frac{1}{u_i \lambda_i} (\lambda_i (P_i(p_i) - P_i(p_i^*)) - u_i \lambda_i w_i^* (p_i - p_i^*)). \end{aligned}$$

The derivative of V_{1_i} along the solution (27) and (10) is (after adding and subtracting w_i from w_i^*):

IV. CONCLUSION

In this paper we have first proven a passivity property for the first-order gradient design of [4] for CDMA power control and, next, used this property to develop more general classes of passive controllers. These extended controllers preserve stability properties of the first-order design. The additional design flexibility can be exploited for several objectives, such as for improved robustness to disturbances (such as unmodeled secondary interference effects from neighboring cells) and to time-delays (propagation delays), as well as for improved performance. As a starting point, in the companion paper [13], we have characterized the robustness of the first-order gradient design against disturbances and time delays. Our next step will be to develop systematic designs of the Zames-Falb filters for the base stations, and of the extended power update laws for the mobiles, to improve these robustness properties.

V. ACKNOWLEDGEMENT

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$$\begin{aligned}
 & V_{1_i}(\xi_i, p_i - p_i^*) \\
 &= \left\{ \sum_{j=1}^{n_i} \frac{c_{ij}}{b_{ij}} \xi_{ij} (-a_{ij} \xi_{ij} + b_{ij} (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)) \right\}_{\xi_{ij}}^+ \\
 &+ \frac{1}{u_i \lambda_i} (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) \left(\sum_{j=1}^{n_i} c_{ij} \xi_{ij} + \delta_i (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i) \right)_{p_i}^+ \\
 &+ (w_i - w_i^*) \dot{p}_i.
 \end{aligned} \tag{29}$$

We first note that

$$\begin{aligned}
 & \frac{c_{ij}}{b_{ij}} \xi_{ij} (-a_{ij} \xi_{ij} + b_{ij} (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i))_{\xi_{ij}}^+ \\
 &= \left(-\frac{a_{ij} c_{ij}}{b_{ij}} \xi_{ij}^2 + c_{ij} \xi_{ij} (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i) \right)_{\xi_{ij}}^+
 \end{aligned} \tag{30}$$

which is immediate if the projection is inactive. If the projection is active, then $\xi_{ij} = 0$, and both sides of the equality are zero. Next, we claim that

$$\begin{aligned}
 & (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) \left(\sum_{j=1}^{n_i} c_{ij} \xi_{ij} + \delta_i (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i) \right)_{p_i}^+ \\
 &\leq (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) \sum_{j=1}^{n_i} c_{ij} \xi_{ij} \\
 &+ \delta_i (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)_{p_i}^+.
 \end{aligned} \tag{31}$$

If the projection is inactive, the inequality holds since

$$\delta_i (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)_{p_i}^+ \leq 0.$$

If the projection is active, then

$$\sum_{j=1}^{n_i} c_{ij} \xi_{ij} + \delta_i (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i) \leq 0,$$

which implies $-\lambda_i P'_i(p_i) + u_i \lambda_i w_i \leq 0$ since $\delta_i > 0$. Thus,

$$d_i (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)_{p_i}^+ = 0,$$

and the left hand side of (31) is zero and the right hand side is non-negative.

Substituting (30) and (31) in (29), we obtain

$$\begin{aligned}
 & V_{1_i}(\xi_i, p_i - p_i^*) \\
 &\leq \sum_{j=1}^{n_i} -\frac{a_{ij} c_{ij}}{b_{ij}} \xi_{ij}^2 + \delta_i (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)_{p_i}^+ \\
 &+ (w_i - w_i^*) \dot{p}_i
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 & \frac{d}{dt} \left(\sum_{i=1}^M V_{1_i} + V_2 \right) \\
 &\leq \sum_{j=1}^{n_i} -\frac{a_{ij} c_{ij}}{b_{ij}} \xi_{ij}^2 + \delta_i (\lambda_i P'_i(p_i) - u_i \lambda_i w_i) (-\lambda_i P'_i(p_i) + u_i \lambda_i w_i)_{p_i}^+.
 \end{aligned}$$

Following the same argument after equation (25), we conclude that the right hand side of the inequality is negative for $(\xi, p) \neq (0, p^*)$ and, hence, the equilibrium $(0, p^*)$ is GAS.