Proceeding of the 2004 American Control Conference
Boston, Massachusetts June 30 - July 2, 2004

Orbit dynamics and kinematics with full quaternions

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Abstract— **Full quaternions constitute a compact notation for describing the generic motion of a body in the space. One of the most important results about full quaternions is that they can be partitioned into a unit quaternion (which describes the orientation with respect to a suitable reference), and a modulus (which represents the translational motion along the direction indicated by the unit quaternion). Since vectors and scalars are also full quaternions, the equations of motion of the body can be rewritten in quaternion form. In this paper the orbit dynamics and kinematics of a point mass moving in the space are transformed in quaternion form. Simple application examples are presented.**

I. Introduction

When dealing with satellite attitude and orbit control, one of the first design issue is the formulation of spacecraft dynamics. According to classical approach, rigid body motion can be decomposed into two parts:

1. *orbital motion*, depending on position and velocity of the satellite Centre of Mass (COM) ;
2. *attitude kinematics and dynamics*, described by Euler parameters (i.e.: unit quaternions) or Euler angles.

This methodology is very well known, has been widely treated in literature (see [1] and [2]), and is commonly used in applications: for example it has been employed in the design of a *drag-free* controller for the European satellite GOCE [3]. In this case, satellite attitude corresponds to the orientation of the body reference frame with respect to a *local orbital* frame, univocally defined by orbit position and velocity. Assuming that the orientation of the body frame with respect to an inertial frame is known, it becomes necessary to parameterize the orientation of the orbital frame with respect to the inertial reference. The problem, apparently straightforward, is transforming the inertial coordinates of the three unit vectors constituting the orbital frame into a set of four Euler parameters. Two alternatives have been considered:

1. to build the rotation matrix and then exploit the well known conversion rules (see [1]) allowing to pass to quaternion parameterization;
2. to associate a *full quaternion* notation (i.e.: non-unitary quaternion) to orbital frame.

The former solution has been employed in attitude determination of the GOCE satellite [3]. The latter one, has been developed to find a direct way to express the motion of the local orbital frame entrained by the COM motion.

A full quaternion can describe the modulus and the orientation of a vector with respect to a given reference frame. This implies, considering the satellite orbit, that *position and velocity can be alternatively denoted with a vector or with the associated full quaternion*. Since that, orbital dynamics and kinematics can be rewritten substituting vector notation with full quaternions. This results in harmonization of motion equations: both orbital dynamics/kinematics and attitude dynamics/kinematics can be rewritten in quaternion form. Then the orientation of the orbital frame can be directly extracted from the related full quaternion at any time.

This paper is devoted to lay down the foundations of this technique with the help of simple applications. First of all, definition and elementary algebra of full quaternions will be introduced in Section II. Next, how full quaternions can represent vector magnifications and finite rotations will be shown. First and second derivatives of full quaternions are then derived in order to rewrite orbital motion equations in quaternion form. This will be explained in Section III, where quaternion kinematics and dynamics will be derived. In Section IV, quaternion kinematic and dynamic equations will be applied to a pair of typical orbital references: the *Local Orbital Reference Frame* and the *Local Vertical – Local Horizontal* frame. In both cases the associated full quaternion will be defined, as well as orbital kinematics and dynamics. Finally, the simple case of uniform circular motion will enlighten the similarities between classical vector form and quaternion expression of orbital motion.

II. Full quaternions

A. Definition

A quaternion $\mathcal{A}$ is defined as a complex number:

$$\mathcal{A} \triangleq a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} = a_0 + \boldsymbol{a}\,. \quad (1)$$

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Quaternions can be also expressed in column vector form with respect to the basis (1, ***i***, ***j*** ,***k***):

$$\mathcal{A}=\begin{bmatrix}a_0 & a_1 & a_2 & a_3\end{bmatrix}^T=\begin{bmatrix}a_0 & \boldsymbol{a}^T\end{bmatrix}^T . \tag{2}$$

Remark. To alleviate notation, the script $\mathcal{A}$ will denote:

1. quaternions in complex number representation (1);
2. and quaternions in column vector form (2).

A vector quaternion is a three-dimensional vector ***b*** represented in quaternion notations, i.e.

$$\mathcal{B}=\begin{bmatrix}0 & \boldsymbol{b}^T\end{bmatrix}^T . \tag{3}$$

In this case the notations $\mathcal{B}$ and ***b*** will have the same meaning. See [4] for further details.

B. Algebra

A brief summary of the full quaternion algebra is provided, leaving the details to the Appendix and [4]. The norm of a quaternion $\mathcal{A}$, denoted by $|\mathcal{A}|$, is a scalar quaternion and is defined in the same way as the Euclidean norm (or l_2 norm) of a general spatial vector:

$$|\mathcal{A}|^2=a_0^2+a_1^2+a_2^2+a_3^2=\left\|\begin{bmatrix}a_0 & a_1 & a_2 & a_3\end{bmatrix}^T\right\|_2^2 . \tag{4}$$

If $|\mathcal{A}|=1$, $\mathcal{A}$ is called a *unit quaternion,* and deserves its own notation $\underline{\mathcal{A}}$. If $\mathcal{A}$ has non-unitary norm, it is called a *full quaternion*.

Remark. Since scalars and vectors are quaternions, scalar and vector algebra applies.

Let $\mathcal{A}=a_0+\boldsymbol{a}$, $\mathcal{B}=b_0+\boldsymbol{b}$ and $\mathcal{C}=c_0+\boldsymbol{c}$ be three quaternions.

1) Multiplication

According to [4], quaternion multiplication is defined as:

$$\mathcal{A}\otimes\mathcal{B}=(a_0+\boldsymbol{a})\otimes(b_0+\boldsymbol{b})=a_0b_0+a_0\boldsymbol{b}+b_0\boldsymbol{a}-\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{b} , \tag{5}$$

where the symbols · and × stand for dot product and cross product. An alternative expression of the norm in (4) can be obtained through quaternion multiplication, namely:

$$|\mathcal{A}|^2=\mathcal{A}\otimes\mathcal{A}^* , \tag{6}$$

where $\mathcal{A}^*=a_0-\boldsymbol{a}$ denotes quaternion conjugate.

The same product in (5) can be expressed in matrix form. First, rewrite the product quaternion $\mathcal{C}$ in vector form:

$$\mathcal{C}=\mathcal{A}\otimes\mathcal{B},\quad \mathcal{C}=\begin{bmatrix}a_0b_0-\boldsymbol{a}\cdot\boldsymbol{b}\\ a_0\boldsymbol{b}+b_0\boldsymbol{a}+\boldsymbol{a}\times\boldsymbol{b}\end{bmatrix}=\begin{bmatrix}c_0\\ \boldsymbol{c}\end{bmatrix} . \tag{7}$$

Then, from matrix expressions for dot and cross products:

$$\boldsymbol{a}\cdot\boldsymbol{b}=\boldsymbol{a}^T\boldsymbol{b},\quad \boldsymbol{a}\times\boldsymbol{b}=\begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\boldsymbol{b}=C(\boldsymbol{a})\boldsymbol{b} , \tag{8}$$

equation (7) can be written as:

$$\begin{bmatrix}c_0\\ \boldsymbol{c}\end{bmatrix}=\begin{bmatrix}a_0 & -\boldsymbol{a}^T\\ \boldsymbol{a} & a_0I+C(\boldsymbol{a})\end{bmatrix}\begin{bmatrix}b_0\\ \boldsymbol{b}\end{bmatrix}=\begin{bmatrix}b_0 & -\boldsymbol{b}^T\\ \boldsymbol{b} & b_0I-C(\boldsymbol{b})\end{bmatrix}\begin{bmatrix}a_0\\ \boldsymbol{a}\end{bmatrix} . \tag{9}$$

Quaternion multiplication is associative and distributive, but not commutative.

2) Commutative property

Although commutative law does not hold in general, the matrix expression (9) shows $\mathcal{A}$ and $\mathcal{B}$ to commute through sign change. Therefore, the following matrix representations of quaternions can be introduced:

$$\mathcal{A}^+=\begin{bmatrix}a_0 & -\boldsymbol{a}^T\\ \boldsymbol{a} & a_0I+C(\boldsymbol{a})\end{bmatrix},\ \mathcal{B}^-=\begin{bmatrix}b_0 & -\boldsymbol{b}^T\\ \boldsymbol{b} & b_0I-C(\boldsymbol{b})\end{bmatrix}, \tag{10}$$

where superscripts + and – denote the sign of the cross product matrix $C(\cdot)$ and I denotes the identity matrix. Using notations defined in (10), the commutative property which is hidden in (9), can be expressed in the compact form:

$$\mathcal{C}=\mathcal{A}^+\mathcal{B}=\mathcal{B}^-\mathcal{A} , \tag{11}$$

where one must pay attention that $\mathcal{A}$ and $\mathcal{B}$ are meant to be in column vector form.

3) Inverse

Each nonzero quaternion $\mathcal{A}$ admits an inverse $\mathcal{A}^{-1}$ such that $\mathcal{A}\otimes\mathcal{A}^{-1}=1$. It is simple to proof that the *inverse quaternion* $\mathcal{A}^{-1}$ of $\mathcal{A}$ holds:

$$\mathcal{A}^{-1}=\mathcal{A}^*/|\mathcal{A}|^2 . \tag{12}$$

Equation (12) states that if $\mathcal{A}$ is a unit quaternion, the inverse equals the conjugate. Instead, if $\mathcal{A}$ is a full quaternion, its norm has to be taken into account.

C. Magnification and finite rotations

As it will be shown below, full quaternions allow to describe at the same time vector rotation as unit quaternions and vector magnification. Consider a unit quaternion $\underline{\mathcal{R}}$ and a quaternion $\mathcal{B}$. A well known method to represent a rotation of $\mathcal{B}$ into $\mathcal{B}'$ by an angle θ around an axis ***u*** is:

$$\mathcal{B}'=\underline{\mathcal{R}}\otimes\mathcal{B}\otimes\underline{\mathcal{R}}^* . \tag{13}$$

Since every unit quaternion admits the *Euler parameters* representation, it is possible to express $\underline{\mathcal{R}}$ in terms of θ and the unit vector ***u***:

$$\underline{\mathcal{R}}=\begin{bmatrix}r_0 & r_1 & r_2 & r_3\end{bmatrix}^T=\begin{bmatrix}\cos(\theta/2) & \sin(\theta/2)\boldsymbol{u}^T\end{bmatrix}^T . \tag{14}$$

By applying (11) and (A.2), the matrix form in (13) ensues:

$$\mathcal{B}'=\underline{\mathcal{R}}^+\mathcal{B}^+\underline{\mathcal{R}}^*=\underline{\mathcal{R}}^+(\underline{\mathcal{R}}^-)^T\mathcal{B} . \tag{15}$$

Employing matrices E^+ and E^- defined in (A.3) yields:

$$\mathcal{B}'=\begin{bmatrix}1 & 0\\ 0 & E^-(\underline{\mathcal{R}}^*)(E^+(\underline{\mathcal{R}}^*))^T\end{bmatrix}\mathcal{B}=\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\mathcal{B}=\boldsymbol{R}\mathcal{B} . \tag{16}$$

The matrix ***R*** represents a 4×4 quaternion transformation in a four-dimension space. Since $\underline{\mathcal{R}}\otimes\underline{\mathcal{R}}^*=1$, matrices $\underline{\mathcal{R}}^+$ and $(\underline{\mathcal{R}}^-)^T$ are orthonormal and ***R*** is a linear operator with the property of leaving invariant quaternion norms. From (16) it is possible to separate a 3×3 rotation matrix:

$$R=E^-(\underline{\mathcal{R}}^*)(E^+(\underline{\mathcal{R}}^*))^T=(r_0^2-\boldsymbol{r}^T\boldsymbol{r})I+2(\boldsymbol{r}\boldsymbol{r}^T+r_0C(\boldsymbol{r})) . \tag{17}$$

Note that the above definition of R is consistent[1] with the definition of direction cosine matrix given in [1].

Now, one can apply the same concepts to a full quaternion $\mathcal{R}$ instead of the unit $\underline{\mathcal{R}}$. In this case equation (13) becomes:

$$\mathcal{B}'=\mathcal{R}\otimes\mathcal{B}\otimes\mathcal{R}^*. \tag{18}$$

Moreover, any full quaternion $\mathcal{R}$ can be factorized into the product of the norm and of the unit quaternion:

$$\mathcal{R}=|\mathcal{R}|\otimes\underline{\mathcal{R}}. \tag{19}$$

Then, using factorization (19), one can rewrite (18) by separating norm and rotational term as follows

$$\mathcal{B}'=|\mathcal{R}|^2\otimes\left(\underline{\mathcal{R}}\otimes\mathcal{B}\otimes\underline{\mathcal{R}}^*\right). \tag{20}$$

Expressing (20) in matrix form makes explicit two operations, norm amplification and rotation as in (15):

$$\mathcal{B}'=\mathcal{R}^+\left(\mathcal{R}^-\right)^T\mathcal{B}=|\mathcal{R}|^2\left[\underline{\mathcal{R}}^+\left(\underline{\mathcal{R}}^-\right)^T\right]\mathcal{B}. \tag{21}$$

Employing matrices E^+ and E^- defined in (A.3) yields:

$$\mathcal{B}'=|\mathcal{R}|^2\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\mathcal{B}=|\mathcal{R}|^2\boldsymbol{R}\mathcal{B}. \tag{22}$$

It is clear from previous equation, that while $\mathcal{B}$ is rotated as in (16), an amplification of the quaternion norm appears. Therefore, in case of full quaternions, product (18) applies two different transformations:

3. a magnification, by the factor $|\mathcal{R}|^2$, of the $\mathcal{B}$ norm;
4. a rotation of $\mathcal{B}$ by an angle θ around the axis $\boldsymbol{u}$ (as stated by Euler Theorem).

In the case $\mathcal{B}$ is a vector quaternion, the factorization (22) reduces to:

$$\begin{bmatrix}0\\ \boldsymbol{b}'\end{bmatrix}=|\mathcal{R}|^2\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\begin{bmatrix}0\\ \boldsymbol{b}\end{bmatrix}\Rightarrow\boldsymbol{b}'=\rho R\boldsymbol{b},\ \rho=|\mathcal{R}|^2. \tag{23}$$

The use of full quaternions allows to generalize the description of the motion of an object in the three-dimensional space: not only rotations but also translations can be parameterized.

III. Quaternion Kinematics and Dynamics

As stated in Section I, the goal of this paper is to rewrite the orbital dynamic and kinematic equations using full quaternions. To this end, first and second derivatives of a quaternion will be determined.

Let $\boldsymbol{r}_i$ and $\boldsymbol{r}_o$ be nonzero vectors which, according to Section II.A, can be considered as vector quaternions. Then, as in (18), it is possible to define a full quaternion $\mathcal{P}$ relating the vector $\boldsymbol{r}_o$ to the reference vector $\boldsymbol{r}_i$ through a rotation and a magnification:

$$\boldsymbol{r}_o=\mathcal{P}\otimes\boldsymbol{r}_i\otimes\mathcal{P}^*\Leftrightarrow\boldsymbol{r}_i=\left(\mathcal{P}\right)^{-1}\otimes\boldsymbol{r}_o\otimes\left(\mathcal{P}^*\right)^{-1}. \tag{24}$$

[1] Actually, the 3×3 matrix R in (17) is the transpose of the direction cosine matrix in [1], because the opposite rotation direction has been used.

A. Kinematics

Differentiating (24) yields:

$$\dot{\boldsymbol{r}}_o=\dot{\mathcal{P}}\otimes\left(\mathcal{P}\right)^{-1}\otimes\boldsymbol{r}_o+\boldsymbol{r}_o\otimes\left(\mathcal{P}^*\right)^{-1}\otimes\dot{\mathcal{P}}^*+\boldsymbol{q}_i,\quad \boldsymbol{q}_i=\mathcal{P}\otimes\dot{\boldsymbol{r}}_i\otimes\mathcal{P}^*. \tag{25}$$

Then, by taking the derivative of the product $\mathcal{P}\otimes\left(\mathcal{P}\right)^{-1}=1$, one can define the quaternion $\mathcal{W}$ as shown below:

$$\dot{\mathcal{P}}\otimes\left(\mathcal{P}\right)^{-1}=-\mathcal{P}\otimes\left(\dot{\mathcal{P}^{-1}}\right)=\mathcal{W}. \tag{26}$$

From the above definition the quaternion kinematic equation follows:

$$\dot{\mathcal{P}}=\mathcal{W}\otimes\mathcal{P}. \tag{27}$$

By factorizing $\mathcal{P}$ as in (19) and by remembering the definition (12), the previous equation develops into:

$$\mathcal{W}=\dot{|\mathcal{P}|}\big/|\mathcal{P}|+\dot{\underline{\mathcal{P}}}\otimes\underline{\mathcal{P}}^*. \tag{28}$$

It is possible to proof that $\dot{\underline{\mathcal{P}}}\otimes\underline{\mathcal{P}}^*$ is a vector quaternion (see the Appendix). Therefore, one can rewrite $\mathcal{W}$ as:

$$\mathcal{W}=\dot{|\mathcal{P}|}\big/|\mathcal{P}|+\dot{\underline{\mathcal{P}}}\otimes\underline{\mathcal{P}}^*=w_0+\boldsymbol{w}=w_0+\boldsymbol{w}_\perp+\boldsymbol{w}_\parallel, \tag{29}$$

where the decomposition of $\boldsymbol{w}$ into normal and parallel components $\boldsymbol{w}_\perp$ and $\boldsymbol{w}_\parallel$ with respect to $\boldsymbol{r}_o$ has been exploited. Substituting (29) into (25) enlightens that the derivative of $\boldsymbol{r}_o$ is unaffected by the parallel component $\boldsymbol{w}_\parallel$:

$$\begin{aligned}\dot{\boldsymbol{r}}_o&=\mathcal{W}\otimes\boldsymbol{r}_o+\boldsymbol{r}_o\otimes\mathcal{W}^*+\boldsymbol{q}_i=2w_0\boldsymbol{r}_o+2\boldsymbol{w}\times\boldsymbol{r}_o+\boldsymbol{q}_i=\\ &=2\mathcal{W}_\perp\otimes\boldsymbol{r}_o+\boldsymbol{q}_i,\ \mathcal{W}_\perp=w_0+\boldsymbol{w}_\perp\end{aligned}. \tag{30}$$

Since $\boldsymbol{r}_o$ is a vector quaternion, equation (23) applies:

$$\begin{bmatrix}0\\ \boldsymbol{r}_o\end{bmatrix}=|\mathcal{P}|^2\begin{bmatrix}1 & 0\\ 0 & R_P\end{bmatrix}\begin{bmatrix}0\\ \boldsymbol{r}_i\end{bmatrix}\Rightarrow\boldsymbol{r}_o=\rho_P R_P\boldsymbol{r}_i,\ \rho_P=|\mathcal{P}|^2, \tag{31}$$

where R_P is a rotation matrix. Comparing (30) with the first derivative of (31):

$$\dot{\boldsymbol{r}}_o=\left\{I\dot{\rho}_P/\rho_P+\left(\dot{R}_P R_P^T\right)\right\}\boldsymbol{r}_o+\rho_P R_P^T\dot{\boldsymbol{r}}_i=\Omega\boldsymbol{r}_o+\rho_P R_P^T\dot{\boldsymbol{r}}_i, \tag{32}$$

yields the following equalities:

$$2w_0=\dot{\rho}_P/\rho_P,\ 2\boldsymbol{w}\times=C\left(2\boldsymbol{w}\right)=\left(\dot{R}_P R_P^T\right)\Rightarrow\dot{R}_P=C\left(2\boldsymbol{w}\right)R_P. \tag{33}$$

One can recognize that $2\boldsymbol{w}$ represents the angular velocity in the three dimensional space (see kinematic equations of motion in [1]) and $2w_o$ represents the translation velocity along the $\boldsymbol{r}_o$ direction. The ensemble $(w_o+\boldsymbol{w})=\mathcal{W}$ forms a full quaternion referred to as *generalized angular velocity*. This term has been chosen because in the traditional attitude representation through unit quaternions the term w_o vanishes, and $\mathcal{W}$ becomes a pure angular velocity.

Rewriting (27) and (30) in matrix notations yields:

$$\begin{aligned}&\dot{\mathcal{P}}=\mathcal{W}^+\mathcal{P} && \dot{\boldsymbol{r}}_o=2\mathcal{W}_\perp^+\boldsymbol{r}_o+\mathcal{P}^+\left(\mathcal{P}^-\right)^T\dot{\boldsymbol{r}}_i\\ &\mathcal{W}^+=\begin{bmatrix}w_0 & -\boldsymbol{w}\\ \boldsymbol{w} & \Omega/2\end{bmatrix} && \mathcal{W}_\perp^+=\begin{bmatrix}w_0 & -\boldsymbol{w}_\perp^T\\ \boldsymbol{w}_\perp & \Omega_\perp/2\end{bmatrix}\end{aligned}, \tag{34}$$

where $\Omega_\perp$ equals Ω under the constraint $\boldsymbol{w}=\boldsymbol{w}_\perp$.

Remark. Quaternion kinematics (27) is more general than vector kinematics (30). Since the angular rate $\boldsymbol{w}$ is unconstrained, the former equation has four degrees of freedom (d.o.f.). In (30), the parallel component $\boldsymbol{w}_\parallel$ disappears, then for describing the $\boldsymbol{r}_o$ rotation, only the normal component $\boldsymbol{w}_\perp$ needs. This is equivalent to state that, in (30), an orthogonality constraint applies to $\boldsymbol{w}$. Then, d.o.f. reduce to three in agreement with classical mechanics. Therefore, equations (30) and (31) can be viewed as output equations of the state equation (27).

B. Dynamics

First define the *generalized angular acceleration* $\mathcal{A}$ as the derivative of the generalized angular rate $\mathcal{W}$:

$$\mathcal{A}=\dot{\mathcal{W}}\Rightarrow a_0+\boldsymbol{a}=a_0+\boldsymbol{a}_\perp+\boldsymbol{a}_\parallel=\dot{w}_0+\dot{\boldsymbol{w}}\,. \tag{35}$$

In (35), the decomposition of $\boldsymbol{a}$ into normal and parallel components with respect to $\boldsymbol{r}_o$ has been exploited.

Remark. Be aware that $\dot{\boldsymbol{w}}_\perp\neq\boldsymbol{a}_\perp$ and $\dot{\boldsymbol{w}}_\parallel\neq\boldsymbol{a}_\parallel$.

Quaternion dynamics follows by taking the derivative of quaternion kinematics (27):

$$\ddot{\mathcal{P}}=\dot{\mathcal{W}}\otimes\mathcal{P}+\mathcal{W}\otimes\dot{\mathcal{P}}=[\mathcal{A}+\mathcal{W}\otimes\mathcal{W}]\otimes\mathcal{P}=\mathcal{D}\otimes\mathcal{P}\,, \tag{36}$$

where the quaternion $\mathcal{D}$ gathers the effect of angular rate and acceleration. Scalar and vector parts of $\mathcal{D}$ are related to the components of $\mathcal{W}$ and $\mathcal{A}$ through:

$$\mathcal{D}=d_0+\boldsymbol{d}=\left(a_0+w_0^2-|\boldsymbol{w}|^2\right)+\left(\boldsymbol{a}+2w_0\boldsymbol{w}\right). \tag{37}$$

The second derivative of $\boldsymbol{r}_o$ can be obtained by exploiting (30) and (36):

$$\begin{aligned}\ddot{\boldsymbol{r}}_o&=\mathcal{D}\otimes\boldsymbol{r}_o+\boldsymbol{r}_o\otimes\mathcal{D}^*+2\mathcal{W}\otimes\boldsymbol{r}_o\otimes\mathcal{W}^*+\boldsymbol{q}=\\&=(2a_0)\boldsymbol{r}_o+(2w_0)^2\boldsymbol{r}_o+2\left[(2w_0)(2\boldsymbol{w})\times\boldsymbol{r}_o\right]+(2\boldsymbol{a})\times\boldsymbol{r}_o+\\&+(2\boldsymbol{w})\times\left[(2\boldsymbol{w})\times\boldsymbol{r}_o\right]+\boldsymbol{q},\quad \boldsymbol{q}=(2w_0)\boldsymbol{q}_i+(2\boldsymbol{w})\times\boldsymbol{q}_i+\dot{\boldsymbol{q}}_i\end{aligned}\,. \tag{38}$$

This expression has a clear similarity with the ordinary equation of the relative motion (see [2] or [5]). Therefore, a physical meaning can be assigned to each term in (38):

1 $\boldsymbol{q}$ represents the acceleration of the reference vector $\boldsymbol{r}_i$;
2 $(2a_0)\boldsymbol{r}_o+(2\boldsymbol{a})\times\boldsymbol{r}_o$ is the apparent acceleration of $\boldsymbol{r}_o$ with respect to $\boldsymbol{r}_i$. In particular: (i) $(2a_0)\boldsymbol{r}_o$ is the apparent acceleration along $\boldsymbol{r}_o$; (ii) $(2\boldsymbol{a})\times\boldsymbol{r}_o$ is the apparent acceleration along a normal direction to $\boldsymbol{r}_o$;
3 $2\left[(2\boldsymbol{w})\times(2w_0)\boldsymbol{r}_o\right]$ is the Coriolis acceleration;
4 $(2\boldsymbol{w})\times\left[(2\boldsymbol{w})\times\boldsymbol{r}_o\right]+(2w_0)^2\boldsymbol{r}_o$ is the centrifugal term.

Further, developing (38) shows that the acceleration of $\boldsymbol{r}_o$ does not depend on the parallel component $\boldsymbol{a}_\parallel$:

$$\begin{aligned}\ddot{\boldsymbol{r}}_o&=2\mathcal{D}_\perp\otimes\boldsymbol{r}_o+2\mathcal{W}\otimes\boldsymbol{r}_o\otimes\mathcal{W}^*+\boldsymbol{q}\\\mathcal{D}_\perp&=\mathcal{A}_\perp+\mathcal{W}\otimes\mathcal{W},\ \mathcal{A}_\perp=a_0+\boldsymbol{a}_\perp\end{aligned}\,. \tag{39}$$

Remark. Quaternion dynamics (36) is more general than vector dynamics (39). Since the angular acceleration $\boldsymbol{a}$ is unconstrained, the former equation has four d.o.f.. In (39), the parallel component $\boldsymbol{a}_\parallel$ disappears, showing an orthogonality constraint on $\boldsymbol{a}$, which corresponds to a d.o.f. reduction. Then, (39) has only three d.o.f., in agreement to with classical mechanics.

IV. Applications

Once obtained the general kinematic and dynamic equations of full quaternions, a step to be done is applying them to orbital motion. Consider a point P with mass m moving in the space, subject to a force $\boldsymbol{F}$. Two kinds of local reference frames, can be attached to the particle:

1 a *Local Orbital Reference Frame* (*LORF*), fixed to the velocity vector $\boldsymbol{v}$;
2 a *Local Vertical Local Horizontal* frame (*LVLH*), fixed to the position vector $\boldsymbol{r}$.

Both frames and their orientation with respect to an inertial reference are shown in Fig. 1. The inertial frame $\mathrm{R}=\{O,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ is a Cartesian reference with origin in O and unit vectors corresponding to $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ already introduced in (1). For each of the two orbital frames, the following problems will be solved:

1. complete definition of the frame axes;
2. assignment of a full quaternion to the frame;
3. formulation of the differential equation of the full quaternion, i.e.: orbital equations in quaternion form.

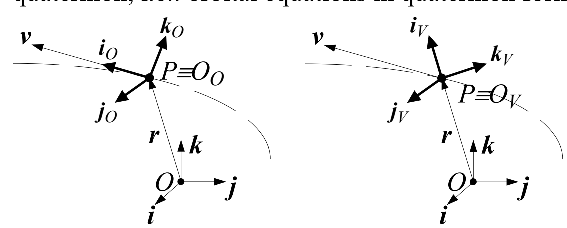

Fig. 1. LORF and LVLH with respect to the inertial frame.

A. LORF Reference Frame

The LORF $\mathrm{R}_O=\{O_O,\boldsymbol{i}_O,\boldsymbol{j}_O,\boldsymbol{k}_O\}$ is a Cartesian reference frame defined as follows:

1. the origin O_O coincides with P;
2. $\boldsymbol{i}_O$ lies along the velocity direction;
3. $\boldsymbol{j}_O$ is normal to the instantaneous orbit plane (defined by position and velocity);
4. $\boldsymbol{k}_O$ completes the frame.

$$\boldsymbol{i}_O=\boldsymbol{v}/|\boldsymbol{v}|,\ \boldsymbol{j}_O=(\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}\times\boldsymbol{v}|,\ \boldsymbol{k}_O=\boldsymbol{i}_O\times\boldsymbol{j}_O\,. \tag{40}$$

The velocity vector and the orientation of the LORF triple can be expressed through the *LORF quaternion* $\mathcal{R}_O$. The definition of $\mathcal{R}_O$ is arbitrary: for example the axis $\boldsymbol{i}$ rotates into $\boldsymbol{i}_O$ and the axis $\boldsymbol{j}$ rotates into $\boldsymbol{j}_O$:

$$\boldsymbol{v}\triangleq\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*,\quad \boldsymbol{j}_O\triangleq\underline{\mathcal{R}}_O\otimes\boldsymbol{j}\otimes\underline{\mathcal{R}}_O^*\,. \tag{41}$$

Because there exists an infinite number of rotations satisfying (41), a further constraint must be introduced: the right equation specifies that the $\boldsymbol{j}$-axis of the inertial frame must be rotated into the orbital plane normal direction.

Factorizing the left equation in (41) as in (20) enlightens the norm $|\mathcal{R}_O|$ of the LORF quaternion to be equal to the square root of the velocity modulus, and the unitary part $\underline{\mathcal{R}}_O$ to represent the orientation of the velocity unit vector with respect to the inertial frame:

$$\begin{gathered}\boldsymbol{v}=v\boldsymbol{n}_v=|\mathcal{R}_O|^2\left(\underline{\mathcal{R}}_O\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_O^*\right),\ \ \|\boldsymbol{n}_v\|=1\ \Rightarrow\\ \Rightarrow|\mathcal{R}_O|=\sqrt{v},\ \ \underline{\mathcal{R}}_O\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_O^*=\boldsymbol{n}_v\end{gathered}. \quad (42)$$

Now, one can apply formula (30) of quaternion kinematics to compute the acceleration $\dot{\boldsymbol{v}}$:

$$\dot{\boldsymbol{v}}=2w_{O,0}\boldsymbol{v}+2\boldsymbol{w}_O\times\boldsymbol{v}=2\mathcal{W}_{O\perp}\otimes\boldsymbol{v},\ \mathcal{W}_{O\perp}=w_{O,0}+\boldsymbol{w}_{O\perp}, \quad (43)$$

where the derivative of $\boldsymbol{i}$, being zero by definition, disappears and $\boldsymbol{w}_O$ has been decomposed into the normal and parallel components $\boldsymbol{w}_{O\perp}$ and $\boldsymbol{w}_{O\|}$ with respect to $\boldsymbol{v}$. The acceleration of the point mass is related to the force $\boldsymbol{F}$ through Newton's Law and remembering that $\boldsymbol{w}_{O\perp}\cdot\boldsymbol{v}=0$:

$$\boldsymbol{F}/m=\dot{\boldsymbol{v}}\Rightarrow w_{O,0}=\left(1/2m|\boldsymbol{v}|^2\right)\boldsymbol{v}\cdot\boldsymbol{F},\ \ \boldsymbol{w}_{O\perp}=\left(1/2m|\boldsymbol{v}|^2\right)\boldsymbol{v}\times\boldsymbol{F}. \quad (44)$$

Since $\boldsymbol{v}$ and $\boldsymbol{F}$ are vector quaternions, a more compact expression for the LORF angular rate can be used:

$$\mathcal{W}_O=\mathcal{W}_{O\perp}+\boldsymbol{w}_{O\|}=-(1/2m)(\boldsymbol{v}\otimes\boldsymbol{v}^*)^{-1}\boldsymbol{F}\otimes\boldsymbol{v}+\boldsymbol{w}_{O\|}. \quad (45)$$

Therefore, the orbital equations for LORF quaternion can be written in quaternion form:

$$\begin{cases}\dot{\mathcal{R}}_O=\mathcal{W}_O\otimes\mathcal{R}_O=\left(\dfrac{\boldsymbol{F}}{2m|\boldsymbol{v}|^2}\otimes\boldsymbol{v}+\boldsymbol{w}_{O\|}\right)\otimes\mathcal{R}_O, & \mathcal{R}_O(0)=\mathcal{R}_{O0}\\ \dot{\boldsymbol{r}}=\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*, & \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}, \quad (46)$$

or in matrix form (exploiting (11) and (A.2)) as follows:

$$\begin{cases}\dot{\mathcal{R}}_O=\mathcal{R}_O^-\mathcal{W}_O=\mathcal{R}_O^-\left[-\left(1/2m|\boldsymbol{v}|^2\right)\boldsymbol{v}^-\boldsymbol{F}+\boldsymbol{w}_{O\|}\right], & \mathcal{R}_O(0)=\mathcal{R}_{O0}\\ \dot{\boldsymbol{r}}=\mathcal{R}_O^+\left(\mathcal{R}_O^-\right)^T\boldsymbol{i}, & \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}. \quad (47)$$

Remark. As stated in Section III.A, the parallel component $\boldsymbol{w}_{O\|}$ does not give contribution to (43). This confirms the existence of an orthogonality constraint to $\boldsymbol{w}_O$, meaning that the four d.o.f. motion of the LORF quaternion is constrained in agreement with the classical mechanics three d.o.f. This is confirmed by Newton's Law, showing $\boldsymbol{w}_{O\|}$ to be completely independent on $\boldsymbol{F}$. The angular rate $\boldsymbol{w}_{O\|}$ affects only (46), and represents an angular rate of the unit vectors $\boldsymbol{j}_O$ and $\boldsymbol{k}_O$ around the axis $\boldsymbol{i}_O$. But if such vectors underwent a rotation, the LORF frame would be lost. Therefore the quaternion constraint $\boldsymbol{w}_{O\|}=0$ follows.

B. LVLH reference frame

The LVLH frame $\mathrm{R}_V=\{O_V,\boldsymbol{i}_V,\boldsymbol{j}_V,\boldsymbol{k}_V\}$ is a Cartesian reference defined as follows:

1. the origin O_V coincides with P;
2. $\boldsymbol{i}_V$ lies along the position direction;
3. $\boldsymbol{j}_V$ is normal to the instantaneous orbit;
4. $\boldsymbol{k}_V$ completes the frame.

$$\boldsymbol{i}_V=\boldsymbol{r}/|\boldsymbol{r}|,\ \ \boldsymbol{j}_V=(\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}\times\boldsymbol{v}|,\ \ \boldsymbol{k}_V=\boldsymbol{i}_V\times\boldsymbol{j}_V. \quad (48)$$

The position vector and the orientation of the LVLH triple can be expressed through the *LVLH quaternion* $\mathcal{R}_V$. In accordance with (41) it can be defined as:

$$\boldsymbol{r}\triangleq\mathcal{R}_V\otimes\boldsymbol{i}\otimes\mathcal{R}_V^*,\quad \boldsymbol{j}_V\triangleq\underline{\mathcal{R}}_V\otimes\boldsymbol{j}\otimes\underline{\mathcal{R}}_V^*. \quad (49)$$

Factorizing the left equation in (49) enlightens the norm $|\mathcal{R}_V|$ of the LVLH quaternion to be equal to the square root of the position modulus, and the unitary part $\underline{\mathcal{R}}_V$ to be the orientation of $\boldsymbol{r}$ with respect to the inertial frame:

$$\begin{gathered}\boldsymbol{r}=r\boldsymbol{n}_r=|\mathcal{R}_V|^2\left(\underline{\mathcal{R}}_V\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_V^*\right),\ \ \|\boldsymbol{n}_r\|=1\ \Rightarrow\\ \Rightarrow|\mathcal{R}_V|=\sqrt{r},\ \ \underline{\mathcal{R}}_V\otimes\boldsymbol{i}\otimes\underline{\mathcal{R}}_V^*=\boldsymbol{n}_r\end{gathered}. \quad (50)$$

Now, one can apply the formula (30) of quaternion kinematics to compute the velocity $\boldsymbol{v}$:

$$\dot{\boldsymbol{r}}=\boldsymbol{v}=2w_{V,0}\boldsymbol{r}+2\boldsymbol{w}_V\times\boldsymbol{r}=2\mathcal{W}_{V\perp}\otimes\boldsymbol{r}, \quad (51)$$

where the decomposition of $\boldsymbol{w}_V$ into normal and parallel components $\boldsymbol{w}_{V\perp}$ and $\boldsymbol{w}_{V\|}$ w.r.t. $\boldsymbol{r}$ has been exploited. The quaternion kinematics of the LVLH follows by (27):

$$\dot{\mathcal{R}}_V=\mathcal{W}_V\otimes\mathcal{R}_V, \quad (52)$$

and the LVLH dynamics follows from (36):

$$\ddot{\mathcal{R}}_V=\left[\mathcal{A}_V+\mathcal{W}_V\otimes\mathcal{W}_V\right]\otimes\mathcal{R}_V=\mathcal{D}_V\otimes\mathcal{R}_V. \quad (53)$$

Then, recalling (38) and (39), the acceleration can be determined as:

$$\begin{gathered}\ddot{\boldsymbol{r}}=\dot{\boldsymbol{v}}=(2a_{V,0})\boldsymbol{r}+(2w_{V,0})^2\boldsymbol{r}+2\left[(2w_{V,0})(2\boldsymbol{w}_V)\times\boldsymbol{r}\right]+\\ +(2\boldsymbol{a}_V)\times\boldsymbol{r}+(2\boldsymbol{w}_V)\times\left[(2\boldsymbol{w}_V)\times\boldsymbol{r}\right]=\\ =2\mathcal{D}_{V\perp}\otimes\boldsymbol{r}+2\mathcal{W}_V\otimes\boldsymbol{r}\otimes\mathcal{W}_V^*,\ \mathcal{D}_{V\perp}=\mathcal{A}_{V\perp}+\mathcal{W}_V\otimes\mathcal{W}_V\end{gathered}. \quad (54)$$

As done for LORF kinematics, one can relate acceleration expression to force $\boldsymbol{F}$ through Newton's Law. Taking the dot product between position and force yields:

$$a_{V,0}=\left\{(\boldsymbol{r}\cdot\boldsymbol{F})/m|\boldsymbol{r}|^2-\left[(2w_{V,0})^2-|2\boldsymbol{w}_{V\perp}|^2\right]\right\}/2. \quad (55)$$

The cross product between position and force brings to:

$$\boldsymbol{a}_{V\perp}=\frac{1}{2}\left\{\frac{\boldsymbol{r}\times\boldsymbol{F}}{m|\boldsymbol{r}|^2}-2(2w_{V,0})(2\boldsymbol{w}_{V\perp})-2\boldsymbol{w}_{V\|}\times2\boldsymbol{w}_{V\perp}\right\}. \quad (56)$$

Expressions (55) and (56) can be compacted into:

$$\begin{gathered}\mathcal{A}_V=\mathcal{A}_{V\perp}+\boldsymbol{a}_{V\|}=a_{V,0}+\boldsymbol{a}_{V\perp}+\boldsymbol{a}_{V\|}=-\left(1/2|\boldsymbol{r}|^2\right)\mathcal{Y}(\boldsymbol{F})+\boldsymbol{a}_{V\|}\\ \mathcal{Y}(\boldsymbol{F})=\frac{\boldsymbol{F}}{m}\otimes\boldsymbol{r}+2|\boldsymbol{r}|^2\mathcal{W}_V\otimes\mathcal{W}_V+2(\mathcal{W}_V\otimes\boldsymbol{r})\otimes(\boldsymbol{r}\otimes\mathcal{W}_V)^*\end{gathered}. \quad (57)$$

Finally, the orbital equations for LVLH quaternion can be written in quaternion form:

$$\begin{cases}\dot{\mathcal{R}}_V=\mathcal{W}_V\otimes\mathcal{R}_V, & \mathcal{R}_V(0)=\mathcal{R}_{V0}\\ \dot{\mathcal{W}}_V=-\left(1/2|\boldsymbol{r}|^2\right)\mathcal{Y}(\boldsymbol{F})+\boldsymbol{a}_{V\|}, & \mathcal{W}_V(0)=\mathcal{W}_{V0}\end{cases}. \quad (58)$$

Remark. As stated in Section III.B, the parallel component $\boldsymbol{a}_{V\|}$ does not give contribution to (54). This confirms the existence of an orthogonality constraint to $\boldsymbol{a}_V$, meaning that the four d.o.f. motion of the LVLH quaternion is constrained in agreement with the classical mechanics three d.o.f. Moreover, from Newton's Law, it follows that $\boldsymbol{a}_{V\parallel}$ is unforced by $\boldsymbol{F}$. The angular acceleration $\boldsymbol{a}_{V\parallel}$ affects only (58), and represents an angular acceleration of the unit vectors $\boldsymbol{j}_V$ and $\boldsymbol{k}_V$ around the axis $\boldsymbol{i}_V$. But if such vectors underwent a rotation, the LVLH frame would be lost. Therefore the constraint $\boldsymbol{a}_{V\parallel}=0$ follows.

C. Uniform Circular Motion

This section ends with a simple example: the uniform circular motion of *P* around *O*, sketched in Fig. 2.

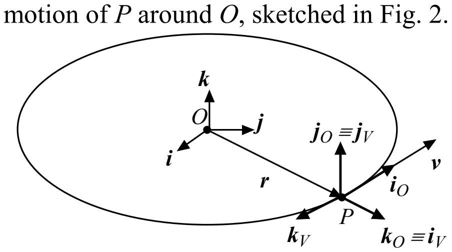

Fig. 2. Uniform circular motion around *O*

LVLH and LORF quaternion definitions are the same as in (49) and (41). First, quaternion kinematics is applied, starting from LVLH case. The generalized angular velocity of the LVLH quaternion is:

$$\mathcal{W}_V=\boldsymbol{w}_{V\perp},\ \boldsymbol{w}_{V\parallel}=0\,. \tag{59}$$

The generalized angular velocity is coincident with the angular rate of *P* around *O*, denoted with $\boldsymbol{\omega}=(\boldsymbol{r}\times\boldsymbol{v})/|\boldsymbol{r}|^2$. This leads to the next result showing quaternion kinematic equation to be similar to classical vector form:

$$\begin{cases}\dot{\boldsymbol{r}}=\boldsymbol{\omega}\times\boldsymbol{r}\\ \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}\quad \begin{cases}\dot{\mathcal{R}}_V=\boldsymbol{w}_V\otimes\mathcal{R}_V\\ \mathcal{R}_V(0)=\mathcal{R}_{V0}\end{cases}, \tag{60}$$

where $\boldsymbol{w}_V=\boldsymbol{\omega}/2$. By applying (36), quaternion dynamic equation can be obtained. Quaternion dynamics, like kinematics, looks similar to classical vector form:

$$\begin{cases}\dot{\boldsymbol{v}}=-|\boldsymbol{\omega}|^2\boldsymbol{r}\\ \boldsymbol{v}(0)=\boldsymbol{v}_0\end{cases}\quad \begin{cases}\ddot{\mathcal{R}}_V=[\boldsymbol{a}_V+\boldsymbol{w}_V\otimes\boldsymbol{w}_V]\otimes\mathcal{R}_V=-|\boldsymbol{w}_V|^2\mathcal{R}_V\\ \mathcal{R}_V(0)=\mathcal{R}_{V0}\end{cases}, \tag{61}$$

where $\boldsymbol{a}_V=0$ by definition of uniform motion.

By using LORF, dynamics is represented by quaternion kinematics, because the quaternion describes point velocity, instead of position. LORF dynamics is:

$$\begin{cases}\dot{\boldsymbol{v}}=\boldsymbol{\omega}\times\boldsymbol{v}\\ \boldsymbol{v}(0)=\boldsymbol{v}_0\end{cases}\quad \begin{cases}\dot{\mathcal{R}}_O=\boldsymbol{w}_O\otimes\mathcal{R}_O\\ \mathcal{R}_O(0)=\mathcal{R}_{O0}\end{cases}. \tag{62}$$

Kinematics follows from definition (41):

$$\begin{cases}\dot{\boldsymbol{r}}=\boldsymbol{v}\\ \boldsymbol{r}(0)=\boldsymbol{r}_0\end{cases}\quad \begin{cases}\dot{\boldsymbol{r}}=\mathcal{R}_O\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*\\ \boldsymbol{r}(0)=\mathcal{R}_O(0)\otimes\boldsymbol{i}\otimes\mathcal{R}_O^*(0)\end{cases}. \tag{63}$$

The last four equations show that the angular velocities of the LORF and LVLH quaternions are the same, namely $\boldsymbol{\omega}/2=\boldsymbol{w}_O=\boldsymbol{w}_V$. This follows from the fact that, for uniform circular motion, position and velocity are always orthogonal, then rotating with the same angular rate.

V. Conclusions and Future Developments

The orbit dynamics and kinematics for the point mass motion has been transformed from the classical vector notation into a new quaternion form. The LORF equations have been tested through MATLAB implementation. Among future developments, the design of quaternion observer and control will cover the most important role.

Appendix

A. Algebra - Conjugate Multiplication properties

When quaternion multiplication involves conjugates, commutative property (11) still hold. Then the product $\mathcal{C}=\mathcal{A}^*\otimes\mathcal{B}^*$ can be written in matrix notation through:

$$\mathcal{C}=(\mathcal{A}^*)^+\mathcal{B}^*=(\mathcal{B}^*)^-\mathcal{A}^*=(\mathcal{A}^+)^T\mathcal{B}^*=(\mathcal{B}^-)^T\mathcal{A}^*\,. \tag{A.1}$$

From previous equation it follows:

$$(\mathcal{A}^*)^+=(\mathcal{A}^+)^T \text{ and } (\mathcal{B}^*)^-=(\mathcal{B}^-)^T\,. \tag{A.2}$$

B. Algebra - Some Interesting Matrices

It is useful to introduce the following matrix notations:

$$\begin{aligned}E^+(\mathcal{X})&=\begin{bmatrix}-\boldsymbol{x} & (x_0 I+C(\boldsymbol{x}))\end{bmatrix}\\ E^-(\mathcal{X})&=\begin{bmatrix}-\boldsymbol{x} & (x_0 I-C(\boldsymbol{x}))\end{bmatrix}\end{aligned}. \tag{A.3}$$

By exploiting the new notation, the matrix expression of quaternions introduced in (10) can be rewritten as:

$$\begin{aligned}\mathcal{A}^+&=\begin{bmatrix}\mathcal{A}^* & (E^-(\mathcal{A}^*))^T\end{bmatrix}^T=\begin{bmatrix}\mathcal{A} & (E^-(\mathcal{A}))^T\end{bmatrix}\\ \mathcal{B}^-&=\begin{bmatrix}\mathcal{B}^* & (E^+(\mathcal{B}^*))^T\end{bmatrix}^T=\begin{bmatrix}\mathcal{B} & (E^+(\mathcal{B}))^T\end{bmatrix}\end{aligned}. \tag{A.4}$$

C. Kinematics - Generalized angular velocity

Rewriting the term $\dot{\underline{\mathcal{P}}}\otimes\underline{\mathcal{P}}^*$ in (26) by using the matrix notation (A.4), yields:

$$\boldsymbol{w}=\dot{\underline{\mathcal{P}}}^+\underline{\mathcal{P}}^*=(\underline{\mathcal{P}}^-)^T\dot{\underline{\mathcal{P}}}=\begin{bmatrix}\mathcal{P}\\ E^+(\underline{\mathcal{P}})\end{bmatrix}\dot{\underline{\mathcal{P}}}=\begin{bmatrix}0\\ E^+(\underline{\mathcal{P}})\dot{\underline{\mathcal{P}}}\end{bmatrix}. \tag{A.5}$$

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